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UNIVERSAL AND PROXIMATELY UNIVERSAL LIMITS

ZVONKO ČERIN and JÓSE M. R. SANJURJO

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Abstract

We present sufficient conditions on an approximate mapping $F : \mathscr{X} \to \mathscr{Y}$ of approximate inverse systems in order that the limit $f : X \to Y$ of F is a universal map in the sense of Holsztyński. A similar theorem holds for a more restrictive concept of a proximately universal map introduced recently by the second author. We get as corollaries some sufficient conditions on an approximate inverse system implying that the its limit has the (proximate) fixed point property. In particular, every chainable compact Hausdorff space has the proximate fixed point property.

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1. Introduction

This paper belongs to a part of topology in which special classes of maps and properties of spaces described by maps are studied. In fact, we consider universal maps in the sense of Holsztyński [2] and their analogue, proximately universal maps [8] in proximate topology of Klee and Yandl [6].

In the late sixties and early seventies, Holsztyński has established many nice properties of universal maps including (see [3]) a result which tries to answer the question: is the inverse limit of universal maps itself universal? We address the same question by looking for conditions under which the limit of an approximate mapping between approximate inverse systems [7] will be either universal or proximately universal.

The approximate inverse systems have been recently introduced with a desire to eliminate some of the unpleasant features of inverse systems that are a consequence of insistence on strict commutativity of all diagrams. This is accomplished by replacing equality of maps with equality up to an open cover. The new concept (whose definition, with all the preliminaries, is recalled in Section 2) has clear advantages over the old. It deserves the attention of a more general audience in spite of a somewhat complicated definition.

The applications of our results are to fixed point theory. We can conclude (Corollary 1) that the limits of special approximate inverse systems have the fixed point property and (Corollary 2) that every chainable compact Hausdorff space has the proximate fixed point property.

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2. Universalities and systems

Unless stated otherwise, by a *space* we mean a compact Hausdorff space and by a *map* we mean a continuous (single-valued) function. For a space X, we let $\widetilde{X}$ denote the family of all open covers of X. If x and y are points of a space X and $\varepsilon \in \widetilde{X}$, we shall use $x \overset{\varepsilon}{=} y$ to mean that some member of ε contains both x and y. For functions $f, g : Z \to X$ we let the relations $f \overset{\varepsilon}{=} g$, $f \approx g$, and $f \overset{\varepsilon}{\approx} g$ mean that $f(z) \overset{\varepsilon}{=} g(z)$ for every $z \in Z$, $f(z) = g(z)$ for some $z \in Z$, and $f(z) \overset{\varepsilon}{=} g(z)$ for some $z \in Z$, respectively.

The following two notions are basic for proximate topology. For an introduction to its motivation and results, the reader should see [6].

Let X and Y be spaces. Let $\alpha \in \widetilde{X}$ and $\beta \in \widetilde{Y}$. A function $f : X \to Y$ is an (α, β)*-function* provided for every $A \in \alpha$ there is a $B \in \beta$ with $f(A) \subset B$. We call f a β*-function* if there is an $\alpha \in \widetilde{X}$ such that f is an (α, β)-function.

We shall now define universal maps (or U-maps) and proximately universal maps (or P-maps) as follows. A map $f : X \to Y$ is a U*-map* provided $f \approx g$ for every map $g : X \to Y$. It is a P*-map* if for every $\varepsilon \in \widetilde{Y}$ there is a $\delta \in \widetilde{Y}$ such that $f \overset{\varepsilon}{\approx} g$ for every δ-function $g : X \to Y$.

When X and Y are compact metric spaces, the last definition is equivalent to the definition of a P-map in [8]. Since a map $g : X \to Y$ is a δ-function for every $\delta \in \widetilde{Y}$, every P-map is a U-map. The converse is not true (see [8]).

Let us now recall basic facts from [7] about approximate inverse systems (or systems) and approximate mappings (or mappings).

Let $(A, \leq)$ be a partially ordered set and let $a \in A$. We let a^+ denote the set of all $b \in A$ with $a \leq b$. For a space X we consider $\widetilde{X}$ partially ordered with the relation of refinement. Hence, for $\sigma \in \widetilde{X}$, σ^+ denotes the collection of all $\pi \in \widetilde{X}$ which refine σ. Also, σ^{+n} stands for the collection of all $\pi \in \widetilde{X}$ such that the nth star $\mathrm{st}^n(\pi)$ of π refines σ, where we define $\mathrm{st}^n(\pi)$ inductively by $\mathrm{st}^1(\pi) = \mathrm{st}(\pi)$ and $\mathrm{st}^n(\pi) = \{\mathrm{st}(H, \pi) : H \in \mathrm{st}^{n-1}(\pi)\}$ for $n > 1$.

In order to avoid repetitions, some statements in the text will be labelled in bold face. It is understood that the statement runs from the first appearance of the label to the first comma or point thereafter.

A *system* $\mathscr{X} = (X_a, \varepsilon_a, p_b^a, A)$ consists of an unbounded directed set $(A, \geq)$, a collection $\{X_a\}_{a\in A}$ of spaces, a collection $\{\varepsilon_a\}_{a\in A}$ of covers $\varepsilon_a \in \widetilde{X_a}$, and a collection $\{p_b^a\}_{(a,b)\in\leq}$ of maps $p_b^a : X_b \to X_a$ such that

(A1) $p_b^a \circ p_c^b \overset{\varepsilon_a}{=} p_c^a$ and $p_a^a = \mathrm{id}$ for every $(a, b, c) \in A \times a^+ \times b^+$,

(A2) for every $(a, \varepsilon) \in A \times \widetilde{X_a}$ there is a $b \in a^+$ with $p_c^a \circ p_d^c \overset{\varepsilon}{=} p_d^a$ for every $(c, d) \in b^+ \times c^+$, and

(A3) for every $(a, \varepsilon) \in A \times \widetilde{X_a}$ there is a $b \in a^+$ with $p_c^a(u) \overset{\varepsilon}{=} p_c^a(v)$ whenever $c \in b^+$ and u and v satisfy $u \overset{\varepsilon_c}{=} v$.

Every system $\mathscr{X}$ determines a subset X of the product $\prod \mathscr{X} = \prod_{a\in A} X_a$ (called the limit of $\mathscr{X}$) consisting of all points (x_a) satisfying the condition (see [7, (1.12)])

(L) for every $(a, \eta) \in A \times \widetilde{X_a}$ there is a $b \in a^+$ with $x_a \overset{\eta}{=} p_c^a(x_c)$ for every $c \in b^+$.

Let $\mathscr{X} = (X_a, \varepsilon_a, p_b^a, A)$ and $\mathscr{Y} = (Y_c, \sigma_c, r_d^c, C)$ be systems with limits X and Y, and for an $a \in A$ and a $c \in C$, let $p^a : X \to X_a$ and $r^c : Y \to Y_c$ denote the restrictions of natural projections $\pi^a : \prod \mathscr{X} \to X_a$ and $\varrho^c : \prod \mathscr{Y} \to Y_c$.

A *mapping* $F : \mathscr{X} \to \mathscr{Y}$ is a pair $F = (\varphi, \{f_c\}_{c\in C})$ consisting of a function $\varphi : C \to A$ and maps $f_c : X_{\varphi(c)} \to Y_c$ such that whenever $c \leq d$ in C there is an $a \geq \varphi(c), \varphi(d)$ in A with $f_c \circ p_b^{\varphi(c)} \overset{\mathrm{st}(\sigma_c)}{=\!=} r_d^c \circ f_d \circ p_b^{\varphi(d)}$ for every $b \in a^+$.

It was shown in [7, (5.8)] that a mapping F induces a map $f : X \to Y$ (called the *limit* of F) such that ([7, (5.6)])

(AM) $$f_c \circ p^{\varphi(c)} \overset{\mathrm{st}^2(\sigma_c)}{=\!=} r^c \circ f \text{ for every } c \in C.$$

Let $F = (\varphi, \{f_c\}_{c\in C})$ be a mapping $F : \mathscr{X} \to \mathscr{Y}$ with the limit $f : X \to Y$. Let $J : \mathscr{X} \to \mathscr{X}$ be the identity mapping $(id_A, \{id_{X_a}\}_{a\in A})$.

For each $b \in A$ we define an embedding $i_b : X_b \to \prod \mathscr{X}$ as follows. Let $y \in \prod \mathscr{X}$. For a $v \in X_b$, put $i_b(v) = w$, where $w_a = p_b^a(v)$ if $a \leq b$ and $w_a = y_a$ otherwise.

LEMMA 1. *Let N be a neighbourhood of X in $\prod \mathscr{X}$. Then there is a $d \in A$ such that $i_e(X_e) \subset N$ for every $e \geq d$.*

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PROOF. For each $x \in X$, choose a finite subset $B(x)$ of A and a $\delta(x,b) \in \widetilde{X}_b$ for each $b \in B(x)$ such that $H(x) = \prod\{\mathrm{st}(x_b, \delta(x,b)) : b \in B(x)\} \times \prod\{X_a : a \in A \setminus B(x)\} \subset N$. Let $\gamma(x,b) \in \delta(x,b)^{+1}$ for each $x \in X$ and $b \in B(x)$. Let $G(x) = \prod\{\mathrm{st}(x_b, \gamma(x,b)) : b \in B(x)\} \times \prod\{X_a : a \in A \setminus B(x)\}$. The collection $\{G(x)\}_{x\in X}$ covers X. Since X is a compact Hausdorff space [7, (4.1)], there is a finite subset Z of X such that the collection $\{G(z)\}_{z\in Z}$ also covers X. Let $B = \bigcup_{z\in Z} B(z)$. For each $b \in B$, let $\gamma(b) \in \bigcap\{\gamma(z, b)^{+2} : z \in Z\}$.

For every $b \in B$, by the condition (A2), there is an $m(b) \geq b$ such that

(1) $$p_r^b \circ p_s^r \overset{\gamma(b)}{=\!=} p_b^s \text{ for every } s \geq r \geq m(b).$$

On the other hand, by [7, (1.14), (1.9), (1.10)] there is an $n(b) \geq b$ such that

(2) $$p_e^b \circ p^e \overset{\gamma(b)}{=\!=} p^b \text{ for every } e \geq n(b).$$

Let $c \in \bigcap\{m(b)^+ \cap n(b)^+ : b \in B\}$. Let $\eta \in \bigcap\{(p_c^b)^{-1}(\gamma(b))^+ : b \in B\}$. Finally, we use [7, (4.2), (2.12), (2.10)] to select a required index $d \geq c$ such that

(3) $$p_e^c(X_e) \subset \mathrm{st}(p^c(X), \eta) \text{ for every } e \geq d.$$

Consider an index $e \geq d$ and a point $y \in X_e$. The relation (3) shows that we can find an $x \in X$ so that $p_e^c(y) \overset{\eta}{=} p^c(x)$. Let $b \in B$. Then

(4) $$p_c^b \circ p_e^c(y) \overset{\gamma(b)}{=\!=} p_c^b \circ p^c(x).$$

Since $e \geq c \geq m(b)$, from (1) we get

(5) $$p_c^b \circ p_e^c(y) \overset{\gamma(b)}{=\!=} p_e^b(y),$$

and since $c \geq n(b)$, from (2) we get

(6) $$p_c^b \circ p^c(x) \overset{\gamma(b)}{=\!=} p^b(x).$$

The relations (4) – (6) together imply

(7) $$\pi^b(x) \overset{\gamma(z,b)}{=\!=\!=} \pi^b(i_e(y)) \text{ for every } z \in Z \text{ and every } b \in B.$$

Choose a $z \in Z$ with $x \in G(z)$. Clearly,

(8) $$\pi^b(z) \overset{\gamma(z,b)}{=\!=\!=} \pi^b(x) \text{ for every } b \in B(z).$$

It follows from (7) and (8) that $\pi^b(i_e(y)) \overset{\delta(z,b)}{=\!=\!=} \pi^b(z)$ for every $b \in B(z)$. Hence, $i_e(y) \in H(z) \subset N$.

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LEMMA 2. *Let* $g : Z \to Y$ *be a map into the limit of a system* $\mathscr{Y}$.

(a) *If for every* $d \in C$ *and every* $\sigma \in \widetilde{Y}_d$ *there is an* $e \geq d$ *such that* $r^d \circ g \overset{\sigma}{\approx} r^d_e \circ h$ *for every map* $h : Z \to Y_e$, *then* g *is a* U*-map.*

(b) *If for every* $d \in C$ *and every* $\sigma \in \widetilde{Y}_d$ *there is an* $e \geq d$ *and an* $\varepsilon \in \widetilde{Y}_e$ *such that* $r^d \circ g \overset{\sigma}{\approx} r^d_e \circ h$ *for every* ε*-function* $h : Z \to Y_e$, *then* g *is a* P*-map.*

PROOF OF (a). It suffices to prove that $g \overset{\varepsilon}{\approx} h$ for every map $h : Z \to Y$ and every $\varepsilon \in \widetilde{Y}$. We first apply [7, (4.2), (2.11), (2.9)] to get an index $c \in C$ and an $\eta \in \widetilde{Y}_c$ such that $(r^c)^{-1}(\eta)$ refines ε. Let $\sigma \in \eta^{+2}$. Then we utilize the property (A2) and [7, (1.14), (1.9), (1.10)] again to get an index $d \geq c$ such that for every $e \geq d$ we have

$$r^c_e \overset{\sigma}{=} r^c_d \circ r^d_e, \quad \text{and} \tag{9}$$

$$r^c \overset{\sigma}{=} r^c_e \circ r^e. \tag{10}$$

Finally, we use the above assumption, to get an $e \geq d$ such that $r^c_d \circ r^d \circ g \overset{\sigma}{=} r^c_d \circ r^d_e \circ k$ for every map $k : Z \to Y_e$. In particular, there is a point $z \in Z$ such that $r^c_d \circ r^d \circ g(z) \overset{\sigma}{=} r^c_d \circ r^d_e \circ r^e \circ h(z)$. With relations (9) and (10) we get $r^c \circ g(z) \overset{\eta}{=} r^c \circ h(z)$. Therefore, $g \overset{\varepsilon}{\approx} h$.

PROOF OF (b). Let $\varepsilon \in \widetilde{Y}$. We shall show the existence of a $\delta \in \widetilde{Y}$ with the property that for every δ-function $h : X \to Y$ there is an $x \in X$ with $g(x) \overset{\varepsilon}{=} h(x)$.

We first choose indices c and d in C and covers η and σ in $\widetilde{Y}_c$ as in the proof of (a). Then we use the above assumption to get an $e \geq d$ and a $\gamma \in \widetilde{Y}_e$ such that for every γ-function $k : X \to Y_e$ there is an $x \in X$ with $r^c_d \circ r^d \circ g(x) \overset{\sigma}{=} r^c_d \circ r^d_e \circ k(x)$. Let $\delta = (r^e)^{-1}(\gamma)$.

Let $h : X \to Y$ be a δ-function. Then $r^e \circ h : X \to Y_e$ is a γ-function. Hence, there is an $x \in X$ with

$$r^c_d \circ r^d \circ g(x) \overset{\sigma}{=} r^c_d \circ r^d_e \circ r^e \circ h(x). \tag{11}$$

It is easy to verify that (9), (10), and (11) imply $r^c \circ g(x) \overset{\eta}{=} r^c \circ h(x)$. Hence, $g(x) \overset{\varepsilon}{=} h(x)$.

A mapping F is a *U-mapping* provided for every $c \in C$ and every $\sigma \in \widetilde{Y}_c$ there is an $a \geq \varphi(c)$ with $g \overset{\sigma}{\approx} f_c \circ p^{\varphi(c)}_b$ for every $b \geq a$ and every map $g : X_b \to Y_c$. It is a *P-mapping* provided for every $c \in C$ and every $\sigma \in \widetilde{Y}_c$ there is a $\pi \in \sigma^+$ and an $a \geq \varphi(c)$ with $g \overset{\sigma}{\approx} f_c \circ p^{\varphi(c)}_b$ for every $b \geq a$ and every π-function $g : X_b \to Y_c$.

A system $\mathscr{X}$ is a *U-system* (a *P-system*) provided the mapping $J : \mathscr{X} \to \mathscr{X}$ is a U-mapping (a P-mapping).

A system $\mathscr{Y}$ is a *V-system* provided that for every $c \in C$ and every $\sigma \in \widetilde{Y}_c$ there is a $d \geq c$ such that for every map $g : W \to Y_d$ defined on a closed subset W of a space Z there is a neighbourhood N of W in Z and a map $h : N \to Y_d$ which satisfies $r^c_d \circ g \stackrel{\sigma}{=} r^c_d \circ h|_W$.

The following lemma implies that every AP-system (that is, a system with the approximate polyhedra as bonding spaces) and therefore every ANR-system is a V-system. Recall [7, (2.3)] that a space X is an approximate polyhedron provided that for every $\sigma \in \widetilde{X}$ there is a finite polyhedron P and maps $u : X \to P$ and $d : P \to X$ with $id \stackrel{\sigma}{=} d \circ u$.

LEMMA 3. *If X is an approximate polyhedron, then for every $\sigma \in \widetilde{X}$ and every map $f : W \to X$ defined on a closed subset W of a space Z there is a neighbourhood N of W in Z and a map $g : N \to X$ with $f \stackrel{\sigma}{=} g|_W$.*

PROOF. Choose a finite polyhedron P and maps $u : X \to P$ and $d : P \to X$ such that $id \stackrel{\sigma}{=} d \circ u$. Let $h = u \circ f$. Since finite polyhedra are absolute neighbourhood extensors for the class of all compact Hausdorff spaces [5], there is a neighbourhood N of W in Z and a map $k : N \to P$ with $h = k|_W$. Let $g = d \circ k$.

A system $\mathscr{Y}$ is a *Q-system* provided that for every $c \in C$ and every $\sigma \in \widetilde{Y}_c$ there is a $d \geq c$ and a $\pi \in \widetilde{Y}_d$ so that for every π-function $g : W \to Y_d$ defined on a closed subset W of a space Z there is a neighbourhood N of W in Z and a function $h : N \to Y_d$ such that the composition $r^c_d \circ h$ is a σ-function and $r^c_d \circ g \stackrel{\sigma}{=} r^c_d \circ h|_W$.

The second of the following two lemmas implies that every PANR-system (that is, a system with proximate absolute neighbourhood retracts as bonding spaces) is a Q-system. Recall [6] that a space X is a proximate absolute neighbourhood retract (PANR) provided for every $\sigma \in \widetilde{X}$ and every space Y which contains X as a closed subset there is a neighbourhood N of X in Y and some σ-function $r : N \to X$ with $id \stackrel{\sigma}{=} r|_X$.

LEMMA 4. *If a space X is an approximate polyhedron, then for every $\xi \in \widetilde{X}$ there is a $\delta \in \widetilde{X}$ such that for every δ-function $f : Z \to X$ there is a map $g : Z \to X$ with $f \stackrel{\xi}{=} g$.*

PROOF. Let $\xi \in \widetilde{X}$. Let $\eta \in \xi^{+1}$. Choose a finite polyhedron P and maps $u : X \to P$ and $d : P \to X$ such that $id \stackrel{\eta}{=} d \circ u$. Let $\beta = d^{-1}(\eta)$. By Theorem 2.1 in [1], there exists a $\gamma \in \widetilde{P}$ such that for every γ-function $h : Z \to P$ there is a map $k : Z \to P$ with $h \stackrel{\beta}{=} k$. Put $\delta = u^{-1}(\gamma)$.

Let $f : Z \to X$ be a δ-function. Let $h = u \circ f$. Then $h : Z \to P$ is a γ-function. Hence, there is a map $k : Z \to P$ such that $h \stackrel{\beta}{=} k$. Put $g = d \circ k$. Clearly, $g : Z \to X$ is continuous and $f \stackrel{\xi}{=} g$.

LEMMA 5. *A space X is a* PANR *if and only if for every* $\xi \in \widetilde{X}$ *there is a* $\delta \in \widetilde{X}$ *so that for every* δ*-function* $f : W \to X$ *defined on a closed subset* W *of a space* Z *there is a neighbourhood* N *of* W *in* Z *and a* ξ*-function* $g : N \to X$ *with* $f \overset{\xi}{=} g|_W$.

PROOF. Suppose X is a PANR. Let $\xi \in \widetilde{X}$. Let $\eta \in \xi^{+2}$. We consider X as a closed subset of a suitable Tychonoff cube K. Since X is a PANR, there is an approximate polyhedron neighbourhood M of X in K and an η-function $r : M \to X$ with $id \overset{\eta}{=} r|_X$. For every $m \in M$, let V_m be an open set in M such that $y \in V_m$ implies $r(y) \overset{\eta}{=} r(m)$. Let $\gamma = \{V_m : m \in M\}$. By Lemma 4, we can find a $\beta \in \widetilde{M}$ such that every β-function into M has a continuous γ-approximation. Let $\delta = \{U \cap X : U \in \beta\}$.

Consider a δ-function $f : W \to X$ defined on a closed subset W of a space Z. Choose a map $h : W \to M$ with $f \overset{\gamma}{=} h$. By Lemma 3 , there is a neighbourhood N of W in Z and a map $k : N \to M$ with $h \overset{\gamma}{=} k|_W$. Put $g = r \circ k : N \to X$.

Then g is obviously a ξ-function. Let $w \in W$. Select $m, n \in M$ such that $h(w), k(w) \in V_m$ and $h(w), f(w) \in V_n$. Hence, we can find $S, T, U, V \in \eta$ with $g(w) \in S$, $r \circ h(w) \in T$, $r(m) \in S \cap T$, $r \circ h(w) \in U$, $f(w) \in V$, and $r(n) \in U \cap V$. It follows that $f \overset{\xi}{=} g|_W$.

The converse implication is obvious.

3. Universality of limits

The previous results will be applied here to prove the following theorem.

THEOREM.

(c) *If* F *is a* U*-mapping of a system* $\mathscr{X}$ *into a* V*-system* $\mathscr{Y}$*, then the limit* $f : X \to Y$ *of* F *is a* U*-map.*

(d) *If* F *is a* P*-mapping of a system* $\mathscr{X}$ *into a* Q*-system* $\mathscr{Y}$*, then the limit* $f : X \to Y$ *of* F *is a* P*-map.*

PROOF OF (c). By Lemma 2 (a), it suffices to show for every $c \in C$ and every $\varepsilon \in \widetilde{Y}_c$ there is a $d \geq c$ such that for every map $g : X \to Y_d$ there is an $x \in X$ with

$$r^c \circ f(x) \overset{\varepsilon}{=} r_d^c \circ g(x). \tag{12}$$

Let $c \in C$ and $\varepsilon \in \widetilde{Y}_c$ be given. Let $\eta \in \varepsilon^{+4}$. We utilize now the property (A3), [7, (1.14), (1.9), (1.10)], the property (A2), and the fact that $\mathscr{Y}$ is a V-system to select indices $p, q, r \in c^+$, an $s \in p^+ \cap q^+ \cap r^+$, and a $d \geq s$ such that

$$st^2(\sigma_m) \in (r_m^c)^{-1}(\eta)^+ \text{ for every } m \geq p, \tag{13}$$

$$r^c \overset{\eta}{=} r_m^c \circ r^m \text{ for every } m \geq q, \tag{14}$$

$$r_n^c \overset{\eta}{=} r_m^c \circ r_n^m \text{ for every } n \geq m \geq r, \tag{15}$$

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and for every map $j : W \to Y_d$ defined on a closed subset W of a space Z there is a neighbourhood N of W in Z and a map $k : N \to Y_d$ such that the relation $r_s^c \circ r_d^s \circ j \overset{\eta}{=} r_s^c \circ r_d^s \circ (k|_W)$ holds.

Consider a map $g : X \to Y_d$. Choose a compact neighbourhood N of X in $\prod \mathscr{X}$ and a map $h : N \to Y_d$ such that

$$r_s^c \circ r_d^s \circ g \overset{\eta}{=} r_s^c \circ r_d^s \circ (h|_X). \tag{16}$$

Let $\xi = (r_d^c)^{-1}(\eta) \in \widetilde{Y}_d$. Select a $\rho \in \widetilde{N}$ such that $v \overset{\rho}{=} w$ in N implies $h(v) \overset{\xi}{=} h(w)$. Let $\mu = (f_d)^{-1}(\xi) \in \widetilde{X}_d$.

We use now Lemma 1 and the assumption that F is a U-mapping to get an index $m \geq \varphi(d)$ such that $i_n(X_n) \subset N$, for every $n \geq m$, and for every $n \geq m$ and every map $t : X_n \to Y_d$ there is a $z \in X_n$ with $t(z) \overset{\eta}{=} f_d \circ p_n^{\varphi(d)}(z)$. Let $n \geq m$. Since $h \circ i_n : X_n \to Y_d$ is well-defined and continuous, there is a point $x_n \in X_n$ with

$$r_d^c \circ h \circ i_n(x_n) \overset{\eta}{=} r_d^c \circ f_d \circ p_n^{\varphi(d)}(x_n). \tag{17}$$

It follows from Lemma 1 that the net $\{i_n(x_n) : n \geq m\}$ has an adherence point x in X. This x is the required point.

Indeed, let M and R be members of μ and ρ which contain $x_{\varphi(d)}$ and x, respectively. Let $V = R \cap (M \times \prod\{X_a : a \in A,\ a \neq \varphi(d)\})$. Choose an $n \geq m$ so that $i_n(x_n) \in V$. Since the $\varphi(d)$-coordinate of $i_n(x_n)$ is $p_n^{\varphi(d)}(x_n)$, it follows that both $x_{\varphi(d)}$ and $p_n^{\varphi(d)}(x_n)$ are in M. Hence,

$$r_d^c \circ f_d \circ p^{\varphi(d)}(x) \overset{\eta}{=} r_d^c \circ f_d \circ p_n^{\varphi(d)}(x_n). \tag{18}$$

Since the points $i_n(x_n)$ and x are in R , we get

$$r_d^c \circ h(x) \overset{\eta}{=} r_d^c \circ h \circ i_n(x_n). \tag{19}$$

From (13) and (AM), we get

$$r_d^c \circ r^d \circ f \overset{\eta}{=} r_d^c \circ f_d \circ p^{\varphi(d)}. \tag{20}$$

From (14), we get

$$r^c \circ f(x) \overset{\eta}{=} r_d^c \circ r^d \circ f(x), \tag{21}$$

while from (15), we get

$$r_d^c \circ h(x) \overset{\eta}{=} r_s^c \circ r_d^s \circ h(x), \tag{22}$$

and

$$r_d^c \circ g(x) \overset{\eta}{=} r_s^c \circ r_d^s \circ g(x). \tag{23}$$

Relations (16) – (23) together imply (12).

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PROOF OF (d). By Lemma 2 (b), it suffices to show that for every $c \in C$ and every $\varepsilon \in \widetilde{Y}_c$ there is a $d \geq c$ and a $\delta \in \widetilde{Y}_d$ such that for every δ-function $g : X \to Y_d$ there is an $x \in X$ with

$$r^c \circ f(x) \overset{\varepsilon}{=} r^c_d \circ g(x) \tag{24}$$

Let a $c \in C$ and an $\varepsilon \in \widetilde{Y}_c$ be given. Let $\eta \in \varepsilon^{+4}$. Select indices $p, q, r, s, d \in C$ and an $a \in A$ and covers $\xi, \delta \in \widetilde{Y}_d$ such that $p, q, r \in c^+$, $s \geq p$, $s \geq r$, $a \geq \varphi(s)$, $d \geq \varphi(s)$,

$$st^2(\sigma_m) \in (r^c_m)^{-1}(\eta)^+ \text{ for every } m \geq p, \tag{25}$$

$$r^c \overset{\eta}{=} r^c_m \circ r^m \text{ for every } m \geq q, \tag{26}$$

$$r^c_n \overset{\eta}{=} r^c_m \circ r^m_n \text{ for every } n \geq m \geq r, \tag{27}$$

and for every $b \geq a$ and every ξ-function $i : X_b \to Y_s$ there is an $x \in X_b$ with $r^c_s \circ i(x) \overset{\eta}{=} r^c_s \circ f_s \circ p^{\varphi(s)}_b(x)$, and for every δ-function $j : W \to Y_d$ defined on a closed subset W of a space Z there is a neighbourhood N of W in Z and a function $k : N \to Y_d$ so that $r^s_d \circ k$ is a ξ-function and $r^c_s \circ r^s_d \circ j \overset{\eta}{=} r^c_s \circ r^s_d \circ (k|_W)$.

Consider a δ-function $g : X \to Y_d$. Choose a compact neighbourhood N of X in $\prod \mathscr{X}$ and a function $h : N \to Y_d$ such that $r^s_d \circ h$ is a ξ-function and

$$r^c_s \circ r^s_d \circ g \overset{\eta}{=} r^c_s \circ r^s_d \circ (h|_X). \tag{28}$$

Let $\nu = (r^c_d)^{-1}(\eta)$. Select a $\rho \in \widetilde{N}$ such that $v \overset{\rho}{=} w$ in N implies $h(v) \overset{\nu}{=} h(w)$. Let $\mu = (f_d)^{-1}(\nu) \in \widetilde{X}_d$.

By Lemma 1, there is an index $m \in A$ such that $m \geq d$ and $i_n(X_n) \subset N$ for every $n \geq m$. Let $n \geq m$. Since $r^s_d \circ h \circ i_n : X_n \to Y_s$ is a well-defined ξ-function, there is an $x_n \in X_n$ with

$$r^c_s \circ f_s \circ p^{\varphi(s)}_n(x_n) \overset{\eta}{=} r^c_s \circ r^s_d \circ h \circ i_n(x_n). \tag{29}$$

The rest of the proof (that is, the selection of a point x and the verification that it has the required property) is analogous to the proof of (c) and we leave it to the reader.

COROLLARY 1.

(i) *A limit X of a system $\mathscr{X}$ that is both a U-system and a V-system has the fixed point property.*

(ii) *A limit X of a system $\mathscr{X}$ that is both a P-system and a Q-system has the proximate fixed point property.*

PROOF OF (i). Since, by assumption, the identity mapping J is a U-mapping, it follows from the theorem that the identity map id_X on X is universal. However, this implies that X has the fixed point property (see Proposition 3 in [3]).

PROOF OF (ii). As above we conclude that id_X is proximately universal. Hence, X has the proximate fixed point property (see [8, Theorem 4]).

COROLLARY 2. *Every chainable compact Hausdorff space X has the proximate fixed point property.*

PROOF. We can assume that the space X is non-degenerate. It is known that X is homeomorphic to a limit of a system $\mathscr{X}$ with each X_a the closed unit segment I and each p_b^a an onto map. Since I is a PANR and every map of I onto itself is a U-map [8], it follows that $\mathscr{X}$ is both a P-system and a Q-system. The conclusion is now a consequence of Corollary 1.

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Kopernikova 7
10020 Zagreb
Croatia
e-mail: cerin@cromath.math.hr

Departamento de Geometría y Topología
Facultad de Matemáticas
Universidad Complutense
28040 Madrid
Spain
e-mail: w745@emducm11.bitnet