Canad. Math. Bull.Vol. **34** (1), 1991 pp. 60–66

ON VALUES OF THE RIEMANN ZETA FUNCTION AT INTEGRAL ARGUMENTS

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ABSTRACT. For each nonnegative integer r,

$$\zeta(r+2) := \sum_{n=1}^{\infty} \frac{1}{n^{r+2}}$$

is represented by a multiple series which is expressed in terms of rational numbers and the special values of the zeta function $\zeta(2h), h = 1, 2, \ldots$. Thus, the set $\{\zeta(2h) \mid h = 1, 2, \ldots\}$ serves as a kind of basis for expressing all of the values $\zeta(s), s = 2, 3, \ldots$.

1. **Introduction.** Classically the Riemann zeta function ζ is defined for each complex number s having real part greater than 1 as follows.

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

As intimated in the title we are here concerned about the values $\zeta(s)$ when s is restricted to $N - \{0, 1\}$, $N = \{0, 1, 2, \ldots\}$. Each such value of ζ, except $\zeta(2)$, is represented by a series different from the defining series. Since these series involve the Bernoulli numbers and a certain doubly indexed sequence of numbers defined in terms of the Bernoulli numbers, we collect these numbers in the following definition.

DEFINITION 1.1. The Bernoulli numbers $B_j, j \in N$, are defined by the generating function:

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j.$$

For each pair $(m, r) \in N \times (N - \{0\})$, we define $A_{2m}(r)$ as follows:

(i) $A_{2m}(1) := B_{2m}$, and

(ii) for $r > 1$,

$$A_{2m}(r) := \sum \frac{\binom{2m}{2i_1, 2i_2, \ldots, 2i_r}}{\{2i_1 + 1\}\{2(i_1 + i_2) + 1\} \cdots \{2(i_1 + i_2 + \cdots + i_{r-1}) + 1\}} \cdot B_{2i_1} B_{2i_2} \cdots B_{2i_r}$$

where the sum is extended over all $(i_1, i_2, \ldots, i_r) \in N^r$ such that $i_1 + i_2 + \cdots + i_r = m$, and $\binom{2m}{2i_1, 2i_2, \ldots, 2i_r}$ is a multinominal coefficient.

We are now prepared to state our main result.

Key words and phrases: Riemann Zeta Function
Received by the editors August 24, 1988.
AMS subject classification: Primary: 10H05, Secondary: 10H08.

THEOREM 1.2. *For each integer $r > 2$,*

$$\zeta(r) = \frac{2^{r-2}}{2^r - 1}\pi^2 \sum_{m=0}^{\infty}(-1)^m A_{2m}(r-2)\frac{\pi^{2m}}{(2m+2)!}. \tag{2}$$

In Section 2 we prove this theorem. We then state and prove a corollary which shows that the values $\zeta(r), r = 2, 3, \ldots$, are intimately related to each other. In fact, we show that the set $\{\zeta(2h) \mid h = 1, 2, \ldots\}$ serves as a kind of basis for expressing all of these values.

Thanks to Euler we know that for each positive integer $n, \zeta(2n)$ is expressible as a rational multiple of π^{2n}. So, the main interest in these matters turns on the possibility of finding formulas for some or all of the values $\zeta(2n+1), n \geq 1$. In this direction Ramanujan [7] discovered (without proof) that: if α and β are positive real numbers such that $\alpha\beta = \pi^2$ and n is a positive integer, then

$$\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k}-1}\right\} = (-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k}-1}\right\}$$
$$- 2^{2n}\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^k.$$

For the particular value $\zeta(3)$ we have the following three formulas due respectively to Grosswald [6], Terras [8] and Apery [1]:

$$\zeta(3) = \frac{7}{180}\pi^3 - 2\sum_{n=1}^{\infty}e^{-2\pi n}\sigma_{-3}(n),$$
$$\zeta(3) = \frac{2}{45}\pi^3 - 4\sum_{n=1}^{\infty}e^{-2\pi n}\sigma_{-3}(n)(2\pi^2 n^2 + \pi n + 1/2),$$
$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

[Of course, $\sigma_{-3}(n) = \sum d^{-3}$, the sum extending over all positive integral divisors d of n.]

In conjunction with other considerations, Apery used his formula to establish the irrationality of $\zeta(3)$. Unfortunately, his method does not seem to generalize to $\zeta(2n+1), n > 1$. Sadly, all of these formulas, including (2) of the present paper, are deficient in some way. E.g., when $r > 3$, (2) becomes rather complicated. It does not seem to be possible to show that the series representation reduces to $\pi^4/90$ when $r = 4$. [Of course, $\zeta(4) = \pi^4/90$, as discovered by Euler.]

2. **Proof of Theorem 1.2.** First of all, we establish a simple lemma.

LEMMA 2.1. *For each positive integer n and each real number t near $t = 0$,*

$$\int t^{-1}(\sin^{-1} t)^n\, dt = C + \sum_{j=0}^{\infty}(-1)^j\frac{2^{2j}B_{2j}}{(2j)!}\frac{(\sin^{-1} t)^{2j+n}}{2j+n}, \tag{3}$$

https://doi.org/10.4153/CMB-1991-010-2 Published online by Cambridge University Press

where C is an arbitrary constant.

PROOF. Under the substitution $\theta = sin^{-1}t$

$$\int t^{-1}(sin^{-1}t)^n\, dt = \int \theta^n \cot\theta\, d\theta .$$

(3) then follows from termwise integration of the identity

$$\theta^n \cot\theta = \sum_{k=0}^{\infty}(-1)^k \frac{2^{2k}B_{2k}}{(2k)!}\theta^{2k+n-1}, \ |\theta| < \pi.$$

Next, since each positive integer n has a unique expression as $n = 2^i(2j+1)$, for some $(i,j) \in N^2$, it follows (by absolute convergence) that

$$\sum_{n=1}^{\infty}\frac{1}{n^r} = \sum_{i=0}^{\infty}\frac{1}{2^{ir}}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^r} = \frac{2^r}{(2^r-1)}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^r}.$$

Hence, it suffices to evaluate $\sum_{j=0}^{\infty}\frac{1}{(2j+1)^r}$, $r \geq 3$.

To evaluate $\sum_{j=0}^{\infty}\frac{1}{(2j+1)^r}$ we modify a method due to Boo Rim Choe [3], and proceed by induction on r.

The Taylor series expansion of $\sin^{-1} t$ near $t = 0$ is given by

$$\sin^{-1} t = t + \sum_{n=1}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\cdot\frac{t^{2n+1}}{2n+1}. \tag{4}$$

With the aid of (3) we multiply both sides of (4) by t^{-1} and integrate both sides of the resulting equation from 0 to $x(|x| < 1)$ to get

$$\sum_{n=0}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\frac{x^{2n+1}}{(2n+1)^2} = \sum_{k=0}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{(2k+1)!}(\sin^{-1} x)^{2k+1}. \tag{5}$$

In (5) let $x = \sin t, |t| \leq \frac{\pi}{2}$, to get

$$\sum_{n=0}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\cdot\frac{\sin^{2n+1} t}{(2n+1)^2} = \sum_{k=0}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{(2k+1)!}\cdot t^{2k+1}.$$

Integrating both sides of the foregoing equation from 0 to $\frac{\pi}{2}$, term by term, we get

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cdot\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\cdot\int_0^{\frac{\pi}{2}}\sin^{2n+1} t\, dt = \frac{\pi^2}{4}\sum_{k=0}^{\infty}(-1)^k B_{2k}\frac{\pi^{2k}}{(2k+2)!}.$$

By Wallis' formula [4, p. 223]

$$\int_0^{\frac{\pi}{2}}\sin^{2n+1} t\, dt = \frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n+1)}, \ n = 1,2,\ldots,$$

the last equation then reduces to

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = \frac{\pi^2}{4} \sum_{k=0}^{\infty} (-1)^k B_{2k} \frac{\pi^{2k}}{(2k+2)!}.$$

Multiplying both sides of the foregoing equation by $\frac{2^3}{(2^3-1)}$, we then get

$$\zeta(3) = \frac{2^{3-2}}{2^3-1} \pi^2 \sum_{m=0}^{\infty} (-1)^m A_{2m}(3-2) \frac{\pi^{2m}}{(2m+2)!}. \tag{6}$$

Fix $r > 3$ and assume inductively that the sequence of operations which led from (4) to (5), (6) has been applied $r-3$ times to yield:

$$\sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \cdot \frac{x^{2n+1}}{(2n+1)^{r-2}} = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} A_{2k}(r-3)}{(2k+1)!} (\sin^{-1} x)^{2k+1}, \tag{7}$$

$$\zeta(r-1) = \frac{2^{r-3}}{2^{r-1}-1} \pi^2 \sum_{m=0}^{\infty} (-1)^m A_{2m}(r-3) \frac{\pi^{2m}}{(2m+2)!}. \tag{8}$$

In (7) we then let $x \longrightarrow t$, multiply the resulting equation by t^{-1}, and antidifferentiate both sides of the last equation with respect to t to get

$$\sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \cdot \frac{t^{2n+1}}{(2n+1)^{r-1}} = C + \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} A_{2k}(r-3)}{(2k+1)!} \int t^{-1} (\sin^{-1} t)^{2k+1}\, dt.$$

Again with the help of (3) we take the definite integral of the foregoing equation from 0 to $x(|x| < 1)$ to get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \cdot \frac{x^{2n+1}}{(2n+1)^{r-1}}
&= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} A_{2k}(r-3)}{(2k+1)!} \sum_{j=0}^{\infty} (-1)^j \frac{2^{2j} B_{2j}}{(2j)!} \cdot \frac{(\sin^{-1} x)^{2j+2k+1}}{2j+2k+1} \\
&= \sum_{m=0}^{\infty} (-1)^m 2^{2m} \frac{(\sin^{-1} x)^{2m+1}}{2m+1} \sum_{(k,j),k+j=m} \frac{1}{(2k+1)!(2j)!} A_{2k}(r-3) B_{2j} \\
&= \sum_{m=0}^{\infty} (-1)^m 2^{2m} \frac{(\sin^{-1} x)^{2m+1}}{(2m+1)!} \sum_{(k,j),k+j=m} \frac{\binom{2m}{2k,2j}}{2k+1} A_{2k}(r-3) B_{2j} \\
&= \sum_{m=0}^{\infty} (-1)^m 2^{2m} A_{2m}(r-2) \cdot \frac{(\sin^{-1} x)^{2m+1}}{(2m+1)!}.
\end{aligned}$$

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That the inner sum in next to the last line equals $A_{2m}(r-2)$ follows from the definition of the sequence $A_{2m}(r)$. Now, in the last equation we let $x = \sin t$, and integrate the resulting equation from 0 to $\frac{\pi}{2}$ to get (in view of Wallis' formula):

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^r} = \frac{\pi^2}{4} \cdot \sum_{m=0}^{\infty} (-1)^m A_{2m}(r-2) \cdot \frac{\pi^{2m}}{(2m+2)!}.$$

Multiplying both sides of the foregoing equation by $\frac{2^r}{(2^r-1)}$, we get the desired result for $\zeta(r)$. Inductively this proves Theorem 1.2.

COROLLARY 2.2. For each $r \in N$, .

(9)
$$\zeta(r+2) = \frac{2^{r-1}\pi^2}{2^{r+2}-1}\left\{1 + \sum_{m=1}^{\infty} \frac{2}{(2m+1)(2m+2)2^{2m}} \sum_{\rho=1}^{r} (-2)^\rho \binom{r}{\rho} \times \sum \frac{\zeta(2h_1)\zeta(2h_2)\cdots\zeta(2h_\rho)}{\{2h_1+1\}\{2(h_1+h_2)+1\}\cdots\{2(h_1+h_2+\cdots+h_{\rho-1})+1\}}\right\}$$

where on the right side the innermost sum is extended over all $(h_1, h_2, \ldots, h_\rho) \in J^\rho, J := N - \{0\}$, such that $h_1 + h_2 + \cdots + h_\rho = m$.

PROOF. For each pair $(m, r) \in N^2$, with $m > 0$, put

$$C(m,r) := \sum_{\rho=1}^{r} (-2)^\rho \sum \binom{r}{\rho} \frac{\zeta(2h_1)\zeta(2h_2)\cdots\zeta(2h_\rho)}{\{2h_1+1\}\{2(h_1+h_2)+1\}\cdots\{2(h_1+h_2+\cdots+h_\rho-1)+1\}},$$

where the range of summation of the innermost sum is as before described. By the vacuous summation convention $C(m, 0) = 0$, and (9) then reduces to $\zeta(2) = \pi^2/6$, a well known formula of Euler. Hence, assume that $r > 0$, and in the definition of the coefficient $A_{2m}(r)$, with $m > 0$ and $r > 0$, *classify* the r-tuples $(i_1, i_2, \cdots, i_r) \in N^r$ such that $i_1 + i_2 + \cdots + i_r = m$ by the number $j \in \{0, 1, \ldots, r-1\}$ of zero-coordinates among $i_1, i_2, \ldots, i_r$. Clearly, for each $j \in \{0, 1, \ldots, r-1\}$, there are $\binom{r}{j}$ ways of assigning 0's to exactly j of the r coordinate spots. And, then for each of these ways the remaining $r-j$ spots are filled with $h_1, h_2, \cdots, h_{r-j} \in J$ in all possible ways. Therefore, we recall another result due to Euler [2, p. 266], viz.,

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}, k \in J,$$

and write:

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$$A_{2m}(r)$$

$$= \sum_{j=0}^{r-1} \binom{r}{j} \sum \frac{\binom{2m}{2h_1, 2h_2, \ldots, 2h_{r-j}} \zeta(2h_1)(-1)^{h_1+1} \frac{2(2h_1)!}{(2\pi)^{2h_1}} \cdots \zeta(2h_{r-j})(-1)^{h_{r-j}+1} \frac{2(2h_{r-j})!}{(2\pi)^{2h_{r-j}}}}{\{2h_1+1\}\{2(h_1+h_2)+1\} \cdots \{2(h_1+h_2+\cdots+h_{r-j-1})+1\}}$$

$$= (-1)^m \frac{(2m)!}{(2\pi)^{2m}} \sum_{j=0}^{r-1} (-2)^{r-j} \binom{r}{j}$$

$$\sum \frac{\zeta(2h_1)\zeta(2h_2)\cdots\zeta(2h_{r-j})}{\{2h_1+1\}\{2(h_1+h_2)+1\} \cdots \{2(h_1+h_2\cdots+h_{r-j-1})+1\}},$$

where the range of summation for the innermost sum is now clear. With a slight change in notation we write (2) as:

$$\zeta(2+r) = \frac{2^r}{2^{r+2}-1}\pi^2 \left\{ \frac{1}{2} + \sum_{m=1}^{\infty} (-1)^m A_{2m}(r) \frac{\pi^{2m}}{(2m+2)!} \right\}$$

$$= \frac{2^{r-1}\pi^2}{2^{r+2}-1} \left\{ 1 + 2\sum_{m=1}^{\infty} (-1)^m A_{2m}(r) \frac{\pi^{2m}}{(2m+2)!} \right\},$$

for each $r \in J$. We now replace $A_{2m}(r)$ by the foregoing finite double sum, and simplify to get

$$\zeta(2+r) = \frac{2^{r-1}\pi^2}{2^{r+2}-1} \left\{ 1 + 2\sum_{m=1}^{\infty} \frac{C(m,r)}{(2m+1)(2m+2)2^{2m}} \right\},$$

for each $r \in J$. This proves our corollary.

REMARK 2.3. Our first observation is that the right side of (9) is conceptually not quite as complicated as it seems. For, the finite double sum which defines $C(m,r)$, $m, r \in J$, has the same range of summation as the right side of

$$(x_1 + x_2 \cdots + x_r)^m = \sum \binom{m}{i_1, i_2, \cdots, i_r} x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r},$$

the multinomial expansion. In fact, upon classifying the $(i_1, i_2, \cdots, i_r) \in N^r$ such that $i_1 + i_2 + \cdots + i_r = m$ by number of zero-coordinates we get a similar finite double sum.

Next, we can replace π^2 by $6\zeta(2)$ on the right side of (9) and realize that the resulting right side is expressed solely in terms of rational numbers and the values $\zeta(2h), h = 1, 2, \ldots$.

Finally, we display (9) for small values of r, say $r \in \{0, 1, 2, 3\}$. As before observed, $\zeta(2) = \frac{\pi^2}{6}$. For $r = 1, 2, 3$ we get the following specializations:

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$$\zeta(3) = \frac{\pi^2}{7}\{1 - 4\sum_{m=1}^{\infty} \frac{\zeta(2m)}{(2m+1)(2m+2)2^{2m}}\},$$

$$\zeta(4) = \frac{2\pi^2}{15}\{1 - 8\sum_{m=1}^{\infty} \frac{\zeta(2m)}{(2m+1)(2m+2)2^{2m}}$$

$$+8\sum_{m=1}^{\infty} \frac{1}{(2m+1)(2m+2)2^{2m}} \sum_{h=1}^{m-1} \frac{\zeta(2h)\zeta(2m-2h)}{2h+1}\},$$

$$\zeta(5) = \frac{4\pi^2}{33}\{1 - 12\sum_{m=1}^{\infty} \frac{\zeta(2m)}{(2m+1)(2m+2)2^{2m}}$$

$$+24\sum_{m=1}^{\infty} \frac{1}{(2m+1(2m+2)2^{2m}} \sum_{h=1}^{m-1} \frac{\zeta(2h)\zeta(2m-2h)}{2h+1}$$

$$-16\sum_{m=1}^{\infty} \frac{1}{(2m+1)(2m+2)2^{2m}} \sum_{\substack{(h,i,j)\\ h+i+j=m}} \frac{\zeta(2h)\zeta(2i)\zeta(2j)}{\{2h+1\}\{2(h+i)+1\}}\}.$$

In [5] the author has given a different derivation of the foregoing series representation of $\zeta(3)$. There it is observed that: since $\zeta(2n) \longrightarrow 1$ as $n \longrightarrow \infty$, the rate of convergence of the series is much faster than that of the defining series $\sum n^{-3}$. However, the rate of convergence is not as good as that of the series in the series representation of $\zeta(3)$ due to Apery [1].

A natural question to ask is: Can irrationally of $\zeta(3)$ be established in terms of the present series representation? For, such a proof, if possible, would likely extend to the values $\zeta(2k+1), k > 1$.

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https://doi.org/10.4153/CMB-1991-010-2 Published online by Cambridge University Press