J. Aust. Math. Soc. **85** (2008), 283–287
doi:10.1017/S1446788708000967

A NOTE ON PERMUTATION GROUPS AND THEIR REGULAR SUBGROUPS

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(Received 17 December 2007; accepted 6 May 2008)

Communicated by Martin W. Liebeck

Dedicated to Cheryl Praeger for her sixtieth birthday

Abstract

In this note we first prove that, for a positive integer $n > 1$ with $n \neq p$ or p^2 where p is a prime, there exists a transitive group of degree n without regular subgroups. Then we look at 2-closed transitive groups without regular subgroups, and pose two questions and a problem for further study.

2000 *Mathematics subject classification*: 20B25, 05C25.

Keywords and phrases: transitive permutation group, regular subgroup, non-Cayley number.

We first define four subsets of positive integers:

$$\begin{aligned}
\mathcal{NR} &= \{n \in \mathbb{N} \mid \text{there is a transitive group of degree } n \text{ without a regular subgroup}\},\\
\mathcal{N}_2\mathcal{R} &= \{n \in \mathbb{N} \mid \text{there is a 2-closed transitive group of degree } n \text{ without a}\\
&\qquad \text{regular subgroup}\},\\
\mathcal{ND} &= \{n \in \mathbb{N} \mid \text{there is a vertex-transitive digraph of order } n \text{ that is non-Cayley}\},\\
\mathcal{NC} &= \{n \in \mathbb{N} \mid \text{there is a vertex-transitive graph of order } n \text{ that is non-Cayley}\}.
\end{aligned}$$

In the literature there has been much work studying the set $\mathcal{NC}$; see [5–9] for example.

Obviously, $\mathcal{NR} \supseteq \mathcal{N}_2\mathcal{R} \supseteq \mathcal{ND} \supseteq \mathcal{NC}$. It is known that $\mathcal{NR} \supsetneq \mathcal{NC}$. For example, $12 \notin \mathcal{NC}$ by [7, Theorem 3], but $12 \in \mathcal{NR}$, since M_{11}, acting on 12 points, has no regular subgroup by [3]. Also it is easy to see that 6 is the smallest number in $\mathcal{NR} \setminus \mathcal{NC}$ since A_6 has no regular subgroups. In the first part of this note, we shall determine the set $\mathcal{NR}$.

It is well known that any prime number p does not belong to any one of the four sets above. Moreover, Marušič [5] proved that $p^2 \notin \mathcal{NC}$. In fact, we have $p^2 \notin \mathcal{NR}$.

PROPOSITION 1. *Any transitive group G of degree p^2 on Ω has a regular subgroup. Hence $p^2 \notin \mathcal{NR}$.*

This work was supported by NSFC (project number 10671114).

PROOF. Take a minimal transitive subgroup P of G. Then P is a p-group and every maximal subgroup M of P is intransitive. For any $\alpha \in \Omega$, we have $|P_\alpha| = |P|/p^2$ and $|M_\alpha| > |M|/p^2$, so $M_\alpha = P_\alpha$. It follows that $P_\alpha \le M$ and hence $P_\alpha \le \Phi(P)$. If $|P : \Phi(P)| = p$, then P is cyclic and is regular. If $|P : \Phi(P)| = p^2$, then $P_\alpha = \Phi(P)$. Since $\Phi(P)$ is normal in P and P_α is core-free, we have $P_\alpha = 1$ and hence $P \cong \mathbb{Z}_p^2$ is regular. □

The following example shows that $p^3 \in \mathcal{NR}$. However, it has been proved that $p^3 \notin \mathcal{NC}$; see [5, 6]. Therefore $p^3 \in \mathcal{NR} \setminus \mathcal{NC}$.

EXAMPLE 2. (1) Let p be an odd prime and let G be the group of order p^4 presented by

$$G = \langle a, b \mid a^{p^2} = b^p = c^p = 1, [a, b] = c, [c, a] = a^p, [c, b] = 1\rangle.$$

Let $H = \langle c\rangle$. Consider the transitive permutation representation φ of G acting on the coset space $[G : H]$. Then $\varphi(G)$ is a transitive group of degree p^3, and $\varphi(G)$ has no regular subgroups.

(2) Let

$$G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^4 = 1, [a, b] = [b, c] = [c, a] = 1, \\ a^d = ab, b^d = bc, c^d = c\rangle.$$

Then $G \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$ has order 2^5. Let $H = \langle b, d^2\rangle$ and φ be the transitive permutation representation of G acting on the coset space $[G : H]$. Then $\varphi(G)$ is a transitive group of degree 2^3 and has no regular subgroup.

PROOF. (1) Since $[c, a] = a^p$, $\langle c\rangle \ntrianglelefteq G$. Since $\operatorname{Ker}\varphi = \operatorname{core}_G(H) = 1$, the action is faithful. So, $\varphi(G) \cong G$. Suppose that $\varphi(G)$ has a regular subgroup, say $\varphi(R)$. Then R is maximal in G, and $RH = G$ by the Frattini argument. But, $H \le G' \le \Phi(G) \le R$, a contradiction.

(2) Similar to (1), we can prove that H is core-free and contained in $\Phi(G)$. The details are omitted. □

Now we are ready to determine the set $\mathcal{NR}$. We first need the following proposition.

PROPOSITION 3. *Let $p < q$ be two primes. Then $pq \in \mathcal{NR}$.*

PROOF. Let $W = \mathbb{Z}_p \wr \mathbb{Z}_q = \langle a\rangle \wr \langle b\rangle$, viewed as an imprimitive group of degree pq. Since the action of b on the base group $\mathbb{Z}_p^q$ is nontrivial, we may take a $\langle b\rangle$-invariant subgroup H of the base group such that the action of b on H is also nontrivial and H is smallest subject to this property. Then b is irreducible on H. Let $G = H \rtimes \langle b\rangle$. Since $p < q$, $|H| = p^k > p$. Take $M \lessdot H$. Consider the transitive permutation representation φ of G acting on the coset space $[G : M]$. Since H is a minimal normal subgroup of G, $\operatorname{core}_G(M) = 1$ and φ is faithful. Since $\langle b\rangle$ is a Sylow q-subgroup and maximal in G by the irreducibility of b on H, G has no subgroup of order pq. Hence $\varphi(G)$ has no regular subgroups. It follows that $pq \in \mathcal{NR}$. □

THEOREM 4. *Let n be a positive integer greater than* 1. *Then $n \in \mathcal{NR}$ unless $n = p$ or p^2 for a prime p.*

PROOF. This theorem follows from Proposition 1, Example 2, Proposition 3 and the fact that, if $m \in \mathcal{NR}$, then $km \in \mathcal{NR}$ for any positive integer k. □

In the second part of this note we look at the set $\mathcal{N}_2\mathcal{R}$. The next proposition shows that $p^3 \notin \mathcal{N}_2\mathcal{R}$, while Marušič [5] proved that $p^3 \notin \mathcal{NC}$.

PROPOSITION 5. *Any* 2*-closed transitive group G of degree p^3 on Ω has a regular subgroup.*

To prove the above proposition, we need the concept of 2-closures of permutation groups introduced by Wielandt [10].

Let G be a permutation group acting on Ω. Suppose that $\Delta_0, \Delta_1, \ldots, \Delta_{r-1}$ are orbits of G acting on $\Omega \times \Omega$. The 2-closure $G^{(2)}$ of G is defined by

$$G^{(2)} = \{x \in \mathrm{Sym}(\Omega) \mid \Delta_i^x = \Delta_i, i = 0, 1, \ldots, r-1\}.$$

Obviously, $G^{(2)} \geq G$; if $G^{(2)} = G$, we say that G is 2-closed. The following lemma is quoted from [10, Exercise 5.28].

LEMMA A. *Suppose that G is a* 2*-closed group and p a prime. Then the Sylow p-subgroup P of G is also* 2*-closed.*

THEOREM B (Wielandt's dissection theorem). *Let G be a permutation group acting on Ω, and H a subgroup of G. Suppose that $\Omega = \Delta \cup \Gamma$, $\Delta \cap \Gamma = \emptyset$, $\Delta \neq \emptyset$, $\Gamma \neq \emptyset$ and $\Delta^H = \Delta$, $\Gamma^H = \Gamma$. If, for any $\delta \in \Delta$, H and H_δ have the same orbits on Γ, then $H^\Delta \times H^\Gamma \leq G^{(2)}$.*

This theorem follows from [10, Theorem 6.5] and the following obvious fact: if $H \leq G$, then $H^{(2)} \leq G^{(2)}$.

PROOF OF PROPOSITION 5. Let $P \in \mathrm{Syl}(G)$. Then P is also transitive on Ω. Take an element $z \in Z(P)$ with $o(z) = p$. Let $\mathcal{B} = \{B_1, \ldots, B_{p^2}\}$ be the set of orbits of $\langle z \rangle$. Then $\mathcal{B}$ is a complete block system of P. Assume that $K = P_{\mathcal{B}}$ is the kernel of P acting on $\mathcal{B}$. Since $K^{B_i} = \mathbb{Z}_p$, K is elementary abelian. Set $\overline{P} = P/K$. Then $\overline{P}$ is a transitive group on $\mathcal{B}$.

Take $1 \neq x \in K$ such that the support $\mathrm{supp}(x)$ of x has the minimum size. We claim that $\mathrm{supp}(x)$ is a block of P. Since K is elementary abelian, x is of order p. If $\mathrm{supp}(x)$ were not a block of P, then we could find an $h \in P$ such that $\mathrm{supp}(x)^h \neq \mathrm{supp}(x)$ and $D = \mathrm{supp}(x) \cap \mathrm{supp}(x)^h \neq \emptyset$. Since every B_i is a block of size p and p a prime, $\mathrm{supp}(x)$, $\mathrm{supp}(x)^h = \mathrm{supp}(x^h)$ and D are unions of several entire blocks of P in $\mathcal{B}$. Set $J = \langle x, x^h \rangle$. Then the nontrivial orbits of J are precisely the blocks contained in $\mathrm{supp}(x) \cup \mathrm{supp}(x^h)$. It is not difficult to check that, for any $g \in \Omega - D$, J and J_g have the same orbits in D. So, by Theorem B, $J^D \times 1^{\Omega - D} \leq P^{(2)}$. In what follows, for brevity, we use J^D to denote $J^D \times 1^{\Omega - D}$. Since P is 2-closed, $P = P^{(2)}$. Then

https://doi.org/10.1017/S1446788708000967 Published online by Cambridge University Press

$J_D \leq P$ and hence $J_D \leq K$. Thus there exists an element $y \in J_D \leq K$ of order p such that $|\mathrm{supp}(y)| < |\mathrm{supp}(x)|$, which contradicts the choice of x.

Now we distinguish three cases, namely: (1) $|\mathrm{supp}(x)| = p$; (2) $|\mathrm{supp}(x)| = p^2$ and (3) $|\mathrm{supp}(x)| = p^3$.

If $|\mathrm{supp}(x)| = p$, then $\mathrm{supp}(x) = B_i$ for some i. Hence $P = \mathbb{Z}_p \wr \overline{P}$. Since $\overline{P}$ is a transitive group of degree p^2, it has a regular subgroup H isomorphic to $\mathbb{Z}_p^2$ or $\mathbb{Z}_{p^2}$. Thus P has a subgroup isomorphic to $\mathbb{Z}_p \wr H$, which has an abelian regular subgroup.

If $|\mathrm{supp}(x)| = p^3$, then $K = \langle z \rangle$ is semiregular. Assume that H/K is a regular subgroup of $\overline{P} = P/K$. Then H is a regular subgroup of P.

Finally we assume that $|\mathrm{supp}(x)| = p^2$. Then $\mathrm{supp}(x)$ is a block of P of length p^2 which is a union of p B_is. Assume that $\mathcal{C} = \{C_1, \ldots, C_p\}$ is a block system of P with $\mathrm{supp}(x) = C_1$. Then $K = K^{C_1} \times \cdots \times K^{C_p} \cong \mathbb{Z}_p^p$.

If $\overline{P} = P/K$ has an element aK of order p^2, then $\langle a, z \rangle$ is a regular subgroup of P. So we may assume that $\exp \overline{P} = p$. Take a regular subgroup $H/K = \langle uK, vK \rangle$ of $\overline{P}$ such that $v \in P_{\mathcal{C}} - K$. (Since $P_{\mathcal{C}}$ has index p in P, this is possible.) We have $H_{\mathcal{C}} = \langle v, K \rangle$ and $u \notin H_{\mathcal{C}}$. Without loss of generality we assume that $C_i^u = C_{i+1}$, for all $i \pmod p$. Define a permutation w of Ω by

$$w = v^{C_1}(v^u)^{C_2}(v^{u^2})^{C_3} \cdots (v^{u^{p-1}})^{C_p}.$$

Since $[v, u] = k \in K$, $v^u = vk$. So

$$\begin{aligned} w &= v^{C_1}(vk)^{C_2}(vkk^u)^{C_3} \cdots (vkk^u \cdots k^{u^{p-2}})^{C_p} \\ &= (v^{C_1}v^{C_2}v^{C_3} \cdots v^{C_p})(1^{C_1}k^{C_2}(kk^u)^{C_3} \cdots (kk^u \cdots k^{u^{p-2}})^{C_p}) \\ &= v\bar{k}, \end{aligned}$$

where $\bar{k} = 1^{C_1}k^{C_2}(kk^u)^{C_3} \cdots (kk^u \cdots k^{u^{p-2}})^{C_p} \in K$, as $K = K^{C_1} \times \cdots \times K^{C_p}$. So $w \in H_{\mathcal{C}}$. Since $u^p \in K$, $v^{u^p} = v$. So u centralizes w, and then $R = \langle u, w \rangle$ is abelian and RK/K is transitive on $\mathcal{B}$. If $|R| > p^2$, then R is regular; if $|R| = p^2$, then $R \times \langle z \rangle$ is regular. This completes the proof of this proposition. □

It is known that not all 2-closed transitive groups are the full automorphism groups of a (di)graph. For example, the regular representation of a finite group that has no graphical regular representation (GRR) or digraphical regular representation (DRR) is such an example since regular groups are obviously 2-closed. (For GRRs and DRRs of finite groups, see [1, 2, 4].) Now we would like to pose the following questions.

QUESTION 1. Determine $\mathcal{N}_2\mathcal{R} \setminus \mathcal{NC}$.

QUESTION 2. Is $\mathcal{ND} = \mathcal{NC}$?

To study Question 1, we should first find nonregular 2-closed groups that are not the full automorphism groups of (di)graphs. We do not know such examples.

To end this note, we propose a problem. We first define one more subset of positive integers:

$\mathcal{PNR} = \{n \in \mathbb{N} \mid$ there is a primitive group of degree n without a regular subgroup$\}$.

PROBLEM 3. Determine the set $\mathcal{PNR}$.

Different from the set $\mathcal{NR}$, we know that $p^n \notin \mathcal{PNR}$ for any prime p and any positive integer n; see [11, Theorem]. Hence, determining the set $\mathcal{PNR}$ should be much harder than $\mathcal{NR}$.

Acknowledgements

The author is grateful to Dr Haipeng Qu and Dr Jing Xu for their valuable discussions on the work in this paper.

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