CONNECTIVITY AND REDUCIBILITY OF GRAPHS

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1. Introduction. Corresponding to every graph, bipartite graph, or directed bipartite graph there exists a directed graph which is connected if and only if the original graph is connected.

In this paper, it is shown that for every directed graph there exists a certain bipartite graph such that the directed graph is connected if and only if the bipartite graph is irreducible. Other connections between reducibility and connectivity are established.

2. Definitions. A *directed graph* D is defined relative to a set V of vertices. An *edge* of D is an ordered pair of vertices (v_1, v_2), $v_1 \neq v_2$, $v_1 \in V$, $v_2 \in V$. If v_1, v_2 is any pair of vertices then there are various possibilities for the two ordered pairs (v_1, v_2) and (v_2, v_1). Either both are edges or one and only one is an edge, or neither is an edge. If (v_1, v_2) is an edge we say that it *connects* v_1 to v_2 (but not v_2 to v_1). A set of edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)$ ordered in such a way that the second member of any edge is the first member of its successor is a *chain* of edges connecting v_1 to v_n. A directed graph D is *connected* if and only if for every pair of vertices v_1, v_2 of D there exists a chain of edges connecting v_1 to v_2 and a chain connecting v_2 to v_1.

A *graph* G is defined relative to a set V of vertices. An edge of G is an unordered pair of elements $[v_1, v_2]$ with $v_1 \in V$, $v_2 \in V$. $v_1 \neq v_2$. A pair of vertices need not be an edge. If $[v_1, v_2]$ is an edge, we say that it *connects* the vertices v_1 and v_2. A set of edges $[v_1, v_2], [v_2, v_3] \ldots [v_{n-1}, v_n]$ ordered in such a way that every consecutive pair of edges has a vertex in common is a *chain* of edges connecting v_1 and v_n. A graph G is *connected* if and only if for every pair v_1, v_2 of vertices there exists a chain connecting v_1 and v_2.

For every graph G we define a corresponding directed graph $D(G)$ by agreeing that $D(G)$ has the same vertex set as G and that if the unordered pair $[v_1, v_2]$ is an edge of G then the ordered pairs (v_1, v_2) and (v_2, v_1) are edges of $D(G)$. Thus $D(G)$ has twice as many edges as G. $D(G)$ is connected if and only if G is connected.

A *bipartite* graph B has two sets of vertices S and T. An edge of B is an unordered pair $[s, t]$, $s \in S$, $t \in T$. An unordered pair $[s, t]$ need not be an edge. If $[s, t]$ is an edge we say that it connects the vertex s of S and the vertex t of T. A set of edges $[s_1, t_1], [s_2, t_1], [s_2, t_2] \ldots [s_n, t_n]$ ordered in such a way that every pair of consecutive edges has a vertex in common is a *chain*

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connecting s_1 and t_n. A bipartite graph B is *connected* if and only if, corresponding to every pair of vertices $s, t(s \in S, t \in T)$ there exists a chain of edges connecting s and t.

For every bipartite graph B we define a corresponding graph $G(B)$. The vertex set V of $G(B)$ is the union of the vertex sets S and T of B. The unordered pair $[s, t]$, $s \in S$, $t \in T$ is an edge of $G(B)$ if and only if $[s, t]$, $s \in S$, $t \in T$ is an edge of B. The graph $G(B)$ is connected if and only if the bipartite graph B is connected. Moreover, the directed graph $D(G(B))$ is connected if and only if B is connected.

A *directed bipartite* graph C has two vertex sets S and T. An edge is an ordered pair (a, b) with one of a and b belonging to S and the other to T. For every pair $s, t(s \in S, t \in T)$ there are various possibilities for the ordered pairs (s, t) and (t, s). Either both are edges of C or one and only one is an edge or neither is an edge. If (a, b) is an edge, it *connects* a to b (but not b to a). A set of edges (s_1, t_1), (t_1, s_2), $(s_2, t_2) \ldots (s_n, t_n)$ ordered in such a way that the last member of any edge is the first member of the next is a *chain* connecting s_1 to t_n. Similarly, a set of edges (t_1, s_1) $(s_1, t_2) \ldots (t_n, s_n)$ is a chain connecting t_1 to s_n. A directed bipartite graph C is *connected* if and only if, corresponding to every pair of vertices $s, t(s \in S, t \in T)$ there exists a chain connecting s to t and a chain connecting t to s.

For every directed bipartite graph C we define a corresponding directed graph $D(C)$. The vertex set V of $D(C)$ is the union of the vertex sets S and T of C. The ordered pair (a, b), $a \in V$, $b \in V$ is an edge of $D(C)$ if and only if (a, b) is an edge of C. The directed graph $D(C)$ is connected if and only if the directed bipartite graph C is connected.

A directed graph D with n vertices is *simply connected* if and only if D contains a subgraph H (that is, every edge of H is an edge of D) consisting of the n edges (v_1, v_2) (v_2, v_3), $(v_3, v_4) \ldots (v_n, v_1)$ in which $v_i \neq v_j$ for $i \neq j$. Such a subgraph H may be called a closed chain of rank n. Each of the n vertices of D is a first member of an edge of H and a second member of another. Every simply connected directed graph is connected.

We use $\bar{K}$ to denote the complement of a set K and $\nu(K)$ to denote the number of elements in K. We continue to use a round bracket ( ) to enclose an ordered pair and a square bracket [ ] to enclose an unordered pair.

In what follows we use the symbol L to denote a graph, a directed graph, a bipartite graph or a directed bipartite graph. A subgraph of the graph L is a graph of the same type as L, having as its vertex set some subset V' of the vertex set V of L and, as its set of edges, some subset of those edges of H for which both vertices belong to V'. We note that a subgraph which has a single vertex is connected, for in such a graph there are no pairs of vertices.

Let V_1 and V_2 be the vertex sets of two subgraphs S_1 and S_2 of the same graph L. The union of S_1 and S_2 is defined as the subgraph of L for which the vertex set is the union of the sets V_1 and V_2 and the set of edges is the union of the sets of edges of S_1 and S_2. The intersection of S_1 and S_2 is similarly defined. The subgraphs S_1 and S_2 are said to be disjoint if and only if V_1 and V_2 are disjoint.

If we select a vertex v of a graph L and form the union of all the connected subgraphs of L which have v as a vertex the resulting connected subgraph is called a maximal connected subgraph of L. Thus any connected graph L is a maximal connected subgraph. At the other extreme, a maximal connected subgraph may have a single vertex (and hence no edges). Any two maximal connected subgraphs of H are disjoint.

Let U be the union of all the maximal connected subgraphs of a graph L. The vertex sets of U and L are identical. If L is a directed graph D or a directed bipartite graph C then L may have edges which are not edges of U. However, if L is a graph G or a bipartite graph B, the sets of edges for L and U are identical and we have $\mathsf{L} = U$.

If a graph L is the union of two disjoint subgraphs S_1 and S_2 then there is no chain of edges connecting a vertex of S_1 to a vertex of S_2 and hence L is not connected. If a bipartite graph B or graph G is disconnected then it is the union of at least two disjoint maximal connected subgraphs. Thus a graph G or a bipartite graph B is connected if and only if it is not the union of two disjoint subgraphs.

As in **(2)** we use $U \times V$ to denote the bipartite graph with vertex sets U and V such that $[u, v]$ is an edge of $U \times V$ for all $u \in U$, $v \in V$.

An induced bipartite graph was defined in **(2**, § 2**)** for any partitioning of the vertex sets.

THEOREM 1. *If the vertex sets S and T of a bipartite graph B are partitioned so that $S = S_1 \cup S_2 \cup \ldots \cup S_k$ and $T = T_1 \cup T_2 \cup \ldots \cup T_k$ and if the subgraphs $(S_i \times T_i) \cap B$ are connected for $i = 1, 2, \ldots, k$, then B is connected if and only if the induced graph B' is connected.*

Proof. If B' is the union of two disjoint subgraphs then B is the union of two disjoint subgraphs.

If B is the union of two disjoint subgraphs B_1 and B_2 then each of the subgraphs $(S_i \times T_i) \cap B$ of B is a subgraph of B_1 or B_2, for, if not, such a subgraph is the union of 2 disjoint subgraphs and hence is not connected. Thus there exist complementary subsets λ, μ of $1, 2, \ldots, k$ such that B_1 has vertex sets $S_1 = \cup_{i\epsilon\lambda} S_i$, $T_1 = \cup_{i\epsilon\lambda} T_i$ and B_2 has vertex sets $S_2 = \cup_{i\epsilon\mu} S_i$, $T_2 = \cup_{i\epsilon\mu} T_i$. Let B_1' and B_2' be the induced graphs of B_1 and B_2 using these partitionings. Clearly B_1' and B_2' are disjoint subgraphs of B'.

Consider a bipartite graph with vertex sets S and T. A pair $[A, B]$ of subsets, $A \subset S$, $B \subset T$ is a *cover* or *exterior* pair of the bipartite graph if, for every edge $[s, t]$ either $s \in A$ or $t \in B$. One of A and B may be the null set ϕ. A cover $[A, B]$ is *minimum* if $\nu(A) + \nu(B)$ is minimum. If there exists a minimum cover $[A, B]$ in which $A \neq \phi$ and $B \neq \phi$, then the bipartite graph is *reducible*. If there exists a minimum cover $[A, B]$ in which either $A = \phi$

or $B = \phi$ and if there is no other minimum cover, then the bipartite graph is *semi-irreducible.* If $[S, \phi]$ and $[\phi, T]$ are minimum covers and if no other cover is minimum, then the bipartite graph is *irreducible.* If a bipartite graph is semi-irreducible, then $\nu(S) \neq \nu(T)$; if it is irreducible then $\nu(S) = \nu(T)$. A disconnected bipartite graph is the union of two disjoint subgraphs and hence is reducible. A minimal semi-irreducible graph was defined in **(2)** as a semi-irreducible bipartite graph which is not the union of two disjoint semi-irreducible subgraphs. If a semi-irreducible bipartite graph B is the union of two disjoint subgraphs, then these subgraphs must be semi-irreducible, for otherwise B would be reducible. Thus a minimal semi-irreducible bipartite graph is not expressible as the union of two disjoint subgraphs. It follows that every irreducible or minimal semi-irreducible graph is connected. Using Theorem 1, we have the following generalization of this result. The generalization refers to the core of the canonical decomposition of a bipartite graph as described in **(1)** and **(2)**.

THEOREM 2. *If B' is the bipartite graph induced by the partitioning of B by means of the vertex sets of the irreducible and minimal semi-irreducible subgraphs which constitute the core of B, then B' is connected if and only if B is connected.*

3. An induced directed graph. Let D be a directed graph with vertex set $V = (v_1, v_2, v_3, \ldots, v_n)$. If we partition V into r disjoint sets $V_1, V_2, \ldots, V_r$ we can define a directed graph D' which is *induced* by this partition. The vertices of D' are $V_1, V_2, \ldots, V_r$. The ordered pair (V_p, V_q); $p \neq q$ is an edge of D' if and only if there exists $v_i \in V_p$, $v_j \in V_q$ such that the ordered pair (v_i, v_j) is an edge of D.

THEOREM 3. *Let the vertex set V of a directed graph D with n vertices be partitioned into disjoint sets $V_1, V_2, \ldots, V_r$ in such a way that for every $p = 1, 2, \ldots, r$ either V_p consists of a single vertex v_j of V or the directed subgraph D_p of D consisting of all edges (v_k, v_h) such that $v_k \in V_p$ and $v_h \in V_p$ is connected. The graph D' induced by such a partitioning is connected if and only if D is connected.*

Proof. Let the vertices of D be $v_1, v_2, \ldots, v_n$. Suppose that D is connected. Consider any two vertices V_{i_1}, V_{i_2} of D'. There exists at least one vertex $v_{j_1} \in V_{i_1}$ and at least one vertex $v_{j_2} \in V_{i_2}$. Since D is connected there exists a chain $(v_{j_1}, v_{k_1})\ (v_{k_1}, v_{k_2}) \ldots (v_{k_p}, v_{j_2})$ connecting v_{j_1} to v_{j_2}. Now replace each vertex v_t which is a first or second member of an edge of this chain by the vertex of D' to which it belongs. This yields the set of ordered pairs

$$(V_{i_1}, V_{h_1})\ (V_{h_1}, V_{h_2}) \ldots (V_{h_p}, V_{i_2}).$$

If both vertices in any of these ordered pairs are identical, delete this ordered pair from the set. The resulting set of ordered pairs is a chain of edges of D'

connecting V_{i_1} to V_{i_2}. Similarly there exists a chain of edges connecting V_{i_2} to V_{i_1}.

Now suppose D' is connected. Let v_{j_1}, v_{j_2} be any pair of vertices of D. We must construct a chain connecting v_{j_1} to v_{j_2}. There exist unique vertices V_{i_1}, V_{i_2} of D' such that

$$v_{j_1} \in V_{i_1}, \qquad v_{j_2} \in V_{i_2}.$$

Since D' is connected there exists a chain

$$(V_{i_1}, V_{h_1}), (V_{h_1}, V_{h_2}) \ldots (V_{h_p}, V_{i_2}).$$

We have $v_{j_1} \in V_{i_1}$. Since (V_{i_1}, V_{h_1}) is an edge of D' there exists

$$v_{x_1} \in V_{i_1}, \qquad v_{y_1} \in V_{h_1}$$

such that (v_{x_1}, v_{y_1}) is an edge of D. If $v_{j_1} = v_{x_1}$ let (v_{x_1}, v_{y_1}) be the first edge of a chain. If not, since D_{i_1} is connected, there exists a chain of edges of D_{i_1} which connects v_{j_1} to v_{x_1}. Add the edge (v_{x_1}, v_{y_1}) to this chain. Since (V_{h_1}, V_{h_2}) is an edge of D' there exists

$$v_{z_1} \in V_{h_1}, \qquad v_{z_2} \in V_{h_2}$$

such that (v_{z_1}, v_{z_2}) is an edge of D. If $v_{y_1} = v_{z_1}$, (v_{y_1}, v_{z_2}) is the next edge of the chain. If not, since D_{h_1} is connected, there exists a chain of edges of D_{h_1} connecting v_{y_1} and v_{z_1}. Add these edges to the chain and then add (v_{z_1}, v_{z_2}). Continuing in this way we construct the chain of edges connecting v_{j_1} to v_{j_2}.

4. Connectivity and irreducibility. Corresponding to every directed graph D with vertex set $V = (v_1, v_2, \ldots, v_n)$ we define a bipartite graph $B(D)$ as follows. The vertex sets of $B(D)$ are $S = (s_1, s_2, \ldots, s_n)$ and $T = (t_1, t_2, \ldots, t_n)$, and $[s_i, t_j]$ is an edge of $B(D)$ if and only if (v_i, v_j) is an edge of D.

The graph $B^*(D)$ is defined as the bipartite graph with the same vertex sets S and T as $B(D)$. Every edge of $B(D)$ is an edge of $B^*(D)$ and in addition the n unordered pairs $[s_i, t_i]$, $i = 1, 2, \ldots, n$ are edges of $B^*(D)$. $B^*(D)$ will be called the *augmented bipartite* graph corresponding to the directed graph D. The edges $[s_i, t_i]$, $i = 1, 2, \ldots, n$ will be called the main diagonal edges of $B^*(D)$.

THEOREM 4. *If $B^*(D)$ is the augmented bipartite graph corresponding to the directed graph D then D is connected if and only if $B^*(D)$ is irreducible.*

Proof. If $B^*(D)$ is reducible there exist proper subsets A of S and B of T such that $[A, B]$ is a minimal exterior pair for $B^*(D)$. Let M be the subset of the indexes $1, 2, 3, \ldots, n$ for which $s_i \in A$ and N the subset of indexes for which $t_i \in B$. Since the edge $[s_i, t_i]$ is covered by $[A, B]$ for $i = 1, 2, 3, \ldots, n$, there exists no i such that $s_i \in \bar{A}$, $t_i \in \bar{B}$. It follows that if $s_i \in \bar{A}$, we have $t_i \in B$ and hence $\bar{M} \subset N$ and $\nu(\bar{A}) \leqslant \nu(B)$. Similarly $\nu(\bar{B}) \leqslant \nu(A)$.

Since $B^*(D)$ contains the subgraph consisting of $[s_i, t_i]$ for all i, its exterior dimension is n. Thus $\nu(A) + \nu(B) = n$ and $\nu(\bar{A}) + \nu(\bar{B}) = n$. Accordingly $\nu(\bar{A}) = \nu(B)$ and $\bar{M} = N$. M and N are complementary subsets of $1, 2, 3, \ldots, n$.

Let s_{i_1} and s_{i_2} be two vertices of $B^*(D)$ such that $s_{i_1} \in \bar{A}$ and $s_{i_2} \in A$. Since $i_1 \neq i_2$, the vertices v_{i_1} and v_{i_2} of D are distinct. If there exists a chain of edges of D,

$$(v_{i_1}, v_{j_1})\ (v_{j_1}, v_{j_2})\ (v_{j_2}, v_{j_3}) \ldots (v_{j_n}, v_{i_2}),$$

connecting v_{i_1} to v_{i_2}, the corresponding set of edges of $B^*(D)$ is

$$[s_{i_1}, t_{j_1}], [s_{j_1}, t_{j_2}], [s_{j_2}, t_{j_3}] \ldots [s_{j_n}, t_{i_2}].$$

Since $s_{i_1} \in \bar{A}$ and $[A, B]$ is a covering, $t_{j_1} \in B$. Since $j_1 \in N$ and $N = \bar{M}$, $s_{j_1} \in \bar{A}$. Repeating the argument we find $t_{j_2} \in B$, $s_{j_2} \in \bar{A}$ and finally $t_{i_2} \in B$ and hence $s_{i_2} \in \bar{A}$. This contradicts s_{i_2} belonging to A. It follows that D is not connected.

If $B^*(D)$ is irreducible, then for every vertex k of S or T, there are at least 2 vertices of the other set connected to k by a single edge. This enables us to form a chain

$$[s_{i_1}, t_{i_1}]\ ,[s_{i_1}, t_{i_2}]\ [s_{i_2}, t_{i_2}]\ [s_{i_2}, t_{i_3}] \ldots$$

in which every other edge is a main diagonal edge. We eventually find an index repeated. Thus we get a closed chain of $2r$ edges which involves r vertices in both S and T. Such a chain has been defined in **(1)** and **(2)** as a cycle of rank r. For definiteness, let the vertices of $B^*(D)$ and D be re-indexed so that the cycle is $[s_1, t_1], [s_1, t_2]\ [s_2, t_2], [s_2, t_3] \ldots [s_r, t_r], [s_r, t_1]$. The off-diagonal edges of this cycle, namely $[s_1, t_2], [s_2, t_3], \ldots, [s_r, t_1]$ correspond in D to the set of edges $(v_1, v_2), (v_2, v_3), (v_3, v_4) \ldots (v_n, v_1)$ which constitutes a closed chain in D.

Let $V_1 = (v_1, v_2, \ldots, v_r)$ and let D_1 be the directed graph which has vertex set V_1 and has as its set of edges those edges (v_i, v_j) of D such that $v_i, v_j \in V_1$. Since D_1 has the closed chain $(v_1, v_2), (v_2, v_3), \ldots, (v_n, v_1)$ as a subgraph, D_1 is simply connected and hence connected. Now consider $B^*(D)$, the corresponding augmented bipartite graph. $B^*(D)$ has vertex sets $S_1 = (s_1, s_2, \ldots, s_r)$ and $T_1 = (t_1, t_2, \ldots, t_r)$ and has as edges those edges $[s_i, t_j]$ of $B^*(D)$ such that $s_i \in S_1$ and $t_j \in T_1$. $B^*(D_1)$ has the cycle of rank r as a subgraph and hence by Theorem 2 of **(2)** is irreducible (in fact, simple irreducible). As outlined in § 4 of **(2)** it may be possible to find other cycles in $B^*(D)$ with half of their edges main diagonal edges, such that no two of these cycles have a vertex in common. Ultimately we find a partitioning of S into disjoint sets $S_1, S_2, \ldots, S_m$, of T into disjoint sets $T_1, T_2, \ldots, T_m$ and of V into disjoint sets $V_1, V_2, \ldots, V_m$ such that

$$\nu(S_1) = \nu(T_1) = \nu(V_1) > 1, \qquad \nu(S_k) = \nu(T_k) = \nu(V_k) \geqslant 1$$

for $k = 2, 3, \ldots, m$, and $s_i \in S_k$ if and only if $t_i \in T_k$ and $t_i \in T_k$ if and only if $v_i \in V_k$. For $k = 2, 3, \ldots, m$ there are two possibilities. The first is that V_k consists of a single vertex v_i in which case S_k and T_k are vertex sets for a bipartite subgraph of D with a single edge $[s_k, t_k]$. The second possibility is that we have a subgraph D_k of D which has V_k as vertex set and has as edges those edges (v_i, v_j) of D such that $v_i \in V_k$, $v_j \in V_k$. The corresponding augmented bipartite graph $B^*(D_k)$ has vertex sets S_k, T_k. D is a connected directed graph and $B^*(D_k)$ is irreducible.

Let D' be the directed graph induced by the partition $V_1, V_2, \ldots, V_k$ and let $[B^*(D)]'$ be the bipartite graph induced by the partitions $S_1, S_2, \ldots, S_k$; $T_1, T_2, \ldots, T_k$. It is immediate that $[B^*(D)]' = B^*(D')$. By Theorem 3 **(2)** since $B^*(D)$ is irreducible, $B^*(D')$ is irreducible. Clearly as in **(2)** there exists an integer s such that, if we carry out the process of partitioning and forming the induced graph s times, the augmented bipartite graph $B^*(D^{(s)})$ is found to contain a cycle of rank equal to the number of vertices in each vertex set. Thus $B^*(D^{(s)})$ is simple irreducible. It follows that $D^{(s)}$ has a closed chain of rank equal to the number of vertices in its vertex set. Thus $D^{(s)}$ is simply connected and hence is connected. By Theorem 1, D is connected if and only if D' is connected, D' is connected if and only if $D^{(2)}$ is connected and finally $D^{(s-1)}$ is connected if and only if $D^{(s)}$ is connected. Since $D^{(s)}$ is connected, D is connected.

Combining the introductory remarks of § 2 with this theorem we see that a graph G is connected if and only if $B^*[\mathrm{D}(G)]$ is irreducible, a directed bipartite graph C is connected if and only if $B^*[D(C)]$ is irreducible and a bipartite graph B is connected if and only if $B^*[D\{G(B)\}]$ is irreducible.

In § 4 of **(2)** the authors outlined a method of determining whether or not a bipartite graph with the same number n of vertices in S and T is irreducible. In this method it is necessary, first of all, to determine the exterior dimension. If the exterior dimension is less than n the graph is reducible. If the exterior dimension is n it is necessary to find a diagonal, that is, a subgraph having n edges and exterior dimension n. For a bipartite graph $B^*(D)$, the exterior dimension is n and a diagonal is $(s_1, t_1), (s_2, t_2), \ldots, (s_n, t_n)$. Starting with this diagonal, and using the method of § 4 **(2)**, a computational procedure can be programmed for determining irreducibility and hence the connectivity of any type of graph which we have considered.

The following examples are interesting.

Example 1. It is possible for $B^*(D)$ to be connected without D being connected. For let D have vertex set $V = (v_1, v_2, v_3)$ and edges (v_1, v_2) (v_1, v_3). D is not connected. $B^*(D)$ has vertex sets $S = (s_1, s_2, s_3)$, $T = (t_1, t_2, t_3)$ and edges $[s_1, t_1]$ $[s_2, t_2]$ $[s_3, t_3]$ $[s_1, t_2]$ $[s_1, t_3]$. $B^*(D)$ is connected. $B^*(D)$ is, of course, reducible.

Example 2. Let the bipartite graph B have vertex sets $V_1 = (v_1, v_2, v_3)$ and $V_2 = (v_4, v_5, v_6)$ and edges $[v_1, v_5]$ $[v_3, v_5]$ $[v_3, v_4]$ $[v_2, v_4]$ $[v_2, v_6]$. B is connected and reducible. $G(B)$ has vertex set $V = (v_1, v_2, v_3, v_4, v_5, v_6)$ and is connected as is $D\{G(B)\}$. $B^*[D\{G(B)\}]$ has vertex sets $S_1 = (s_1, s_2, s_3, s_4, s_5, s_6)$ and $T = (t_1, t_2, t_3, t_4, t_5, t_6)$ and edges $[s_1, t_5]$ $[s_5, t_1]$ $[s_3, t_5]$ $[s_5, t_3]$ $[s_3, t_4]$ $[s_4, t_3]$ $[s_2, t_4]$ $[s_4, t_2]$ $[s_2, t_6]$ $[s_6, t_2]$ together with the edges $[s_i, t_i]$, $i = 1, 2, 3, 4, 5, 6$. B^* is irreducible.

Example 3. Let the directed bipartite graph C have vertex sets $V_1 = (v_1, v_2)$ and $V_2 = (v_3)$ and edges (v_1, v_3) (v_3, v_1) (v_2, v_3). C is not connected. The directed graph $D(C)$ has vertex set $V = (v_1, v_2, v_3)$ and is not connected. $B^*[D(C)]$ has vertex sets $S = (s_1, s_2, s_3)$, $T = (t_1, t_2, t_3)$ and edges (s_1, t_3) (s_3, t_1) (s_2, t_3) (s_1, t_1) (s_2, t_2) (s_3, t_3). $B^*[D(C)]$ is reducible.

5. The class of D(B) graphs. Let us call a bipartite graph with n vertices in each vertex set, an $n \times n$ bipartite graph. Corresponding to an $n \times n$ bipartite graph B with vertex sets $S = (s_1, s_2, \ldots, s_n)$ and $T = (t_1, t_2, t_3, \ldots, t_n)$ we define a directed graph $D(B)$ with vertex set $V = (v_1, v_2, \ldots, v_n)$ by agreeing that (v_i, v_j) is an edge of $D(B)$ if and only if $[s_i, t_j]$ is an edge of B and $i \neq j$. $D(B)$ is not uniquely determined by B but is also a function of the indexing of the vertices of S and T. We see at once that, for any directed graph D, we have $D\{B^*(D)\} = D$. Also, for any $n \times n$ bipartite graph B, we have $B^*\{D(B)\} = B$ if and only if $[s_i, t_i]$ is an edge of B for $i = 1, 2, \ldots, n$. However, since every edge of B is an edge of $B^*\{D(B)\}$ it follows that if B is irreducible, $B^*\{D(B)\}$ is irreducible and hence $D(B)$ is connected.

If we re-index the vertex set $S = (s_1, s_2, \ldots, s_n)$ of an $n \times n$ bipartite graph B leaving T unaltered, the irreducibility and connectivity of B are unchanged but the structure and connectivity of $D(B)$ may be altered. For consider the 4×4 bipartite graph B with edges $[s_1, t_2]$ $[s_4, t_3]$ $[s_3, t_4]$ $[s_2, t_1]$. B is disconnected and reducible and $D(B)$ is disconnected. However, if we re-index S so that $s_1' = s_1$, $s_2' = s_4$, $s_3' = s_3$, $s_4' = s_2$ then B has edges $[s_1', t_2]$ $[s_2', t_3]$, $[s_3', t_4]$ and $[s_4', t_1]$ and $D(B)$ is connected.

If the exterior dimension of an $n \times n$ bipartite graph is n, there exists a diagonal consisting of n edges and we can index B in such a way that this diagonal is the main diagonal consisting of the n edges $[s_i, t_i]$, $i = 1, 2, \ldots, n$. With this indexing $B = B^*\{D(B)\}$ and hence B is irreducible if and only if $D(B)$ is connected.

We have proved the following theorem.

THEOREM 5. *If B is an irreducible $n \times n$ bipartite graph then $D(B)$ is connected for every indexing of B.*

If B is an $n \times n$ bipartite graph then either the exterior dimension of the graph is less than n so that B is reducible, or it is possible to index the vertex sets S and T in such a way that B is irreducible if and only if $D(B)$ is connected.

6. The canonical decomposition. In this section we consider the canonical decomposition of the bipartite graph $B^*(D)$ corresponding to a directed graph D Let the k disjoint irreducible subgraphs in the core of $B^*(D)$ be denoted by $G_1, G_2, \ldots, G_k$. Corresponding to G_i, $i = 1, 2, \ldots, k$ there exists a unique subgraph D_i of D such that $B^*(D_i) = G_i$. These directed subgraphs $D_1, D_2, \ldots, D_k$ are disjoint. By Theorem 3, since $B^*(D_i)$ is irreducible, the directed graph D_i is connected for $i = 1, 2, \ldots, k$. To see that $D_1, D_2, \ldots, D_k$ are maximal connected, consider any connected subgraph E of D. The set of edges of the main diagonal of $B^*(E)$ is a subset of the main diagonal of $B^*(D)$. Since $B^*(E)$ is irreducible it is a subgraph of G_i for some i and hence E is a subgraph of D_i for some i. Accordingly, we have the following theorem.

THEOREM 6. *For any directed graph D, let the core of $B^*(D)$ consist of $G_1, G_2, \ldots, G_k$. If D_i, $i = 1, 2, \ldots, k$ is the unique subgraph of D such that $B^*(D_i) = G_i$, then $D_1, D_2, \ldots, D_k$ are the disjoint maximal connected subgraphs of D.*

If $B^*(D)$ has no inadmissible edges, D is the union of $D_1, D_2, \ldots, D_k$ and $B^*(D)$ is the union of $G_1, G_2, \ldots, G_k$. In such cases, the method outlined in § 4 of **(2)** can be applied to $B^*(D)$, using the main diagonal, to construct the disjoint maximal connected subgraphs $D_1, D_2, \ldots, D_k$ of D. There are many such cases, for if $B^*(D)$ is symmetrical about the main diagonal (that is, if for all pairs (i, j), $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, $i \neq j$, (v_i, v_j) is an edge of D if and only if (v_j, v_i) is an edge of D) then, considering the canonical decomposition of $B^*(D)$, we see that $B^*(D)$ has no inadmissible edges. Such symmetry occurs if D is of the type $D(G)$ or $D\{G(B)\}$.

Consider any bipartite graph B and let $G_1, G_2, \ldots, G_k$ be the irreducible subgraphs of B^* which constitute its core. To each G_i there corresponds a unique subgraph B_i of B such that $G_i = B^*[D\{G(B_i)\}]$. By Theorem 3, the subgraphs $B_1, B_2, \ldots, B_k$ are connected. They are the disjoint maximal connected subgraphs of B. If B_i has no inadmissible edges then it is identical with one of the irreducible or minimal semi-irreducible subgraphs which constitute the core of B. If B_i has inadmissible edges than these edges are inadmissible in B also, and the core of B_i is the union of two or more of the irreducible or minimal semi-irreducible subgraphs of the core of B.

Finally, consider an augmented bipartite graph of the type $B^*(D)$ or $B^*[D(C)]$. Such a B^* is not in general symmetrical about the main diagonal, and hence, although its core consists of k disjoint irreducible subgraphs, these subgraphs need not be symmetrical about the main diagonal. Moreover such a B^* may have inadmissible edges. As in § 4, let the vertex set of D (or $D(C)$) be $V = (v_1, v_2, \ldots, v_n)$ and let the vertex sets of B^* be $S = (s_1, s_2, \ldots, s_n)$ and $T = (t_1, t_2, \ldots, t_n)$. As in **(2)** § 3, let the k irreducible subgraphs of the core of B^* have vertex sets S_i and T_i, $i = 1, 2, \ldots, k$ and index the subgraphs D_i of D so that D_i corresponds to the irreducible subgraph $G_i = (S_i \times T_i) \cap B^*$ of the core of B^*. Since the edges of the main diagonal of B^* belong to the core of B^*, we see that if S_i consists of $s_{i_1}, s_{i_2}, \ldots, s_{i_p}$ then T_i consists of $t_{i_1}, t_{i_2}, \ldots, t_{i_p}$. The set of inadmissible edges of B^* is the union of the sets of edges of the subgraphs $(S_i \times T_j) \cap B^*$ for all i, j such that $i > j$. Let D_{ij} $(i > j)$ be the subgraph of D (or $D(C)$) corresponding to the subgraph $(S_i \times T_j) \cap B^*$ of B^*. If $S_i = (s_{i1}, s_{i2}, \ldots, s_{ip})$ and $T_j = (t_{j1}, t_{j2}, \ldots, t_{jp})$ then D_{ij} has $p + q$ distinct vertices

$$v_{i1}, v_{i2}, \ldots, v_{ip}, \; v_{j1}, v_{j2}, \ldots, v_{jq}.$$

Every edge of D_{ij} connects one of

$$v_{i1}, v_{i2}, \ldots, v_{ip}$$

to one of

$$v_{j1}, v_{j2}, \ldots, v_{jq}.$$

The subgraphs $D_{ij}(i > j)$ define a partial ordering of the subgraphs $D_1, D_2, \ldots, D_k$ if we agree that $D_i < D_j$ provided D_{ij} has at least one edge. This partial ordering is consistent with the total ordering $D_1 < D_2 < \ldots < D_k$.

7. A connection with ordering relations. As an example consider a finite set $A = (a_1, a_2, a_3, \ldots, a_n)$ and a binary relation $R(a_i, a_j)$ connecting certain ordered pairs (a_i, a_j). We say that R is consistent with an ordering relation if for every $r = 2, 3, 4, \ldots, n$, there do not exist elements $a_{q1}, a_{q2}, \ldots, a_{q_r}$ such that

$$R(a_{q1}, a_{q2}), R(a_{q2}, a_{q3}), \ldots, R(a_{q_{r-1}}, a_{q_r}), R(a_{q_r}, a_{q1})$$

all hold. Any relation consistent with an ordering relation can be extended to a total ordering.

Define D as the directed graph with vertex set $V = (v_1, v_2, \ldots, v_n)$ such that (v_i, v_j) is an edge of D if and only if $R(a_i, a_j)$. It is easily seen that R is consistent with an ordering if and only if the core of $B^*(D)$ consists of n irreducible subgraphs each consisting of a single edge (s_i, t_i) $i = 1, 2, \ldots, n$. For if R is consistent with an ordering and one of the blocks of the core contained at least two edges, then D would have a connected subgraph with at least two vertices v_i and v_j. This implies there is a chain connecting v_i and v_j viz. $v_i = v_{q1}, v_{q2}, \ldots, v_{qt} = v_j$ and a chain connecting v_j and v_i namely

$$v_j = v_{qt}, v_{qt+1}, \ldots, v_{q_r} = v_{q1} = v_i.$$

But then

$$R(a_{q1}, a_{q2}), R(a_{q2}, a_{q3}), \ldots, R(a_{q_r}, a_{q1}),$$

a contradiction. Conversely, let the core of $B^*(D)$ consist of n irreducible subgraphs (s_i, t_i). In the notation introduced in **(2)** to describe the canonical decomposition, if $G_p = (S_p \times T_p) \cap B^*(D)$ consists of the single edge (s_{i_p}, t_{i_p}) for $p = 1, 2, \ldots, n$ then (s_{i_p}, t_{i_q}) is an edge of $B^*(D)$ only if $q > p$. The total ordering

$$a_{i1} < a_{i_2} < a_{i3} \ldots < a_{i_n}$$

is consistent with the relation R, that is, whenever $R(a_i, a_j)$, then $a_i < a_j$.

More generally, let R be any relation and let the canonical decomposition of $B^*(D)$ consist of k irreducible subgraphs $G_p = (S_p \times T_p) \cap B^*(D)$ for $p = 1, 2, 3, \ldots, k$ and let n_p be the number of vertices in each vertex set of G_p. If all the edges (s_i, t_j) of G_p which are below the diagonal are deleted from the graph $B^*(D)$ and if the relation R is modified by deleting the corresponding $R(a_i, a_j)$ then the modified relation R is consistent with an ordering. Thus minimal number of pairs which must be deleted from an arbitrary binary relation R to make it consistent with an ordering is not more than

$$\sum_{p=1}^{k} \frac{n_p^2 - n_p}{2}.$$

The actual minimum number of such pairs could be found as follows. In each subgraph G_p permute the rows and columns in such a way that the main diagonal is unaltered but the number of edges below the main diagonal is minimal. Let this minimum number be m_p. The minimum number of pairs which must be deleted to convert R to a relation consistent with an ordering is $\sum_{p=1}^{k} m_p$.

8. Applications. The results of this paper can be used in the construction of an algorithm which yields the complete canonical decomposition of an n by m bipartite graph. This algorithm can be applied to the optimal assignment problem to determine the dimension of the space of dual solutions and to reduce the problem of finding all dual solutions to the case when there is only one primal solution.

The results can be applied also in investigating the structure of powers of matrices with non-negative elements and in determining properties of characteristic roots of such matrices with particular reference to stochastic matrices. Again, stochastic matrices form a semi-group under multiplication. Graphical concepts can be used to study the ideals of this semi-group and in particular the structure of finitely generated ideals.

Another application is to the study of matrices with non-negative entries with assigned row and column sums. Such matrices form a convex set. An algorithm can be found for expressing any such matrix as an average of vertex matrices.

This work will be described in subsequent publications by two of the present authors.

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