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NEAR-RINGS OF MAPPINGS ON FINITE TOPOLOGICAL GROUPS

GORDON MASON

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Abstract

When G is a topological group, the set $N(G)$ of continuous self-maps of G, and the subset $N_0(G)$ of those which fix the identity of G, are near-rings. In this paper we examine the (left) ideal structure of these near-rings when G is finite. $N_0(G)$ is shown to have exactly two maximal ideals, whose intersection is the radical. In the final section we investigate subnear-rings of $N_0(G)$ determined by certain continuous elements of the endomorphism near-ring.

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1. Introduction

Let G be a topological group written additively. We denote by $N(G)$ the set of continuous maps f: $G \to G$, and by $N_0(G)$ those elements of $N(G)$ for which $f(0) = 0$. When no confusion can arise, the notation will not indicate the topology used, but we will reserve $T(G)$ and $T_0(G)$ for the case when G has the discrete topology. $N(G)$ and $N_0(G)$ are near-rings with identity under pointwise addition and composition of functions. For general results on near-rings, the reader is referred to Pilz [8] and in this paper all near-rings will be right near-rings. Unless otherwise stated G will denote a finite group, and in the next section we will apply some ideas of Hofer [3] to obtain information about the (left) ideals in $N(G)$ and $N_0(G)$. In the third section we look at two subnear-rings of $N_0(G)$ determined by

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endomorphisms, namely the intersection of $N_0(G)$ with the endomorphism near-ring $E(G)$, and the near-ring distributively generated by continuous elements of $E(G)$. In particular, the orders of these near-rings are obtained for classes of near-rings for which the order of $E(G)$ is known (see [1], [5], [6], [7]).

2. Ideals in $N(G)$ and $N_0(G)$

If G is a finite topological group, the topology is determined by a normal subgroup. That is, G has a topology if and only if H is a normal subgroup such that a basis for the open sets of G consists of the cosets of H (see for example [2]). Then G is disconnected, the connected component of 0 being H. Now, if G is an infinite Hausdorff group and C is the connected component of 0, Hofer [3] defined $M_0 = \{f \in N_0(G) | f^{-1}(0)$ contains a clopen set about $0\}$, $P = P(C) = \{f \in N(G) |$range of $f \subseteq C\}$ and $P_0 = P \cap N_0$ and observed that P is an ideal in $N(G)$ and P_0 and M_0 are ideals in $N_0(G)$ such that $M_0 \setminus P_0 \neq \varnothing$. In our case, although G is not T_2, these results are still true. Moreover, H is the smallest clopen set about 0 so $M_0 = \{f | f(H) = 0\}$ which is sometimes written $(0 : H)$ or Ann H. In fact, it is known that every left ideal of $T_0(G)$ is of the form Ann S for some $S \subset G$ ([8] Corollary 7.28), and that these all intersect down to left ideals in $N_0(G)$. We observe that M_0 is one of these intersections and is, in fact, an *ideal*, although $T_0(G)$ has no ideals.

THEOREM 2.1. *If H is a subgroup of index 2 in a group G of order $2n$ then $|N(G)| = 4 \cdot n^{2n}$, $|N_0(G)| = 2 \cdot n^{2n-1}$, $|P| = n^{2n}$, $|P_0| = n^{2n-1}$ and $|M_0| = 2 \cdot n^n$ where, as above, $P = \{f \in N | f(G) \subseteq H\}$ and $P_0 = P \cap N_0$.*

PROOF. The only non-trivial open sets are H and $g + H$ $(g \notin H)$. Therefore, f: $G \to G$ is continuous if and only if it is one of the following: (a) $f^{-1}(H) = H$ and $f^{-1}(g + H) = g + H$, (b) $f^{-1}(H) = g + H$ and $f^{-1}(g + H) = H$, (c) $f^{-1}(H) = \varnothing$ and $f^{-1}(g + H) = G$, (d) $f^{-1}(H) = G$ and $f^{-1}(g + H) = \varnothing$. There are n^{2n} maps in each case. In $N_0(G)$ since $f^{-1}(0) \supseteq (0)$ only maps from (a) and (d) are allowed and there are now n^{2n-1} choices in each case. Clearly $|M_0| = 2 \cdot n^n$. As for P and P_0, more generally if $|H| = k$ and $|G| = m$, then any map f: $G \to G$ whose range is in H is continuous so $|P| = k^m$, and it is easy to see that $|P_0| = k^{m-1}$.

As a corollary note that for H of index 2, P_0 is maximal in N_0 being also of index 2.

Now Hofer has shown that when G is T_2, $P(C)$ is often the unique maximal ideal in $N(G)$, for example when $|G/C| = n > 2$. Actually the proof in that case is independent of the T_2 condition. It uses the canonical map ψ: $N(G) \to N(G/C)$ given by $\psi(f)(x + c) = f(x) + C$ whose kernel is $P(C)$ and which is onto when C is open. Now the isomorphism $N(G/C) \simeq N(G)/P$ induces $N_0(G/C) \simeq (N(G)/P)_0$ but the latter is not *a priori* isomorphic to $N_0(G)/P_0$. In other words a near-ring surjection $N \to R$ induces a map $N_0 \to R_0$ which may not be onto. However, examining ψ more closely, let ν: $G \to G/C$ be the canonical map and k: $G/C \to G$ a map for which νk is the identity. Then for every $g \in N(G/C)$, $g = \psi(f)$ where $f(x) = k(g(x + C))$. In our case $C = H$ is open, G/H is finite and so discrete and ψ restricted to $N_0(G)$ produces a near-ring homomorphism with kernel P_0. Moreover, we can choose k so that $k(H) = 0$ so the preimage f of $g \in N_0(G/C)$ is actually in $N_0(G)$ as required. We have

THEOREM 2.2. *P_0 is a maximal ideal in $N_0(G)$.*

Clearly, however, N_0 does not have a unique maximal ideal since M_0 is contained in some maximal ideal(s) and $M_0 \not\subset P_0$. We shall see below that for G finite, M_0 is maximal and that P_0 and M_0 are the only maximal ideals. For now we simply observe (even for G infinite) that Zorn's lemma applies to $S = \{\text{ideals } I \mid I \setminus P_0 \neq 0\}$ and the maximal elements so obtained must be maximal ideals.

In Lemma 2.12 of [3] it was proved that if G is T_2 and I is an ideal of $N_0(G)$ such that $I \setminus P_0 \neq \varnothing$ then I contains all functions whose range is finite. The conclusion is false for general non-Hausdorff groups; for example when G is finite this would say I is all of $N_0(G)$, but we know M_0 is a proper ideal satisfying $M_0 \setminus P_0 \neq \varnothing$. However, the following weaker statement is true, and in fact it is a valid replacement for Lemma 2.12 in [3, Theorem 3.3 and Theorem 3.8(b)] (see next Corollary). Let $R(f)$ denote the range of f and call its order the rank of f.

PROPOSITION 2.3. *Let G be any disconnected group and C the connected component of* 0. *(G need not be Hausdorff or finite.)*

(a) *If I is an ideal in $N_0(G)$ such that $I \setminus P_0 \neq \varnothing$, then I contains all functions f with $R(f) = \{0, a\}$ where $a \notin C$. Moreover, I does not contain all functions of rank* 2, *in the case G is finite.*

(b) *If $|G/C| > 2$ and I is an ideal in $N(G)$ with $I \setminus P \neq \varnothing$ then I contains all functions f with $R(f) = \{c, a\}$ where $c \in C$, $a \notin C$.*

PROOF. The proof given in [3, Lemma 2.12] remains valid except at one point. It is noted that when G is T_2, if $R(g) = \{0, a\}$ then $g^{-1}(0)$ and $g^{-1}(a)$ are clopen. For arbitrary G, this will be true if $a \notin C$. The result follows.

Suppose I contains all functions of rank 2. We show I contains P_0 which is a contradiction. Proceeding by induction let $R(f) = \{0, h_2, \ldots, h_n\} \subset H$. Put $f_1(x) = 0$ if $f(x) = 0$, $f_1(x) = h_2$ otherwise. Then f_1 is continuous and by hypothesis it is in I. Also by induction $f - f_1 \in I$ so $f \in I$.

COROLLARY. *For G as in the proposition, if $|G/C| = n > 2$ then $P(C)$ is the unique maximal ideal of $N(G)$ (see* [3, *Theorem* 3.8b]).

We now show M_0 is maximal, first recording the following characterization of continuity.

LEMMA 2.4. *If $\psi \in T_0(G)$ then $\psi \in N_0(G)$ if and only if $\psi(g) \in x + H \Rightarrow \psi(g + H) \subseteq x + H$.*

PROOF. This is simply the observation that ψ is continuous if and only if the inverse images of basic open sets (cosets) are open.

THEOREM 2.5. *There is a near-ring isomorphism between $T_0(H)$ and $N_0(G)/M_0$.*

PROOF. Map ψ: $N_0(G) \to T_0(H)$ by restriction, noting that by the lemma $\psi|_H$ *is* in $T_0(H)$. This is easily seen to be a near-ring homomorphism. Moreover, it is onto since any $f \in T_0(H)$ can be extended to a continuous map on G by, for example, setting $f(x) = 0$ for all $x \notin H$. Finally, $\ker \psi = \{f \mid f|_H = 0\} = M_0$.

Since by [8, Theorem 7.30] $T_0(H)$ is simple we have

COROLLARY. *M_0 is a maximal ideal in N_0.*

We refer the reader to [8, Chapter 5] for definitions of the radicals.

THEOREM 2.6. *All the radicals J_i of $N_0(G)$ coincide and equal $M_0 \cap P_0$.*

PROOF. $J = M_0 \cap P_0$ is nilpotent since $f \in J$ implies $f(H) = 0$ and $f(G) \subset H$. Therefore for all $f, f_1 \in J$ $ff_1(G) \subset f(H) = 0$. Hence $J \subseteq J_1$. On the other hand by [8, Theorem 5.42] $J_1 = J_2 = \cap$ all maximal ideals $\subseteq J$. Hence $J = J_1$ and by [8, Theorem 5.48] $J = J_0 = J_{1/2}$ also.

THEOREM 2.7. *M_0 and P_0 are the only maximal ideals of N_0.*

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PROOF. Suppose I is maximal, $I \neq P_0$. We show $I \supseteq M_0$. By the previous Theorem $I \supset M_0 \cap P_0$ so we show I contains all f with $f(H) = 0$ and $R(f) \not\subset H$. Proceeding by induction, it is true for rank $f = 2$ by Proposition 2.3. Suppose $f \in M_0$ has rank n, $R(f) \not\subset H$, that is, $R(f) = \{0, a_2, \ldots, a_n\}$ with $a_2 \notin H$. Define $f_1(x) = 0$ if $x \in f^{-1}(H)$, $f_1(x) = a_2$ otherwise. Then f_1 is continuous, $\operatorname{rank}(f - f_1) = n - 1$ and $f - f_1 \in I$ either because $R(f - f_1) \subset H$ or by induction. Also $f_1 \in I$ by Proposition 2.3 so $f \in I$ as required.

To exploit Theorem 2.5 further we note as mentioned earlier that for any finite group G the left ideals in $T_0(G)$ are precisely of the form Ann S for $S \subset G$, so maximal ones are obtained for $S = \{g\}$ and minimal ones for $S = G - \{g\}$. Following Pilz [8] we will denote the latter by L_g and put $\overline{L_g} = L_g \cap N_0$. By [8, 7.18] we have $L_g = T_0(G)e_g$ where e_g is the idempotent given by $e_g(g) = g$, $e_g(x) = 0$ for all $x \neq g$. Note that G is an $N(G)$-group and an $N_0(G)$-group under canonical action.

LEMMA 2.8. *${}_{N(G)}G$ is strongly monogenic and ${}_{N_0(G)}G$ is monogenic.*

PROOF. All constant maps are in $N(G)$ so $N(G) \cdot g = G$ for every g. In the case of $N_0(G)$ if $g \notin H$ then for all x there is $f \in N_0$ with $f(g) = x$ so for those g, $N_0(G) \cdot g = G$.

THEOREM 2.9. (a) *e_g is continuous if and only if $g \in H$ and then $\overline{L_g} = N_0(G)e_g$.*

(b) *$\overline{L_h} \simeq N_0(G)/\text{Ann } h$ for all $h \in H$ and $\overline{L_g} \simeq H$ for all g.*

(c) *$G \simeq N(G)/\text{Ann } g$ (for all $g \in G$) as $N(G)$-groups. If $g \notin H$ $G \simeq N_0(G)/\text{Ann } g$ as $N_0(G)$-groups.*

(d) *$\overline{L_g}$ is a minimal $N_0(G)$-subgroup of N_0 for all g and* Ann h *is a maximal $N_0(G)$-subgroup for all $h \in H$. (So they are respectively minimal and maximal left ideals also.)*

PROOF. (a) Clearly e_h is continuous if $h \in H$ and on the other hand if $g \notin H$, $e_g^{-1}(H) = G \setminus \{g\}$ is not open. Moreover, $\overline{L_g} = L_g \cap N_0 = T_0(G)e_g \cap N_0(G) = N_0(G)e_g$.

(b) Map α: $N_0(G) \to \overline{L_h} = N_0(G)e_h$ by $\alpha(f) = fe_h$. This is an $N_0(G)$-epimorphism whose kernel is Ann h. Now let β: $H \to \overline{L_g}$ be given by $\beta(h_0) = f$ where $f(g) = h_0$, $f(x) = 0$ for all $x \neq g$. Then β is an $N_0(G)$-homomorphism which is $1 - 1$. It is also onto for if $f \in \overline{L_g}$, $f(x) = 0$ except when $x = g$ and then $f(g) = a$, where by continuity $a \in H$. Thus $f = \beta(a)$ as required.

(c) By Lemma 2.8 and [8, Proposition 3.4] we have $G \simeq N(G)/\text{Ann } g$ for all g and $G \simeq N_0(G)/\text{Ann } g$ for all $g \notin H$.

(d) We show $\overline{L_h}$ is minimal by showing for all $0 \neq f \in \overline{L_h}$ $N_0(G)f = \overline{L_h}$. Now $f(x) = 0$ $\forall x \neq h$ and $f(h) \neq 0$. Define g by $g(f(h)) = h$, $g = 0$ otherwise. Then

$g \in N_0$ and $e_h = gf \subset N_0 f$. Hence $\overline{L_h} = N_0(G)e_h \subset N_0 f$ as required. Clearly $\overline{L_g} \simeq \overline{L_h}$ for all g so that $\overline{L_g}$ are minimal also. Applying (b) we have Ann h is a maximal N_0-subgroup for all $h \in H$.

Having seen the Ann h are maximal N_0-subgroups we turn our attention to Ann g, $g \notin H$.

LEMMA 2.10. *The only N_0-subgroup of G is H.*

PROOF. If K is an N_0-subgroup of G, it is a subgroup of $(G, +)$ such that $\psi(K) \subseteq K$ for all ψ in N_0. This is certainly true of H. On the other hand, if $H \setminus K \neq \varnothing$ let $h \in H \setminus K$, $h \neq 0$ and define ψ by $\psi(g) = h$ for all $g \neq 0$, and $\psi(0) = 0$. Then ψ is continuous but $\psi(K) \not\subseteq K$. Finally if $H \subsetneqq K$, $K = \bigcup_S (g + H)$ for some set S of coset representatives. There exists $g \in S \setminus H$ so define ψ_1 by $\psi_1(g + H) = y + H$ for any $y \notin S$, and ψ_1 the identity on the rest of G. Then again ψ_1 is continuous but $\psi_1(K) \not\subset K$.

Put $S_g = \{ f \in N_0 | f(g + H) \subseteq H \}$ for $g \notin H$.

PROPOSITION 2.11. *S_g is a maximal left ideal and maximal N_0-subgroup properly containing* Ann x *for all $x \in g + H$. Moreover $S_g \simeq S_x$ for all $x, g \notin H$ and $\bigcap_{g \notin H} S_g = P_0$.*

PROOF. S_g is a normal subgroup of $(N_0(G), +)$ since H is normal in G. Moreover for all $\psi, \alpha \in N_0$ and $f \in S_g$ let $x = [\psi(\alpha + f) - \psi\alpha](g + h) = \psi(\alpha(g + h) - h_1) - \psi\alpha(g + h)$ where $f(g + h) = h_1 \in H$. By Lemma 2.4, $x \in H$, so S_g is a left ideal. By Theorem 2.8(c) $G \simeq N_0(G)/\text{Ann}\, g$ and under this isomorphism (which comes from the evaluation map) S_g corresponds to H. Thus the S_g are all isomorphic (in fact $S_g = S_x$ if $x \in g + H$). By Lemma 2.9, the only $N_0(G)$-subgroup of G is H so the S_g are the only $N_0(G)$-subgroups of N_0 which contain Ann g. Finally $P_0 \subset S_g$ for all g and if $f \in \bigcap S_g$ then $f(g + H) \subseteq H$ for all g, that is, $f \in P_0$.

3. Subnear-rings of $N_0(G)$

In this section, G is again a finite topological group with topology determined by a normal subgroup H and we will write $N_0(G)$ as N_H. Let I, A and E be the near-rings distributively generated by Inn G, Aut G and End G which are respectively the groups of inner automorphisms, automorphisms and endomorphisms of

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G. There is an extensive literature on these near-rings for various classes of finite groups (see for example [1], [4], [5], [6], [7]) and using these results we will examine two kinds of subnear-rings of N_H. The first is $E_H = N_H \cap E$, the near-ring of continuous maps in E, and the second is C_H, the near-ring distributively generated by the continuous elements in End G. (E_H, although a subnear-ring of E, is not necessarily distributively generated.) Since every inner automorphism is continuous in all topologies we have for all H

(1) $$I \subseteq C_H \subseteq E_H \subseteq E.$$

We shall see later that for $G = D_8$, the dihedral group of order 8, we have in fact a chain of maximal proper inclusions $I \subsetneqq A \subsetneqq C_H \subsetneqq E_H \subsetneqq E$. At the other extreme, it may happen that $I = E$ (for example G dihedral of order $2n$, n odd ([6]) or $G = S_n$ the symmetric group, for $n \geqslant 5$ ([1])). In such a case $E \subset N_H$ and the same is true whenever H is fully invariant, in view of the next result.

LEMMA 3.1. *If* $\psi \in \operatorname{End} G$, $\psi \in E_H$ *if and only if* $\psi(H) \subseteq H$. *Also if* $\psi \in \operatorname{Aut} G$, $\psi \in E_H$ *if and only if* $\psi H = H$.

PROOF. Apply Lemma 2.4.

Thus for characteristic subgroups H the sequence (1) can be modified to include $I \subseteq A \subseteq C_H$. To complete the example for S_n, $n = 3$ or 4, in each case the only normal subgroups are members of the derived series [9, page 112]) and these are fully invariant so again $E \subset N_H$.

PROPOSITION 3.2. $J_i(N_H) \cap E \subseteq J_i(E)$ *for all radicals* J_i.

PROOF. From [4] we know all radicals of E coincide. Since $J(N_H)$ is nilpotent so is $J(N_H) \cap E$ and the result follows.

From the remarks following Theorem 16 in [4] we find that if G has a unique fully invariant subgroup H then $J(E)$ is precisely (in our notation) $M_0 \cap P_0 \cap E$. Thus in this case, equality holds in Proposition 3.2. (An example will be given later where equality does not hold.) To complete our discussion of S_n, $n \geqslant 5$, it is mentioned in [1] that $E(S_n)$ is close to being all of $T_0(S_n)$. We know $E \subseteq N_H \subseteq T_0$.

PROPOSITION 3.3. $E(S_n) = N_H(S_n)$ *when* $H = A_n$.

PROOF. From [1] $E = N + (T_0(H) \oplus Z_2)$ is a semi-direct sum where $N = M_0 \cap P_0$ (again in our notation). Moreover $T_0(H)$ is a direct sum of $n!/2$ subgroups, each isomorphic to H. Thus $|E| = |N_H|$ using Theorem 2.1.

To obtain information about C_H and E_H we use the following decomposition procedure from, for example, [7]. If R is a d.g. near-ring, one decomposes the generators r_i by an idempotent e_1 to obtain elements of the form $r_i - e_1 r_i$, and of the form $e_1 r_i$, the latter generating a group M_1. The elements of the first form are conjugated by elements of M_1 and these conjugates generate A_1. Choose a second idempotent $e_2 \in A_1$ and again form the conjugates of all $x - e_2 x$ $(x \in A_1)$ by elements of the group generated by the $e_2 x$. These conjugates generate D and $R = D + A_1 + M_1$. The procedure may be iterated.

In particular if $G = D_{2n}$ is the dihedral group of order $2n$ for n even with presentation $G = \langle a, b | a^n = b^2 = abab = e \rangle$, following [7] we denote an endomorphism ψ by $[s, t]$ where $\psi(a) = s$ and $\psi(b) = t$, and denote any map by the images of $(e, a, a^2, \ldots, a^{n-1} | b, ab, \ldots, a^{n-1}b)$ in that order. The endomorphisms are of six types:

(1) $[a^y, a^x b]$, $\quad 0 \leqslant y, x \leqslant n-1$,

(2) $[d, e]$
(3) $[d, d]$ $\Big\}$ d *any element of order* 2,

(4) $[e, a^{n/2}]$,

(5) $[a^x b, a^{x+n/2} b]$
(6) $[a^x b, a^{n/2}]$ $\Big\}$ $0 \leqslant x \leqslant n-1$.

Let $H = \langle a \rangle$ be the cyclic normal subgroup of index 2. We now restrict to $n = 4$ for ease of calculation. Then ([7]) $E = D + A_1 + M_1$ where $D = \{(e, e, a^2, a^2 | e, e, a^2, a^2) \oplus (e, a^2, a^2, e | e, e, a^2, a^2)\}$, $A_1 = \{(e, g, e, g | e, g, e, g) | g \in G\}$ and $M_1 = \{(e, e, e, e | g, g, g, g) | g \in G\}$. Using Lemma 3.1 we see all 8 elements of M_1, all 4 elements of D and the 4 elements of A_1 for which $g \in H$, are continuous. Thus $|E_H| \geqslant 128$. But since $|E| = 256$ and there are endomorphisms which are not continuous, $|E_H| = 128$. If on the other hand we topologize G by $K = \{e, b, a^2, a^2 b\}$ then there are 32 maps in E_K which are a sum of continuous maps in each of D, A_1 and M_1 so $|E_K| \geqslant 32$. As we shall see later however the order of E_K is actually 128.

Now we can obtain C_H by applying the procedure outlined above. First, there are 29 continuous endomorphisms, namely all those from (1), those from (2) and (3) with $d = a^2$, (4), and the identity $[e, e]$. Using the idempotent $\gamma_1 = [e, b]$ we get one form for $\alpha - \gamma_1 \alpha$, namely $\beta = (e, a^y, a^{2y}, a^{3y} | e, a^y, a^{2y}, a^{3y})$ for $0 \leqslant y \leqslant 3$. The elements $\gamma_1 \alpha$ are $[e, a^x b]$, $0 \leqslant x \leqslant 3$, $[e, e]$, and $[e, a^2]$ which generate $M_1 = \{(e, e, e, e | g, g, g, g)\}$ as before. Conjugating β by the $\gamma_1 \alpha$ gives β and $\beta_1 = (e, a^y, a^{2y}, a^{3y} | e, a^{-y}, a^{-2y}, a^{-3y})$. Choose the idempotent $\gamma_2 = (e, a, a^2, a^3 | e, a, a^2, a^3)$ to get the single form $\gamma_2 \beta = \gamma_2 \beta_1 = \beta$ so A_1 has 4 elements. Then $\beta - \gamma_2 \beta = [e, e]$ and $\beta_1 - \gamma_2 \beta_1 = (e, e, e, e | e, a^{2y}, e, a^{2y})$ so conjugating these by $\gamma_2\beta = \beta$ gives only $\beta_1 - \gamma_2\beta_1$ again and hence D has 2 elements. Thus $|C_H| = 64$.

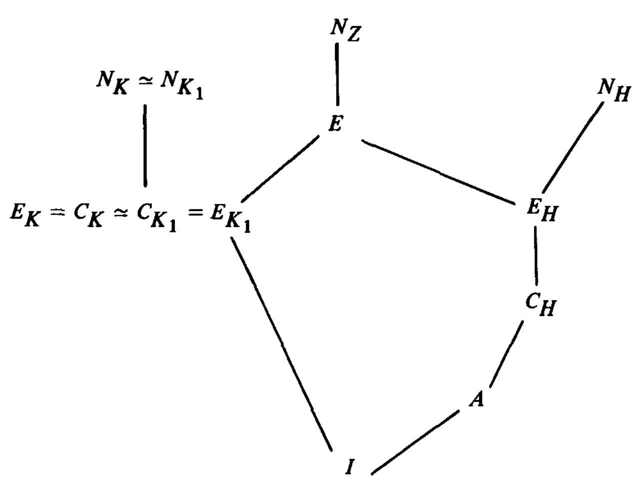
THEOREM 3.4. *If n is even and $H = \langle a \rangle$, $|C_H(D_{2n})| = n^3$. Moreover for $n = 4$, $I \subsetneq A \subsetneq C_H \subsetneq E_H \subsetneq E$ where the order of each near-ring is twice the preceding one.*

PROOF. For general D_{2n} the procedure outlined above can be generalized, giving $C_H = D + A_1 + M_1$ where $|M| = 2n$, $|A_1| = n$ and $|D| = n/2$. The second statement comes from the above discussion and [7].

REMARK. The elements of C_H are of the form $\Sigma \pm f_i$ where the f_i are the continuous endomorphisms. In general, all possible ordered sums must be calculated. As an interesting consequence of the computer programme we used to exhibit the elements of $C_H(D_8)$ we found that (i) of the 20 continuous endomorphisms, only the 8 automorphisms plus the endomorphism $(e, e, e, e|b, b, b, b)$ were needed, (ii) only elements Σf_i (all +) were needed, and (iii) all elements could be obtained from one particular ordering of the 9 generators. Also in producing the 32 elements of $A(D_8)$, (ii) and (iii) were true.

To complete the discussion of D_8, let $K = \{e, a^2, b, a^2b\}$. Then there are 24 continuous endomorphisms and the standard decomposition using the idempotents $\gamma_1 = [e, b]$ and $\gamma_2 = (e, ab, e, ab|e, ab, e, ab)$ show that $|C_K| = 128$. Since there are endomorphisms which are not continuous, $|E_K| = 128$ too. The only other normal subgroups are (i) $K_1 = \{e, ab, a^2, a^3b\}$ and (ii) the centre $Z = \{e, a^2\}$. By symmetry $C_{K_1} \cong C_K$, and every endomorphism is Z-continuous.

THEOREM 3.5. *The continuous subnear-ring structure of $T_0(D_8)$ is given by*



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For $G = D_{2n}$, n odd, we have already seen that $I = E \subset N_H$ for all H. In [6] it is shown that $J(E)$ contains an element ψ for which $\psi(a) = a^w$. For $H = \langle a \rangle$ then, $\psi \notin M_0$ so the inequality of Proposition 3.2 is strict.

Finally let $G = Q_n$ be the generalized quaternion group of order 2^n, $n > 3$ with presentation $Q_n = \left\langle a, b | a^{2^n - 1} = b\,ab^{-1}a = a^{2^{n-2}}b^2 = e \right\rangle$. Then (see [5]) the normal subgroups are precisely the subgroups of $H = \langle a \rangle$, or $K_1 = \langle a^2, b \rangle$ or $K_2 = \langle a^2, ab \rangle$ and the automorphisms are of the form $[a^y, a^x b]$ for $0 \leqslant x, y \leqslant 2^n - 1$ and y odd. Moreover $|I| = 2^{3n-5}$ and $|A| = |E| = 2^{3n-4}$.

Theorem 3.6. *For $G = Q_n$, $C_L = E_L = A = E$ for all $L \leqslant H$ and $C_{K_i} = E_{K_i} = I$.*

Proof. Since I has index two in E, for every normal A, C_A and E_A will equal I or E. Invoking Proposition 3.2 and looking at the form of the automorphisms we see every automorphism is L-continuous for all $L \leqslant H$. On the other hand, only half are K_i-continuous, namely those $[a^y, a^x b]$ with y odd and x even. As shown in [5] the map $(e, e, e, e, \text{--}|a, a, \text{--}, a)$ is in A but it is not K_i-continuous. The result follows.

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Department of Mathematics & Statistics
University of New Brunswick
Fredericton, N. B.
Canada