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SOME APPLICATIONS OF GROUP OBSTRUCTIONS

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Abstract

We show that the classical interpretation of $H^3(G, A)$ is equivalent to Taylor's solutions of compound extensions of groups. It is also equivalent to the exactness to an eight term sequence. Only halves of the equivalences are fully shown in the paper but the other halves are clear.

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In [10], an invariant proof of the obstruction theorems is given, (see Section 1 for statements of results). In this note, we will point out that if obstruction theory is viewed in this way, some old problems can be proved readily. The first problem we consider is Taylor's compound extension problem [7] which asks for conditions to complete the following diagram

$$\begin{array}{ccccc} A & = & A & & \\ \downarrow & & \downarrow & & \\ E & \rightarrowtail & ? & \twoheadrightarrow & Q \\ \downarrow & \Sigma & \downarrow & & \| \\ N & \rightarrowtail & G & \twoheadrightarrow & Q \end{array}$$

so that rows and columns are exact. He showed that the problem can be reduced to the case where A is central in E. Taylor's original solution of this reduced problem was very lengthy. This is probably because he did not recognize that his problem is exactly the osbstruction problem and had to modify the whole obstruction theory developed by Eilenberg-Mac Lane. We identify the two problems once we observe that the square Σ is a pullback.

Another equivalent form of the obstruction theorem can be stated as follows: Let $1 \to N \to G \to Q \to 1$ be an exact sequence. The center A of N can be endowed with an Q-module structure [5]. The obstruction theorem (actually part (a) of it (see Theorem 2a)) is equivalent to the exactness of the sequence

$$0 \to \mathrm{Der}(Q, A) \to \mathrm{Der}(G, A) \to \mathrm{Hom}_Q(N_{ab}, A) \to H^2(Q, A)$$
$$\to H^2(G, A) \to \mathrm{Sext}^1_G(N, A) \to H^3(Q, A) \to H^3(G, A)$$

where $\mathrm{Der}(Q, A)$ is the group of derivations, N_{ab} the abelianization of N and $\mathrm{Sext}^1_G(N, A)$ the group of equivalence classes of some short exact sequences, (see Section 1). This sequence is not new [3, 4, 6, 8] but the available proofs are fairly lengthy.

Finally, we should point out that results considered in this note are clearly valid over many other algebraic categories, such as the category of Lie algebras over a field.

1. Preliminaries

Let G be a group. A G-group E is a group together with an action σ: $G \to \mathrm{Aut}(E)$. Clearly G is a G-group with conjugation as an action and if π: $G \to Q$ is a group homomorphism, then Q is a G-group and π is a G-homomorphism. If A is a Q-module, we recall that a special one-fold extension [10] of A by Q is an exact sequence

(1.1) $$\xi\colon 0 \to A \to G \xrightarrow{\pi} Q \to 1$$

of G-groups where the action of G on A is induced by π. Special one-fold extensions are also called abelian extensions [5].

A special two-fold extension is an exact sequence

(1.2) $$\xi\colon 0 \to A \xrightarrow{i} E \xrightarrow{\phi} G \xrightarrow{\pi} Q \to 1$$

of G-groups and G-homomorphisms with the following properties:

(i) The G-action on A is given by the Q-action on A via π,

(ii) The E-action on E by conjugation coincides with the E-action induced by ϕ.

Special two-fold extensions are also called *crossed modules* (with prescribed ends). It is clear that in a special two-fold extension (1.2), A is central in E.

We define equivalence relations on special one-fold and two-fold extensions analogous to that on module extensions and denote the sets of equivalence clsses by $\mathrm{Sext}^1(Q, A)$ and $\mathrm{Sext}^2(Q, A)$ respectively.

If we fix a free presentation $1 \to R \to F \to Q \to 1$, then every equivalence class in $\mathrm{Sext}^2(Q, A)$ has a canonical representative of the form

(1.3) $$\xi_{F\tau}\colon 0 \to A \to E \to F \to Q \to 1$$

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The equivalence class is zero if and only if the F-sequence $0 \to A \to E \to R \to 1$ has a right F-splitting.

Since every equivalence class of special two-fold extensions comes from an abstract Q-kernel [10], we call the special two-fold extension ξ in (1.2) *extendable* if and only if there exists an exact sequence of H-groups, $1 \to E \to H \to Q \to 1$ such that the following diagram is commutative

$$(1.4)\qquad \begin{array}{lccccccccc} \bar{\xi}: & & & 1 & \longrightarrow & E & \rightarrowtail & H & \twoheadrightarrow & Q & \longrightarrow & 1 \\ & & & & & \| & & \downarrow & & \| & & \\ \xi: & 0 & \longrightarrow & A & \longrightarrow & E & \rightarrowtail & G & \twoheadrightarrow & Q & \longrightarrow & 1 \end{array}$$

and the G-action on E comes from H. (Note that the kernel of $H \to G$ is necessarily A and H defines a G-action on E because A is central in E.) If this is the case, we shall say that $\bar{\xi}$ induces ξ. The following theorems are proved in [10].

THEOREM 1. $H^3(A, A) \overset{Nat}{\cong} \mathrm{Sext}^2(Q, A)$.

THEOREM 2. a) *The special two-fold extension* ξ *is extendable if and only if* $[\xi] = 0$.

b) *If* $[\xi] = 0$, *then the set of equivalence classes* $\bar{\xi}$ *inducing* ξ *is in one-one correspondence with* $H^2(Q, A)$.

(Theorem 2 is stated slightly different from that in [10] to suit our purpose but the proof is exactly the same. The same statement can also be found in [2].)

In this paper we also deal with short exact sequence of G-groups. If N is a normal subgroup of G, and A is an N-module, then $\mathrm{Sext}^1_G(N, A)$ will denote the group of equivalence classes of extensions of A by N

$$\xi_G\colon 0 \to A \to E \xrightarrow{\phi} N \to 1$$

which are both an E-sequence as well as a G-sequence. Moreover, the action of N on E is given by $\phi(e) \cdot e_1 = e e_1 e^{-1}$, $e, e_1 \in E$.

2. Compound extensions

In [7], Taylor investigates conditions of filling the following diagram with exact rows and columns,

$$(2.1)\qquad \begin{array}{ccccc} A & = & A & & \\ \downarrow & & \downarrow & & \\ E & \rightarrowtail & ? & \twoheadrightarrow & Q \\ \downarrow & & \downarrow & & \| \\ N & \rightarrowtail & G & \twoheadrightarrow & Q \end{array}$$

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He observed that problem can be reduced to a problem where A is central in E.

We will discuss this problem assuming A is central in E. We will say that (2.1) is *weakly extendable*† if there is a group H such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E & \rightarrowtail & H & \twoheadrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow & \Sigma & \downarrow \theta & & \| & & \\ 1 & \longrightarrow & N & \rightarrowtail & G & \twoheadrightarrow & Q & \longrightarrow & 1 \end{array} \tag{2.2}$$

is commutative with exact rows. Note that the diagram Σ is necessarily a pullback diagram and the kernel of θ is A. We also note that if the diagram (2.1) is weakly extendable, then we can define a G-action on E via θ, that is $\theta(h)e = heh^{-1}$. This is a well-defined action because A is central in E. With such a G-action on E, we can easily see that the sequence

$$\xi_H: 0 \to A \to E \to G \to Q \to 1$$

is a special two-fold extension. Therefore we may discuss Taylor's problem of (2.1) with the additional assumption that $0 \to A \to E \to G \to Q \to 1$ is a special extension. Thus we make the following definitions.

A special two-fold extension

$$\xi: 0 \to A \to E \to G \to Q \to 1 \tag{2.3}$$

is *extendable* if the associated compound extension problem (2.1) is weakly extendable *and* if the G-action on E is obtained from the extension as described in the paragraph following (2.2). The special two-fold extension (2.3) is *weakly extendable* if the associated compound extension (2.1) is weakly extendable. From Section 1, it is therefore clear that Taylor extension problem follows from the obstruction theorem (Theorem 2a).

The weak extension problem of (2.1), that is, we just ask when we can fill in the diagram without insisting on the special structure constraint on E, is more interesting. To solve the weak extension problem, we first have to investigate relations among various special extension structures on a given sequence

$$\xi: 0 \to A \to E \to G \to Q \to 1$$

of groups. This is given in the next lemma.

LEMMA 4. *Let* $1 \to N \to G \to Q \to 1$ *be an exact sequence. Let* $0 \to A \to E \to N \to 1$ *be an exact sequence where* A *is a* Q*-module. Let* $\sigma, \tau: G \to \mathrm{Aut}(E)$ *be two actions of* G *on* E *so that*

$$\xi: 0 \to A \xrightarrow{i} E \xrightarrow{\phi} G \xrightarrow{\pi} Q \to 1$$

†Weak extendability is really not a very good choice but in order not to cause confusion with the *extendability* of special extensions, we probably cannot call it "extendable," see Definition (2.3).

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is a special two-fold extension with either action. Then there is a homomorphism f: $Q \to \mathrm{Hom}(N, A)$ *so that any two of* σ, τ, f *uniquely determine the third.*

PROOF. Let $g \in G$, $e \in E$. Then since ϕ is a G-map,

$$\sigma(g)(e) = \tau(g)(e) + f(g)(e), \qquad f(g)(e) \in A.$$

Clearly since the G-module structure of A is fixed, $\sigma(g)(a) = \tau(g)(a)$. Thus $f(g)(a) = 0$ and we may regard f: $G \to \mathrm{Hom}(N, A)$. If $g \in N$, then $g = \phi(e')$. By definition of special extensions, $e' \cdot e = \sigma(g)(e) = \tau(g)(e)$. Hence $f(g) = 0$ and f: $Q \to \mathrm{Hom}(N, A)$.

Recall that $\mathrm{Hom}(N, A)$ can be made into a Q-module with action given by $(\pi(g) \cdot \alpha)(n) = g \cdot (\alpha(g^{-1} \cdot n))$ for $g \in G$, $\alpha \in \mathrm{Hom}(N, A)$ and $n \in N$. Also if we interpret $H^2(N, A)$ to be the sete of equivalence classes of central extensions of A by N, we have an obvious map u: $\mathrm{Sext}^1_G(N, A) \to H^2(N, A)$.

THEOREM 5. *Let* $[\xi] \in H^2(N, A)$. *Then there is a bijective map* Ψ_ξ: $H^1(Q, \mathrm{Hom}(N, A)) \to u^{-1}([\xi])$.

PROOF. Let $0 \to A \to E \to N \to 1$ with action τ: $G \to \mathrm{Aut}(E)$ represent an element of $u^{-1}([\xi])$. If σ: $G \to \mathrm{Aut}(E)$ is another action, Lemma 4 says that σ is obtained from τ by adding f. Let d_f: $Q \to \mathrm{Hom}(N, A)$ be defined by

$$d_f(\pi(g))(n) = f(\pi(g))(g^{-1} \cdot n).$$

Simple calculation shows that d_f is a derivation. It is also clear that there is a bijective correspondence between such f's and d_f's. Thus we have a surjective homomorphism from $\mathrm{Der}(Q, \mathrm{Hom}(N, A))$ to $u^{-1}([\xi])$. If the two actions σ, τ of G on E give equivalent extensions in $\mathrm{Sext}^1_G(N, A)$, then there is a G-isomorphism θ: $E \to E$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \rightarrowtail & E & \twoheadrightarrow & N & \longrightarrow & 1 \\ & & \| & & \downarrow\theta & & \| & & \\ 0 & \longrightarrow & A & \rightarrowtail & E & \twoheadrightarrow & N & \longrightarrow & 1 \end{array}$$

commutative. Thus $\theta(e) = e - \alpha(\phi(e))$ for some $\alpha \in \mathrm{Hom}(N, A)$. If $\sigma(g)(e) = \tau(g)(e) + f(g)(\phi(e))$, then $f(g)(\phi(e)) = \alpha(g \cdot \phi(e)) - g \cdot \alpha(\phi(e))$. Therefore $d_f(g)(n) = \alpha(n) - (g \cdot \alpha)(n)$ or d_f is a principal derivation. This shows that we have a bijection map Ψ_ξ: $H^1(Q, \mathrm{Hom}(N, A)) \to u^{-1}([\xi])$.

COROLLARY 5 (Ratcliffe [6]). *The sequence*

$$0 \to H^1(Q, \mathrm{Hom}(N, A)) \xrightarrow{\Psi_0} \mathrm{Sext}^1_G(N, A) \to H^2(N, A)$$

is exact.

PROOF. We only have to verify Ψ_0 is a homomorphism. Suppose that σ, τ: $G \to \mathrm{Aut}(E)$ are two actions on $E = A \times N$ (as groups) which give two representatives of elements of $u^{-1}(0)$.

$$\xi_\sigma\colon 0 \to A \to E \to N \to 1,$$
$$\xi_\tau\colon 0 \to A \to E \to N \to 1.$$

By Lemma 4, $\sigma(g)(a, n) = (g \cdot a + f_1(g)(n), g \cdot n)$ and $\tau(g)(a, n) = (g \cdot a + f_2(g)(n), g \cdot n)$ for some f_1, f_2: $Q \to \mathrm{Hom}(N, A)$. The class $[\xi_\sigma] + [\xi_\tau]$ is represented by the extension $\xi_{\sigma+\tau}$ of the following diagram

$$\begin{array}{ccccccccccc}
 & 0 & \longrightarrow & A \times A & \rightarrowtail & E \times E & \twoheadrightarrow & N \times N & \longrightarrow & 1 \\
 & & & \| & & \beta\uparrow & \Sigma & \uparrow\Delta & & \\
 & 0 & \longrightarrow & A \times A & \rightarrowtail & E' & \twoheadrightarrow & N & \longrightarrow & 1 \\
 & & & \nabla\downarrow & \Sigma' & \downarrow\gamma & & \| & & \\
\xi_{\sigma+\tau}\colon & 0 & \longrightarrow & A & \rightarrowtail & E & \twoheadrightarrow & N & \longrightarrow & 1
\end{array}$$

where Δ is the diagonal map, ∇ the codiagonal map, $E' = \{(a_1, a_2, n) | a_1, a_2 \in A\}$ with G-action $g^*(a_1, a_2, n) = (a_1 + f_1(g)(n), a_2 + f_2(g)(n), n)$, $\beta(a_1, a_2, n) = ((a_1, n), (a_2, n))$, $\gamma(a_1, a_2, n) = (a_1 + a_2, n)$, Σ a pullback diagram and Σ' a "pushout" diagram. It is clear from this construction that the G-action on E in $\xi_{\sigma+\tau}$ is given by $f_1 + f_2$. Hence Ψ_0 is a homomorphism.

THEOREM 6. *Let*

$$\xi_\sigma\colon 0 \to A \to E \to G \to A \to 1$$

be a special two-fold extension with a G-action σ on E. Then ξ_σ is weakly extendable if and only if there is a derivation d such that the action τ: $G \to \mathrm{Aut}(E)$ given by $\tau(g)(e) = \sigma(g)(e) + d(g)(g \cdot \sigma(e))$ makes $0 \to A \to E \to G \to Q \to 1$ a representative of the zero element in $\mathrm{Sext}^2(Q, A)$.

If it is extendable, then the equivalence classes $1 \to E \to H \to G \to 1$ in the solutions are in one-one correspondence with $H^2(Q, A)$.

3. An exact sequence

Let $1 \to N \xrightarrow{i} G \xrightarrow{j} Q \to 1$ be an exact sequence of groups. Then $N_{ab} = N/[N, N]$ is a Q-module. Let A be a Q-module.

THEOREM 7. *The following sequence is exact,*

$$\begin{aligned}
&0 \to \mathrm{Der}(Q, A) \to \mathrm{Der}(G, A) \to \mathrm{Hom}_Q(N_{ab}, A) \to H^2(Q, A) \\
&\xrightarrow{j^*} H^2(G, A) \xrightarrow{i^*} \mathrm{Sext}^1_G(N, A) \xrightarrow{c} H^3(Q, A) \xrightarrow{j^*} H^3(G, A)
\end{aligned} \tag{3.1}$$

The sequence is natural in the sequence $1 \to N \to G \to G \to 1$. Exactness of

$$0 \to \mathrm{Der}(Q, A) \to \mathrm{Der}(G, A) \to \mathrm{Hom}_Q(N_{ab}, A) \to H^2(Q, A) \to H^2(G, A)$$

was proved by Hochschild and Serre (see [5]). A conceptual proof was first given by Barr and Rinehart [1]. Since our approach has no cocycles, we state this theorem as in [1]. The exactness of the remaining terms were proved by Rinehart [8], Huesbchmann [3], Loday [4], and Ratcliffe [6].

Proof. We only prove the exactness of the last five terms. The naturality becomes clear once we describe the maps.

We will regard $H^2(G, A) = \mathrm{Sext}^1(G, A)$, $H^3(G, A) = \mathrm{Sext}^2(G, A)$, and so on. Then the maps i^*, j^* are just pulling back along i and j. The connecting homomorphism c is the splicing map.

Exactness at $H^2(G, A)$. Let ξ_G: $0 \to A \to E_G \to G \to 1$ represent an element in $H^2(G, A)$. Suppose $[i^*(\xi_G)] = 0$,

$$\begin{array}{rcccccccc} & 0 & \longrightarrow & A & \rightarrowtail & E_G & \overset{\pi_G}{\twoheadrightarrow} & G & \longrightarrow & 1 \\ & & & \| & & \tilde{i}\uparrow & & \uparrow i & & \\ i^*(\xi_G): & 0 & \longrightarrow & A & \rightarrowtail & E_N & \overset{\pi_N}{\twoheadrightarrow} & N & \longrightarrow & 1 \end{array}$$

that is, there is a splitting homomorphism s_N: $N \to E_N$ such that $\pi_N s_N = 1_N$. Then $\tilde{i}s_N(N)$ is normal in E_G. Let $E_Q = \mathrm{Cok}\ \tilde{i}s_N$.

$$\begin{array}{ccccccccc} 1 & \to & N & \overset{\tilde{i}s_N}{\rightarrowtail} & E_G & \twoheadrightarrow & E_Q & \to & 1 \\ & & \| & & \downarrow & \Sigma & \downarrow & & \\ 1 & \to & N & \rightarrowtail & G & \twoheadrightarrow & Q & \to & 1 \end{array}$$

Then the square Σ is a pullback diagram since $E_G \to E$ and $G \to Q$ have the same kernels. Thus $0 \to A \to E_Q \to Q \to 1$ is an exact sequence. Indeed ξ_Q: $0 \to A \to E_Q \to Q \to 1$ represents an element in $H^2(G, A)$ and $j^*(\xi_Q) = \xi_G$. This shows $\mathrm{Im}\, j^* \supseteq \mathrm{Ker}\ i^*$. It is trivial to see that $i^*j^* = 0$; hence $\mathrm{Im}\, j^* = \mathrm{Ker}\ i^*$.

Exactness at $\mathrm{Sext}^1_G(N, A)$. Let $[\xi_N] \in \mathrm{Sext}^1_G(N, A)$,

$$\xi_N\colon 0 \to A \to E_N \to N \to 1$$

Then $c[\xi_N] = [c\xi_N]$,

$$c(\xi_N)\colon 0 \to A \to E_N \to G \to Q \to 1$$

is zero if and only if there is E_G such that

$$\text{(i)}\qquad \begin{array}{ccccccccc} & & A & = & A & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & E_N & \rightarrowtail & E_G & \twoheadrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow & \Sigma & \downarrow & & \| & & \\ 1 & \longrightarrow & N & \rightarrowtail & G & \twoheadrightarrow & Q & \longrightarrow & 1 \end{array}$$

is commutative and (ii) the G-action on E_N (hence A) comes from E_G. Condition (i) says that Σ is a pullback. This coupled with (ii) implies that $0 \to A \to E_N \to N \to 1 = i^*(0 \to A \to E_G \to G \to 1)$. This proves the exactness at $\mathrm{Sext}^1_G(N, A)$.

Exactness at $H^3(Q, A)$. Let $[\xi_N] \in \mathrm{Sext}^1_G(N, A)$.

$$\xi_N\colon 0 \to A \to E_N \to N \to 1$$

Then

$$\begin{array}{lccccccccccc}
c\xi_N\colon & 0 & \longrightarrow & A & \rightarrowtail & E_N & \longrightarrow & G & \twoheadrightarrow & Q & \longrightarrow & 1 \\
 & & & \| & & \| & & \tilde{j}\uparrow & \Sigma & \uparrow j & & \\
j^*(c\xi_N)\colon & 0 & \longrightarrow & A & \rightarrowtail & E_N & \longrightarrow & N \times_s B & \twoheadrightarrow & G & \longrightarrow & 1
\end{array}$$

where $\tilde{j}(n, g) = g$. Since

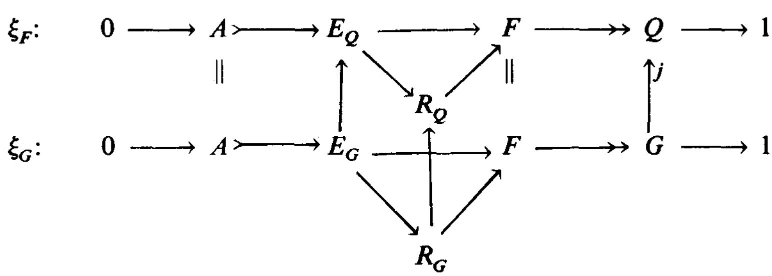
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \rightarrowtail & E_N & \longrightarrow & N \times_s G & \twoheadrightarrow & G & \longrightarrow & 1 \\
 & & & & \| & & \uparrow & & \| & & \\
 & & 1 & \longrightarrow & E_N & \longrightarrow & E_N \times_s G & \twoheadrightarrow & G & \longrightarrow & 1
\end{array}$$

commutes, $[j^*(c\xi_N)] = 0$ by Theorem 2.

Conversely, let $1 \to R_G \to F \to G \to 1$ be a free presentation of G. Then $1 \to R_Q \to F \to Q \to 1$ is a free presentation of Q for some R_Q. Let $[\xi_F]$ be an element of $H^3(Q, A)$,

$$\xi_F\colon 0 \to A \to E_Q \to F \to Q \to 1$$

Construct the following diagram.



where $E_G \to F = E_G \to E_Q \to F$ is a pullback diagram. It is very easy to see that $[\xi_G] = [(j^*\xi_F)]$. Suppose $[\xi_G] = 0$. This means that there is a F-splitting map $R_G \to E_G$ such that $R_G \to E_G \to R_G = 1_{R_G}$. We also note that since

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R_Q & \longrightarrow & F & \twoheadrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & \Sigma & \downarrow & & \| & & \\
1 & \longrightarrow & N & \longrightarrow & G & \twoheadrightarrow & Q & \longrightarrow & 1
\end{array}$$

commutes, Σ is necessarily a pullback diagram. Thus

$$\mathrm{Ker}(R_Q \to N) = \mathrm{Ker}(F \to G) = R_G.$$

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Thus R_G is normal in R_Q. It follows from this that $\text{Im}(R_G \to E_G \to E_Q)$ is normal in E_Q. Let E_N be the $\text{Cok}(R_G \to E_Q)$. We have a commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R_G & \rightarrowtail & E_Q & \twoheadrightarrow & E_N & \longrightarrow & 1 \\ & & \| & & \downarrow & \Sigma' & \downarrow & & \\ 1 & \longrightarrow & R_G & \rightarrowtail & R_Q & \twoheadrightarrow & N & \longrightarrow & 1 \end{array}$$

Again, since Σ' is a pullback, $0 \to A \to E_N \to N \to 1$ is exact. It is easy to check that $[0 \to A \to E_N \to G \to Q \to 1] = [\xi_F]$. This completes the proof.

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