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CHARACTERIZATIONS OF A GENERALIZED NOTION OF COMPACTNESS

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Abstract

This paper gives theorems which encompass known characterizations of many of the generalized compactness properties.

Introduction

Several generalized compactness properties of topological spaces have been characterized by analogues of the following theorems for compactness:

*) A space X is compact if and only if the projection $\pi_y: X \times Y \to Y$ is closed for every space Y, Mrówka (1959), Hanai (1961), Franklin and Sorgenfrey (1966), Scarborough (1969).

**) A T_1 space Y is compact if and only if each function into Y with a closed graph (closed-graph function) is continuous, Franklin and Sorgenfrey (1966), Scarborough (1969), Kasahara (1973).

It has also been shown that m-compact spaces have characterizations with these forms, Hanai (1961), Franklin and Sorgenfrey (1966).

More recently, the generalized compactness properties, Lindelöf, m-Lindelöf, H-closed and nearly-compact have been shown to have characterizations with these forms, Joseph (submitted for publication), Herrington and Long (1975), Joseph (to appear).

Also of recent interest has been the class of (m, n)-compact spaces (Hodel and Vaughan (1974), Vaughan (1975)). In this paper, we establish that (m, n)-compact spaces have characterizations with forms (*) and (**). The class of (m, n)-spaces dates back to the work of Aleksandrov and Urysohn (1929) and has been extensively studied by Smirnov (1950) and independently by Gál (1957, 1958).

Preliminary definitions and remarks

A space is defined to be (m, n)-compact for the cardinals m and n if each open covering θ of the space with $|\theta|$ (the cardinality of θ) at most n contains a subcovering of the space with cardinality at most m. Using the convention that 1 stands for "finitely many", ∞ stands for "arbitrary cardinality" and ω stands for the first infinite cardinal, we see readily that the $(1, \infty)$-compact, $(1, m)$-compact, (ω, ∞)-compact, and (m, ∞)-compact spaces are respectively the compact, m-compact, Lindelöf and m-Lindelöf spaces. Gál (1958) called a filter on a set X an (m, n)-filter if the filter has a base with cardinality at most n and with the m-intersection property; he proved that a space X is (m, n)-compact if and only if every (m, n)-filter adheres to some point in X. We make the following definition.

DEFINITION. *A topological space is an* (m, n)-space *if* 1) *at each point of the space, there is an open set base of cardinality at most n, and* 2) *the intersection of any collection of at most m open subsets of the space is open.*

We will denote the closure of a subset K of a topological space by $\mathrm{cl}(K)$.

Main results

We now give our main results.

THEOREM 1. *A topological space X is an (m, n)-compact space if and only if $\pi_y: X \times Y \to Y$ is a closed function for every (m, n)-space Y.*

PROOF. Let $K \subset X \times Y$ be closed; let $y \in \mathrm{cl}[\pi_y(K)]$. Let θ be an open set base at y with $|\theta| \leq n$. Then $\{\mathrm{cl}[\pi_x(K \cap (X \times V))]: V \in \theta\}$ is a base for an (m, n)-filter on X. Let $x \in X$ satisfy $x \in \mathrm{cl}[\pi_x(K \cap (X \times V))]$ for each $V \in \theta$. It is not difficult to show that $(x, y) \in \mathrm{cl}(K) = K$ and consequently that $y \in \pi_y(K)$ to complete one part of the proof. For the converse, let $\mathscr{B}$ be a base for an (m, n)-filter on X and let $y_0 \notin X$. Let $Y = X \cup \{y_0\}$ with the topology generated by $\{\{y\}: y \in X\} \cup \{B \cup \{y_0\}: B \in \mathscr{B}\}$. Then Y is an (m, n)-space. Let $K = \{(y, y): y \in X\}$. Then $y_0 \in \mathrm{cl}(\pi_y(K)) = \pi_y[\mathrm{cl}(k)]$. Let $x \in X$ with $(x, y_0) \in \mathrm{cl}(K)$ and let V be open in X about x. Then $[V \times (Y - V)] \cap K = \emptyset$, so $Y - V$ is not open in Y. Thus $B \cap V \neq \emptyset$ is satisfied for each $B \in \mathscr{B}$ and $\mathscr{B}$ adheres to x. This completes the proof.

THEOREM 2. *A T_1 space Y is (m, n)-compact if and only if all closed-graph functions (bijections) from (m, n)-spaces to Y are continuous.*

PROOF. Let X be an (m, n)-space, suppose Y is (m, n)-compact and assume $g: X \to Y$ is a closed-graph function. If $G(g)$ is the graph of g and $K \subset Y$ is closed, then $\pi_y^{-1}(K) \cap G(g)$ is closed in $X \times Y$ and $g^{-1}(K) = \pi_x[\pi_y^{-1}(K) \cap G(g)]$ which is closed in X by Theorem 1; so one part of the theorem is proved. For the converse, suppose all closed-graph functions from (m, n)-spaces to the T_1 space Y are continuous, let $x_0 \in Y$, and let $\mathscr{B}$ be a

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base for an (m, n)-filter on Y which fails to adhere to any point in $Y - \{x_0\}$. Let $X = Y$ with the topology generated by $\{\{x\}: x \in X = \{x_0\}\} \cup \{B \cup \{x_0\}: B \in \mathscr{B}\}$ as base. Let $i: X \to Y$ be the identity function and let $(x, y) \in (X \times Y) - G(i)$. If $x \neq x_0$, then $\{x\}$ is open in X, $Y - \{x\}$ is open about y in Y and $(\{x\} \times (Y - \{x\})) \cap G(i) = \varnothing$, so $(x, y) \notin \mathrm{cl}(G(i))$. If $x = x_0$, then $y \neq x_0$, so there is a V open in Y about y and a $B \in \mathscr{B}$ satisfying $B \cup \{x_0\} \subset X - V$. So $X - V$ is open in X about x_0 and $((X - V) \times V) \cap G(i) = \varnothing$; thus $(x, y) \notin \mathrm{cl}(G(i))$, i is a closed-graph bijection and i is continuous since X is an (m, n)-space. Let V be open in Y about x_0. Then $i^{-1}(V)$ is open in X about x_0, so there is a $B \in \mathscr{B}$ satisfying $B \subset V$ and $\mathscr{B} \to x_0$. This completes the proof.

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