Proceedings of the Edinburgh Mathematical Society (1996) **39**, 525–533 ©

OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS

by BING LIU and JURANG YAN

(Received 9th November 1994)

In this paper we are dealing with oscillatory and asymptotic behaviour of solutions of second order nonlinear difference equations of the form

$$\Delta(r_{n-1}\Delta x_{n-1})+F(n,x_n)=G(n,x_n,\Delta x_n),\, n\in N(n_0). \tag{E}$$

Some sufficient conditions for all solutions of (E) to be oscillatory are obtained. Asymptotic behaviour of nonoscillatory solutions of (E) is considered also.

1991 *Mathematics subject classification*: Primary 39A10.

1. Introduction

Recently, there has been a lot of interest in the oscillation and nonoscillation of second order difference equations. See, for example, [1–6] and the references cited therein. In this paper, we consider the second order nonlinear difference equation of the form

$$\Delta(r_{n-1}\Delta x_{n-1})+F(n,x_n)=G(n,x_n,\Delta x_n), \tag{E}$$

where $n\in N(n_0)=\{n_0,n_0+1,n_0+2,\ldots\}$ (n_0 is a fixed non-negative integer) and Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$. Moreover, F and G are real-valued functions with $x\colon N(n_0)\to\mathbf{R}$, $r\colon N(n_0)\to(0,+\infty)$, $F\colon N(n_0)\times R\to R$ and $G\colon N(n_0)\times\mathbf{R}^2\to\mathbf{R}$.

The purpose of this paper is to establish some new results on the oscillatory and asymptotic behaviour of solutions of (E). Our results differ greatly from those in [1–6] and the known literature.

As is customary (see [3], [4] and [6]), a nontrivial solution $\{x_n\}$ of (E) is said to be oscillatory if for every $N>0$ there exists a $k\geqq N$ such that $x_kx_{k+1}\leqq 0$. Otherwise the solution is called nonoscillatory.

In this paper, we further assume that the following conditions hold:

(H) There exist sequences $\{f(n)\}$, $\{g(n)\}$ and ratio m of two odd integers such that for all sufficiently large n

$$\frac{F(n,u)}{u^m} \geqq f(n) \quad \text{for } u \neq 0,$$

and

$$\frac{G(n,u,v)}{u^m} \leqq g(n) \quad \text{for } u \neq 0.$$

2. Asymptotic behaviour of nonoscillatory solutions

In this section, we assume that

$$\sum_{k=n_0}^{\infty} [f(k)-g(k)] = \infty. \tag{1}$$

Theorem 1. *Let conditions (H) and (1) hold, then any nonoscillatory solution of (E) must belong to one of the following two types:*

$$A_c : x_n \to C \neq 0, \ n \to \infty,$$
$$A_0 : x_n \to 0, \ n \to \infty.$$

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (E), then x_n is eventually positive or negative. Thus, from (*E*), we have

$$\begin{aligned}\Delta\left(\frac{r_{n-1}\Delta x_{n-1}}{x_{n-1}^m}\right) &= \frac{r_n \Delta x_n}{x_n^m} - \frac{r_{n-1}\Delta x_{n-1}}{x_{n-1}^m} \\ &= \frac{x_{n-1}^m r_n \Delta x_n - x_n^m r_{n-1}\Delta x_{n-1}}{x_n^m x_{n-1}^m} \\ &= \frac{\Delta(r_{n-1}\Delta x_{n-1})}{x_n^m} - \frac{\Delta x_{n-1} m \cdot r_{n-1}\Delta x_{n-1}}{(x_{n-1}x_n)^m} \\ &\leqq -[f(n)-g(n)] - \frac{\Delta x_{n-1}^m \cdot r_{n-1}\Delta x_{n-1}}{(x_{n-1}x_n)^m}. \end{aligned} \tag{2}$$

By the mean value theorem

$$\Delta x_{n-1}^m = m\xi_n^{m-1}\Delta x_{n-1}, \tag{3}$$

where $x_{n-1} < \xi_n < x_n$ or $x_n < \xi_n < x_{n-1}$. Thus from (2), (3) we have

$$\Delta\left(\frac{r_{n-1}\Delta x_{n-1}}{x_{n-1}^m}\right) \leqq -[f(n)-g(n)] - \frac{m\xi_n^{m-1} \cdot r_{n-1}(\Delta x_{n-1})^2}{(x_{n-1}x_n)^m}$$

$$\leqq -[f(n)-g(n)]. \tag{4}$$

Summing (4) from n_0+1 to n, we get

$$\frac{r_n \Delta x_n}{x_n^m} \leqq \frac{r_{n0} \Delta x_{n0}}{x_{n0}^m} - \sum_{k=n0+1}^{n} [f(k)-g(k)]. \tag{5}$$

If x_n is eventually positive, then there exists $n_1 \in N(n_0)$ such that $x_n > 0$ for $n \in N(n_1)$, thus from (5) and (1) we have

$$\Delta x_n < 0 \quad \text{for } n \in N(n_1).$$

Hence x_n is monotone decreasing, and $\lim_{n\to\infty} x_n = C \geqq 0$, where C is a constant.

If x_n is eventually negative, then there exists $n_2 \in N(n_0)$ such that $x_n < 0$ for $n_2 \in N(n_0)$, thus from (5) and (1) we have

$$\Delta x_n > 0 \quad \textit{for } n \in N(n_2).$$

Hence x_n is monotone increasing, then $\lim_{n\to\infty} x_n$ exists and $\lim_{n\to\infty} x_n = C \leqq 0$.

Thus any nonoscillatory solution of (E) must belong to the following two types: A_c or A_0. The proof of Theorem 1 is complete.

Theorem 2. *Let conditions* (H) *and* (1) *hold.*

(*i*) *If* $m=1$, *then a necessary condition for equation* (E) *to have a nonoscillatory solution* $\{x_n\}$ *which belongs to* A_c *is that*

$$\sum_{k=n_1+1}^{k} \frac{1}{r_k} \sum_{i=n_1+1}^{\infty} [f(i)-g(i)] < \infty, \tag{6}$$

where $n_1 \in N(n_0)$ *is sufficiently large.*

(*ii*) *If* $0<m<1$, *then a necessary condition for equation* (E) *to have a nonoscillatory solution* $\{x_n\}$ *which belongs to* A_0 *or* A_c *is also* (6).

Proof. (i) if $m=1$, let $\{x_n\}$ be a nonoscillatory solution of (E) which belongs to A_c. If $C>0$, then x_n is eventually positive. From the proof of Theorem 1, we have that Δx_n is eventually negative and from (1), there exists $n_1 \in N(n_0)$ such that $x_n > 0$, $\Delta x_n < 0$, and $\sum_{i=n_1+1}^{n} [f(i)-g(i)] > 0$ for $n \in N(n_1)$. Summing (4) from n_1+1 to n, it follows that

$$\frac{r_n \Delta x_n}{x_n} \leqq \frac{r_{n_1} \Delta x_{n_1}}{x_{n_1}} - \sum_{i=n_1+1}^{n} [f(i)-g(i)] \leqq - \sum_{i=n_1+1}^{n} [f(i)-g(i)],$$

this is,

$$\frac{\Delta x_n}{x_n} \leqq -\frac{1}{r_n} \sum_{i=n_1+1}^{n} [f(i)-g(i)]. \tag{7}$$

Let $q(t)=x_n+(t-n)\Delta x_n$, $n\leqq t\leqq n+1$. Then $q'(t)=\Delta x_n<0$, and $0<x_{n+1}\leqq q(t)\leqq x_n$ for $n<t<n+1$. Hence

$$\begin{aligned}\sum_{k=n_1+1}^{n}\frac{\Delta x_k}{x_k}&=\sum_{k=n_1+1}^{n}\int_k^{k+1}\frac{q'(t)}{x_k}dt\geqq\sum_{k=n_1+1}^{n}\int_k^{k+1}\frac{q'(t)}{q(t)}dt\\&=\sum_{k=n_1+1}^{n}[\log q(k+1)-\log q(k)]\\&=\sum_{k=n_1+1}^{n}[\log x_{k+1}-\log x_k]\\&=\log x_{n+1}-\log x_{n_1+1}.\end{aligned}\tag{8}$$

Thus from (7) and (8), we have

$$\begin{aligned}&\sum_{k=n_1+1}^{n}\frac{1}{r_k}\sum_{i=n_1+1}^{k}[f(i)-g(i)]\\&\leqq\log x_{n_1+1}-\log x_{n+1},\end{aligned}$$

from which letting $n\to\infty$ and noting $\lim_{n\to\infty}x_n=C>0$, we obtain (6).

(ii) If $0<m<1$, let $\{x_n\}$ be a solution of (E) which belongs to A_0 or A_c. As shown in the proof of case $m=1$, we can obtain

$$\frac{\Delta x_n}{x_n^m}\leqq-\frac{1}{r_n}\sum_{i=n_1+1}^{n}[f(i)-g(i)]\tag{9}$$

and

$$\sum_{k=n_1+1}^{n}\frac{\Delta x_k}{x_k^m}\leqq(1-m)[x_{n_1+1}^{1-m}-x_{n+1}^{1-m}].\tag{10}$$

From (9) and (10) we have

$$\sum_{k=n_1+1}^{n}\frac{1}{r_k}\sum_{i=n_1+1}^{k}[f(i)-g(i)]\leqq(1-m)(x_{n_1+1}^{1-m}-x_{n+1}^{1-m}),$$

from which letting $n\to\infty$, and noting $0<m<1$ and $\lim_{n\to\infty}x_n=0$ or $\lim_{n\to\infty}x_n=C>0$, we obtain (6), that is.

$$\sum_{k=n_1+1}^{\infty}\frac{1}{r_k}\sum_{i=n_1+1}^{k}[f(i)-g(i)]<\infty.$$

If $\{x_n\}$ is eventually negative, similarly we can show that (6) holds. Thus the proof Theorem 2 is complete.

3. Oscillation of solutions

Theorem 3. *Let conditions* (H), (1) *and the following condition hold,*

$$\sum_{k=n_1+1}^{\infty}\frac{1}{r_k}=\infty. \tag{11}$$

Then all solutions of (E) *are oscillatory.*

Proof. Suppose on the contrary that there exists a nonoscillatory solution $\{x_n\}$. Without loss of generality, we assume that x_n is eventually positive. From the proof of Theorem 1, we have that Δx_n is eventually negative and from (1), there exists $n_1\in N(n_0)$ such that $x_n>0$, $\Delta x_n<0$ for $n\in N(n_1)$ and

$$\sum_{i=n_1+1}^{n}[f(i)-g(i)]\geqq 0 \quad \text{for } n\in N(n_1).$$

Summing (E) from n_1+1 to n, we have

$$\begin{aligned} r_n\Delta x_n &= r_{n_1}\Delta x_{n_1}-\sum_{i=n_1+1}^{n}[F(k,x_k)-G(k,x_k,\Delta x_k)]\\ &\leqq r_{n_1}\Delta x_{n_1}-\sum_{k=n_1+1}^{n}x_k^m[f(k)-g(k)]\\ &= r_{n_1}\Delta x_{n_1}-x_n^m\sum_{k=n_1+1}^{n}[f(k)-g(k)]+\sum_{k=n_1+1}^{n-1}\Delta x_k^m\sum_{i=n_1+1}^{k}[f(i)-g(i)]\\ &= r_{n_1}\Delta x_{n_1}-x_n^m\sum_{k=n_1+1}^{n}[f(k)-g(k)]+\sum_{k=n_1+1}^{n-1}(m\xi_k^{m-1}\Delta x_k)\sum_{i=n_1+1}^{k}[f(i)-g(i)] \end{aligned} \tag{12}$$

where $x_{k+1}<\xi_k<x_k$.

From $x_n>0$, $\Delta x_n<0$ for $n\in N(n_1)$ and (12), we have

$$r_n\Delta x_n\leqq r_{n_1}\Delta x_{n_1}.$$

Thus

$$\Delta x_n\leqq\frac{1}{r_n}r_{n_1}\Delta x_{n_1}. \tag{13}$$

Summing (13) from n_1+1 to $n-1$, we get

$$x_n\leqq x_{n_1+1}+r_{n_1}\Delta x_{n_1}\sum_{k=n_1+1}^{n-1}\frac{1}{r_k} \tag{14}$$

from (14), letting $n \to \infty$ and using (11) and $\Delta x_{n_1} < 0$, we have $x_n \to -\infty$, which contradicts $x_n > 0$. Thus Theorem 3 is proved.

Theorem 4. *Let conditions (H) with $m=1$, (11) and the following conditions hold,*
(i) There exists a sufficiently large $n_1 \in N(n_0)$ such that for $n \in N(n_1)$, $f(n) - g(n) \geqq 0$ and

$$\sum_{k=n_1+1}^{\infty} [f(k) - g(k)] < \infty. \tag{15}$$

(ii) There exists positive sequence $\{C_n\}$ such that

$$\sum_{k=n_1+1}^{\infty} C_k [f(k) - g(k)] = \infty \tag{16}$$

and

$$\sum_{k=n_1+1}^{\infty} \frac{(\Delta C_{k-1})^2}{C_k \left(\frac{1}{r_{k-1}} \sum_{i=k}^{\infty} [f(i) - g(i)] \right)} < \infty. \tag{17}$$

Then all solutions of (E) are oscillatory.

Proof. Suppose that there exists a nonoscillatory solution $\{x_n\}$. Without loss of generality, we assume that $x_n > 0$ for $n \in N(n_1)$. Hence (4) holds. Now, we show that $\Delta x_n < 0$ for sufficiently large n and that this will lead to a contradiction.

Case (a). If there exists $n_2 \in N(n_1)$ such that $\Delta x_{n_2} = 0$, then summing (4) from $n_2 + 1$ to n, we have

$$\frac{r_n \Delta x_n}{x_n} \leqq \frac{r_{n_2} \Delta x_{n_2}}{x_{n_2}} - \sum_{k=n_2+1}^{n} [f(k) - g(k)]$$

$$= - \sum_{k=n_2+1}^{n} [f(k) - g(k)].$$

Thus from (15), we have $\Delta x_n < 0$ for $n \in N(n_2)$. Hence summing (E) from $n_3 \in N(n_2)$ to n, we can obtain that

$$\lim_{n \to \infty} x_n = -\infty$$

which contracts $x_n > 0$.

Case (b) If $\Delta x_n > 0$ for $n \in N(n_1)$. Similarly to (4) we have

$$\Delta\left(\frac{r_{n-1}\Delta x_{n-1}}{x_{n-1}}\right) < -[f(n)-g(n)]. \tag{18}$$

Summing (18) from $n+1$, $n\in N(n_1)$, to N and letting $N\to\infty$, we have

$$0 \leqq \lim_{N\to\infty}\frac{r_N\Delta x_N}{x_N} \leqq \frac{r_n\Delta x_n}{x_n} - \sum_{k=n+1}^{\infty}[f(k)-g(k)].$$

Thus

$$\sum_{k=n+1}^{\infty}[f(k)-g(k)] \leqq \frac{r_n\Delta x_n}{x_n}.$$

From $\Delta x_n > 0$ for $n\in N(n_1)$, we have

$$\frac{1}{r_n}\sum_{k=n+1}^{\infty}[f(k)-g(k)] \leqq \frac{1}{x_{n_1}}\Delta x_n. \tag{19}$$

Hence

$$\begin{aligned}
\Delta\left(\frac{r_{n-1}C_{n-1}\Delta x_{n-1}}{x_{n-1}}\right) &= \frac{r_nC_n\Delta x_n}{x_n} - \frac{r_{n-1}C_{n-1}\Delta x_{n-1}}{x_{n-1}}\\
&= \frac{C_n(r_n\Delta x_n - r_{n-1}\Delta x_{n-1})}{x_n} + \frac{C_nr_{n-1}\Delta x_{n-1}}{x_n} - \frac{r_{n-1}C_{n-1}\Delta x_{n-1}}{x_{n-1}}\\
&= C_n\frac{G(n,x_n,\Delta x_n)-F(n,x_n)}{x_n} - \frac{C_nr_{n-1}(\Delta x_{n-1})^2}{x_nx_{n-1}} + \frac{\Delta C_{n-1}r_{n-1}\Delta x_{n-1}}{x_{n-1}}\\
&\leqq -C_n[f(n)-g(n)] - \frac{r_{n-1}x_n}{x_{n_1}}\left[\frac{\sqrt{C_n}\Delta x_{n-1}}{x_n} - \frac{\Delta C_{n-1}}{2\sqrt{C_n}}\right]^2 + \frac{r_{n-1}(\Delta C_{n-1})^2}{4C_n}\cdot\frac{x_n}{x_{n-1}}\\
&\leqq -C_n[f(n)-g(n)] + \frac{r_{n-1}(\Delta C_{n-1})^2}{4C_n}\cdot\frac{x_n}{x_{n-1}}\\
&\leqq -C_n[f(n)-g(n)] + \frac{r_{n-1}(\Delta C_{n-1})^2}{4C_n}\\
&= -C_n[f(n)-g(n)] + \frac{r_{n-1}\Delta x_{n-1}\cdot(\Delta C_{n-1})^2}{4C_n\cdot\Delta x_{n-1}}.
\end{aligned} \tag{20}$$

Summing the following inequality from n_1+1 to $n+1$,

$$r_{k-1}\Delta x_{k-1} \leqq -x_k[f(k)-g(k)],$$

we find that

$$r_{n-1}\,\Delta x_{n-1} \leqq r_{n_1}\,\Delta x_{n_1} - \sum_{k=n_1+1}^{n-1} x_k[f(k)-g(k)]$$
$$\leqq x_{n_1}\,\Delta x_{n_1} = M_0. \tag{21}$$

Using (21), (19), and (20) we have

$$\Delta\left(\frac{r_{n-1}C_{n-1}\,\Delta x_{n-1}}{x_{n-1}}\right)$$
$$\leqq -C_n[f(n)-g(n)] + \frac{M_0\cdot(\Delta C_{n-1})^2}{4x_{n_1}\cdot C_n\left(\dfrac{1}{r_{n-1}}\displaystyle\sum_{k=n}^{\infty}[f(k)-g(k)]\right)}$$
$$= -C_n[f(n)-g(n)] + M\cdot\frac{(\Delta C_{n-1})^2}{C_n\left(\dfrac{1}{r_{n-1}}\displaystyle\sum_{k=n}^{\infty}[f(k)-g(k)]\right)}, \tag{22}$$

where $M = M_0/4x_{n_1}$. Summing (22) from n_1+1 to n, we have

$$\frac{r_nC_n\,\Delta x_n}{x_n} \leqq \frac{r_{n_1}C_{n_1}\,\Delta x_{n_1}}{x_{n_1}} - \sum_{k=n_1+1}^{n} C_k[f(k)-g(k)]$$
$$+M\sum_{k=n_1+1}^{n}\frac{(\Delta C_{n-1})^2}{C_k\left(\dfrac{1}{r_{k-1}}\displaystyle\sum_{i=k}^{\infty}[f(i)-g(i)]\right)}.$$

Letting $n\to\infty$ and noting (16), (17), we get

$$\lim_{n\to\infty}\frac{r_nC_n\,\Delta x_n}{x_n} = -\infty.$$

Thus there exists $n_2\in N(n_1)$ such that $\Delta x_n<0$ for $n\in N(n_2)$, which contradicts $\Delta x_n>0$ for $n\in N(n_1)$.

Thus from Cases (a) and (b) we can show that there exists $n_3\in N(n_1)$ such that $\Delta x_{n_3}<0$. Summing (4) from n_3+1 to n we have

$$\frac{r_n\,\Delta x_n}{x_n} \leqq \frac{r_{n_3}\,\Delta x_{n_3}}{x_{n_3}} - \sum_{k=n_3+1}^{n}[f(k)-g(k)].$$

Hence $\Delta x_n<0$ for $n\in N(n_3)$. Similarly to the last part of the proof of Theorem 3 and from (11) we have $\lim_{n\to\infty} x_n = -\infty$, which contradicts $x_n>0$. Theorem 4 is proved.

For the purpose of illustration we consider the following example.

Example. Consider the difference equation

$$\Delta\left(\frac{1}{2n^{1+\delta}}\Delta x_{n-1}\right)+\frac{1}{n^{1+\delta}}x_n+\frac{1}{4(n+1)^{1+\delta}}(\Delta x_n)^2=0,\ n\in N(n_0),\ n_0\geqq 1$$

where $0<\delta<1$. Let $C_n=n$, $f(n)=1/n^{1+\delta}$ and $g(n)=0$, $n\in N(n_0)$, then we find that conditions (H), (11), and (15)–(17) are satisfied. Thus from Theorem 4 all solutions of (E) are oscillatory. In fact, $\{x_n\}=\{(-1)^n\}$ is such a solution. We believe that the conclusion is not deducible from the oscillation criteria in [**3**, **4**, **6**] and the known literature.

Acknowledgements. The authors thank the referee for many valuable suggestions.

REFERENCES

1. Sui–Sun Cheng and Horng-Jaan Li, Bounded and zero convergent solutions of second order difference equations. *J. Math. Anal. Appl.* **141** (1989), 463–483.

2. Andrzej Drozdowicz and Jerzy Popenda, Asymptotic behavior of solutions of difference equation of second order, *J. Comput. Appl. Math.* **47** (1993), 141–149.

3. Hue-Zhong He, Oscillatory and asymptotic behavior of second order nonlinear difference equations. *J. Math. Anal. Appl.* **175** (1993), 482–498.

4. Zdzislaw Szafranski and Blazej Szmanda, Oscillatory behavior of difference equations of second order, *J. Math. Anal. Appl.* **150** (1990), 414–424.

5. William F. Trench, Asymptotic behavior of solutions of Emden—fowler difference equations with oscillating coefficients, *J. Math. Anal. Appl.* **179** (1993), 135–153.

6. B. G. Zhang, Oscillation and asymptotic behavior of second order difference equations, *J. Math. Anal. Appl.* **173** (1993), 58–68.

Department of Mathematics
Hubei Normal University
Huangshi, Hubei 435002
People's Republic of China

Department of Mathematics
Shanxi University
Taiyuan, Shanxi 030006
People's Republic of China