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ON LATTICE ANALOGUES OF ABSOLUTELY SUMMING CONSTANTS*

BY

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ABSTRACT. Let E be a Banach lattice,

$$1 \le p \le \infty,\ 1/p + 1/p' = 1,\ (\sum' |x_i|^p)^{1/p} = \sup\{\sum a_i x_i : a_i \in \mathbb{R}, \sum |a_i|^{p'} \le 1\},$$
$$w_p(\{x_i\}) = \sup\{(\sum |\langle x', x_i\rangle|^p)^{1/p} : x' \in E', \|x'\| \le 1\},$$

where $x_1, \ldots, x_n \in E$. We study properties of constants

$$\varphi_p(E) = \sup \|(\sum |x_i|^p)^{1/p}\| / w_p(\{x_i\}).$$

A characterization of AM-spaces is obtained which generalizes the result of Abramocič, Positselskiĭ, Yanovskii. Asymptotic estimates of φ_p for some classical finite dimensional lattices are given.

Introduction. Let E be a Banach lattice. In this paper we introduce and study constants $\varphi_p(E)$ which are analogues of Y. Gordon's p-absolutely summing constants ([3]). In the case $p = 1$, $\varphi_1(E)$ is the inverse of a constant studied by Y. Abramovič, E. Positselskiĭ, and L. Yanovskiĭ ([2]). Our results concerning φ_p simplify studying relationships between series $(\sum |\langle x', x_n\rangle|^p)^{1/p}$ and $(\sum |x_n|^p)^{1/p}$, the topic investigated in [9]. In particular for $p = 1$ one obtains (cf [2]) a simple proof of the Schlotterbeck conjecture: unconditional convergence of series $\sum x_n$ in E implies the convergence of $\sum |x_n|$ if and only if E is isomorphic to an AM-space in the sense of Kakutani ([4]).

1. **Preliminaries.** Throughout this paper E denotes a real Banach lattice, E' stands for its norm dual. We say E is an AM-space (cf Kakutani [4]) if $\|\sup(|x|, |y|)\| = \max(\|x\|, \|y\|)$. Let $1 \le p \le \infty$ and let $p' = p/(p-1)$ be the exponent dual to p. Following J. Krivine we define the function

$$E^n \ni (x_1, \ldots, x_n) \to \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \in E$$

putting

$$\left(\sum_{i=1}^n |x_i|^p\right)^{1/p} = \sup \sum_{i=1}^n a_i x_i,$$

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where supremum is taken over all choices of real numbers $a_1, \ldots, a_n$ for which $\sum |a_i|^{p'} \le 1$ (see [5] for details). As usual $(\sum |x_i|^p)^{1/p} = \sup|x_i|$, if $p = \infty$.

DEFINITION. $\varphi_p = \varphi_p(E) = \sup\|(\sum |x_i|^p)^{1/p}\|/w_p(\{x_i\})$, where supremum is taken over all $x_1, \ldots, x_n \in E$ and

$$w_p(\{x_i\}) = \sup\left\{\left(\sum |\langle x', x_i\rangle|^p\right)^{1/p} : x' \in E', \|x'\| \le 1\right\},$$

we always mean $0/0 = 0$.

It is worth mentioning here that Y. Abramovič ([1]) proved that E can be renormed to become an AM-space iff $\varphi_\infty(E) < \infty$. In this situation $\|x\|_M \le \|x\| \le \varphi_\infty \|x\|_M$, where $\|\cdot\|_M$, where $\|\cdot\|_M$ is the new AM-norm. In [2] it is also shown that $\varphi_\infty^{1/2} \le \varphi_1 \le \varphi_\infty$ and the Schlotterbeck theorem follows immediately from the above inequality.

We also need in this paper a useful notion of lattice moments of random vectors which for a Bochner p-integrable random vector X with values in E are defined as follows:

$$(E\,|X|^p)^{1/p} = \sup\{EfX\},$$

where f are real random variables with $E\,|f|^{p'} \le 1$. For this notion to be correct some additional properties of X or E are required (cf. [8]). However for finite rank random vectors the definition makes sense and in the special case when $X = \sum x_i f_i$ and f_i are independent p-stable $(1 \le p \le 2)$ random variables we have for all q, $1 \le q < p$, that

$$(E\,|X|^q)^{1/q} = c_{p,q}\left(\sum |x_i|^p\right)^{1/p}. \tag{1}$$

Moreover, if f_i are standard gaussian (i.e. 2-stable) then for all p, $1 \le p < \infty$

$$(E\,|X|^p)^{1/p} = c_p\left(\sum |x_i|^2\right)^{1/2} \tag{2}$$

(cf [5]).

2. **Properties of** φ_p.

Directly from the definition we obtain the following:

LEMMA 1. *Denote* $w_p(X) = \sup\{(E\,|\langle x', X\rangle|^p)^{1/p} : \|x'\| \le 1\}$, *then*

$$\varphi_p(E) = \sup\|(E\,|X|^p)^{1/p}\|/w_p(X),$$

where supremum is taken over all finite rank random vectors X.

Now recall that a linear operator $U: G \to E$, where G is a Banach space, is said to be majorizing if $\|\sup |Ux_i|\,\| \le K \sup \|x_i\|$ for all finite sets $\{x_i\} \subset G$. Let $M(U)$ denote the smallest constant K in the above inequality. U is majorizing iff it admits a factorization $G \xrightarrow{V} M \xrightarrow{W} E$, where M is an AM-space and $W \ge 0$ (cf [7]).

LEMMA 2. $\varphi_p(E) = \sup M(U)/\|U\|$, *where supremum is taken over all compact linear operators* $U: l^{p'} \to E$.

Proof. For fixed n and $x_1, \ldots, x_n \in E$ consider the expression $\|(\sum |x_i|^p)^{1/p}\|/w_p(\{x_i\})$. We define the linear operator $U: l_n^{p'} \to E$ as

$$U\{a_i\}_{i=1}^n = \sum_{i=1}^n a_i x_i.$$

By definition $M(U) = \|(\sum |x_i|^p)^{1/p}\|$. In fact, since U acts on the finite dimensional space (the compactness of the unit ball is essential)

$$\begin{aligned} M(U) &= \|\sup\{|Uz|: z \in l_n^{p'}, \|z\| \le 1\}\| \\ &= \left\|\sup\left\{\sum a_i x_i : \sum |a_i|^{p'} \le 1\right\}\right\| \\ &= \left\|\left(\sum |x_i|^p\right)^{1/p}\right\|. \end{aligned}$$

Also $\|U\| = w_p(\{x_i\})$. U easily can be extended to $\tilde{U}: l^{p'} \to E$ preserving the both norms $M(U)$ and $\|U\|$. Hence

$$\varphi_p(E) \le \sup M(U)/\|U\|.$$

The converse inequality follows by the routine argument: first one considers finite rank operators, then an arbitrary compact operator is approximated by finite rank operators.

REMARK 1. If $\dim E = n$ then clearly

$$\varphi_p(E) = \sup\{M(U)/\|U\|: U: l_n^{p'} \to E\}.$$

Now let us formulate our main result.

THEOREM 1. *Let E be a finite dimensional Banach lattice, say $E = [e_i]_{i=1}^n$, where e_i are disjoint normalized vectors. Let $K = K(E)$ be the smallest constant for which $\|x\|_E \le K\|x\|_{l_n^2}$* i.e.

$$K = \sup \left\|\sum_{i=1}^n a_i e_i\right\| \Big/ \left(\sum |a_i|^2\right)^{1/2} = w_2(\{e_i\}).$$

Then the following estimates are valid:

(i) *for all p, $1 \le p \le \infty$,*

$$\varphi_p \le \varphi_\infty;$$

(ii) *for all q, p such that $1 \le q < p \le 2$*

$$\varphi_p \le \varphi_q;$$

(iii) *for all p, $2 \le p \le \infty$*

$$\varphi_2 \le \varphi_p \le K\varphi_2.$$

Proof. (i) follows from the mentioned Abramovič result and the fact that in AM-spaces $(\sum |x_i|^p)^{1/p} = w_p(\{x_i\})$.

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(ii) is a consequence of (1) and Lemma 1. In fact,

$$\varphi_q = \sup \|(E\,|X|^q)^{1/q}\|/w_q(X) \geq \sup \|(E\,|\sum x_i f_i|^q)^{1/q}\|/w_q(\sum x_i f_i)$$

where f_i are independent p-stable random variables. By (1) the latter is equal to

$$\sup c_{p,q}\|(\sum |x_i|^p)^{1/p}\|/c_{p,q}w_p(\{x_i\}) = \varphi_p$$

In (iii), the left hand side inequality can be obtained by the same procedure as above, however using (2) instead (1). In order to get the right hand side inequality we apply Lemma 2 and the factorization property of majorizing operators. More precisely, let $U: l_n^{p'} \to E$, $\|U\| = 1$; U possesses the factoring: $l_n^{p'} \xrightarrow{J} l_n^2 \xrightarrow{\tilde{U}} E$, where J is the natural injection map and $\tilde{U}\{a_i\}_{i=1}^n = U\{a_i\}_{i=1}^n$, so $\|J\| = 1$, $\|\tilde{U}\| = K$. Hence $M(U) \leq M(\tilde{U}) \leq K\varphi_2(E)$ which gives the required condition.

Corollary 1. *For all p, $1 \leq p \leq \infty$*

$$\varphi_p \leq \varphi_\infty^{1/2}\varphi_2.$$

Proof. Indeed, let us note $K \leq \varphi_\infty^{1/2}$ as

$$K = w_2(\{e_i\}) \leq \|\sum e_i\|^{1/2} = \varphi_\infty^{1/2}.$$

Hence, by (iii) one obtains the estimate for all p, $2 \leq p \leq \infty$ For other p the estimate follows from the inequality $\varphi_p \leq \infty$

Corollary 2. *For all p, $1 \leq p \leq \infty$*

$$\varphi_\infty \leq K\varphi_p.$$

Corollary 3. *For all p, $1 \leq p \leq \infty$*

$$\varphi_\infty^{1/2} \leq \varphi_p \leq \varphi_\infty.$$

Note that Corollary 3 is a generalization of the Abramovič–Positselskiĭ–Yanovskiĭ theorem. Since for arbitrary Banach lattice we have that $\varphi_p(E) = \sup\{\varphi_p(E_0)$: $E_0 \subset E$ is finite dimensional$\}$ thus in view of the Abramovič characterization of AM-spaces we can rephrase Corollary 3 as follows (see [9]):

Theorem 2. *E is isomorphic to* an AM-space *iff* $\varphi_p(E) < \infty$.

Remark 2. Theorem 2 permits us to give the formula for $\varphi_p(E)$ without restrictions of Lemma 1. Namely,

$$\varphi_p(E) = \sup\{\|(E\,|X|^p)^{1/p}\|/w_p(X)\text{: } X \text{ is Bochner } p\text{-integrable r.v.}\}$$

Indeed, if $\varphi_p(E) = \infty$ the statement is immediate. So let $\varphi_p(E) < \infty$. Hence by Theorem 2, E is isomorphic to an AM-space and $\|(E\,|X|^p)^{1/p}\| \leq C(E\,\|X\|^p)^{1/p}$

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for all Bochner integrable X. The existence of $(E|X|^p)^{1/p}$ follows from Kakutani's representation E as a function space $C(K)$, in this case $(E|X|^p)^{1/p}(t) = (E|X(t)|^p)^{1/p}$ for all $t\in K$ (cf [5], [8]). Now the proof follows from routine approximation an arbitrary random vector X by a sequence of finite rank random vectors.

3. Estimates of $\varphi_p(l_n^q)$. The estimates follow immediately from Theorem 1 in almost all cases. To take care of exceptional cases we need some complementary facts. Let us recall the notion of (the converse of) Y. Gordon's constant

$$\gamma_q = \gamma_q(E) = \sup(\sum \|x_i\|^q)^{1/q}/w_q(\{x_i\}).$$

LEMMA 3. *For fixed n and all $1\le p,\ q\le\infty$ we have that*

(i) $\varphi_p(l_n^q)\le\gamma_q(l_n^p)$;
(ii) $\varphi_1(l_n^q)\le\gamma_1(l_n^1)\le cn^{1/2}$.

Proof. If $U:G\to E$ is a linear operator of the form $U=\sum_{i=1}^n g_i\otimes e_i$, where $g_i\in G'$ and $\{e_i\}_{i=1}^n\subset E$ is a sequence of normalized disjoint vectors then $M(U)=\|\sum\|g_i\|\,e_i\|$. Indeed,

$$\begin{aligned} M(U) &= \sup\{\|\sup_{1\le k\le n}|\sum\langle g_i, z_k\rangle e_i|\,\|: z_k\in G, \|z_k\|\le 1\}\\ &\le \|\sum\|g_i\|\,e_i\|. \end{aligned}$$

On the other hand let z_k, $k=1,\ldots,n$, be such that $\|z_k\|=1$, $\langle g_k, z_k\rangle=\|g_k\|$. Hence

$$\begin{aligned} \|\sum\|g_i\|e_i\| &= \left\|\sum \sup_k |\langle g_i, z_k\rangle|\, e_i\right\|\\ &= \left\|\sup_k \sum\langle g_i, z_k\rangle e_i\right\| \le M(U) \end{aligned}$$

by disjointness of $\{e_i\}$.

We apply Lemma 2. Let $U:l_n^{p'}\to l_n^q$, $U=\sum_{i=1}^n g_i\otimes e_i$, where $g_i\in l_n^p$ and $\{e_i\}$ forms the standard basis of l_n^q. Thus

$$M(U)/\|U\| = (\sum\|g_i\|^q)^{1/q}/w_q(\{g_i\})\le\gamma_q(l_n^p),$$

which yields (i). Now since for $g_i\in l_n^1$, $(\sum\|g_i\|^q)^{1/q}\le\|(\sum|g_i|^q)^{1/q}\|$ we have by (i) that

$$\varphi_1(l_n^q)\le\gamma_q(l_n^1)\le\varphi_1(l_n^1)=\gamma_1(l_n^1).$$

The last term is the inverse of the known Mcphail's constant of l_n^1, so it is majorized by $cn^{1/2}$ ([6]). The proof is completed.

THEOREM 3 (see [2] for the case $p=1$). *For fixed p and q we have*

$$\varphi_p(l_n^q) \sim \begin{cases} n^{1/2} & \text{if } 1 \le p \le 2, 1 \le q \le 2, \\ n^{1/q} & \text{if } 1 \le p \le \infty, 2 \le q \le \infty \\ & \text{or } 2 \le p \le \infty, q' \le p, \\ n^{1/p'} & \text{if } 2 \le p \le \infty, q' \ge p. \end{cases}$$

Proof. If $1 \le p \le 2$, $1 \le q \le 2$, the upper estimate follows from Lemma 3, since $\varphi_p \le \varphi_1$; the lower one—from Corollary 2 as

$$\varphi_p \ge \varphi_\infty / K = n^{1/q} / n^{1/q-1/2} = n^{1/2}.$$

If $1 \le p \le \infty$, $2 \le q \le \infty$, then $\varphi_p \le \varphi_\infty = n^{1/q}$. On the other hand by Corollary 2

$$\varphi_p \ge \varphi_\infty / K = n^{1/q}.$$

Let $2 \le p \le \infty$. By Lemma 2, $\varphi_p \ge M(J)/\|J\|$, where J is the formal identity map $J: l_n^{p'} \to l_n^q$ so if $q' \le p$ then $n^{1/q} = \varphi_\infty \ge \varphi_p \ge n^{1/q}$ and if $q' \ge p$ then

$$\varphi_p \ge n^{1/q} / n^{1/q-1/p} = n^{1/p}.$$

The upper estimate in the case $2 \le p \le \infty$, $q' \ge p$ follows from Lemma 3 and [3] (Th. 5) as

$$\varphi_p(l_n^q) \le \gamma_q(l_n^p) \le cn^{1/p'}$$

and the proof is complete.

4. Additional results. By the standard approximation method one can prove the following version of Lemma 2:

LEMMA 2a. $\varphi_p = \sup\{M(U)/\|U\|: U: L^{p'} \to E\}$ *for any infinite dimensional space* $L^{p'}$.

Hence we obtain a partial generalization of Theorem 3:

THEOREM 4. *Let* K_2 *be* 2-*concavity constant of* E i.e.

$$K_2 = \sup(\sum \|x_i\|^2)^{1/2} / \|(\sum |x_i|^2)^{1/2}\|.$$

Then for all p, $1 \le p \le 2$, $\varphi_p \le K_2 \varphi_2$. *Hence* $\le K_2 cn^{1/2}$, *if* $\dim E = n$.

Proof. Let $U: l^{p'} \to E$ be a compact operator. By J. Krivine's theorem ([5]) U possesses a factorization $U: l^{p'} \xrightarrow{J} L^2 \xrightarrow{\tilde{U}} E$, where $\|\tilde{U}\| \le K_2$, $\|J\| \le 1$ and $\tilde{U} \ge 0$. Hence $M(U) \le M(\tilde{U}) \le K_2 \varphi_2$ by Lemma 2a. Now $\varphi_2(E) \le \varphi_1(E) \le cn^{1/2}$ by Lemma 2.2 in [2] and Mcphail estimation ([6]).

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