INFINITE CAPACITY STORAGE PROCESSES

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1. Introduction

In this paper we shall discuss infinite capacity storage processes in which periods of input and output alternate. The length of a period of input and the length of the period of output immediately following may be statistically dependent and the change in storage level during an input or output period may depend on the length of the time interval in a rather general manner. However, we do not exploit either of these facts in the present paper.

We shall obtain a generating function for a transform of the storage level, or content, W_n, at the discrete times t_n which denote the beginnings of periods of input. After giving an example we show how certain asymptotic results may be obtained.

The investigation of storage processes has been carried on for several years and the monograph by P. A. P. Moran [1] and the review article by N. U. Prabhu [2] are recommended surveys of the subject.

2. Notation and basic formulae

The storage systems considered in this paper are those for which periods of input and output alternate, that is, the storage content is non-decreasing for an interval of time and is non-increasing for a period of time immediately there-after. The moments at which periods of input begin will be denoted by t_n $(n = 0, 1, 2)$. To fix the time scale we shall suppose $t_0 = 0$. We shall let $T_{j+1} = t_{j+1} - t_j$, $(j = 0, 1, 2, \cdots)$, and for convenience will refer to the time interval (t_j, t_{j+1}) as the period T_{j+1}.

The storage content at time t_j will be denoted by W_j. The change in storage content during the input portion of the period T_j will be denoted by X_j and the change during the output portion will be max $[W_{j-1}+X_j-Y_j, 0]$. We assume the triples (X_j, Y_j, T_j) are independently and identically

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distributed and we shall denote the joint distribution of the random variables X_j, Y_j and T_j by $F(x, y, t)$. We let

$$\phi(s, v) = E\big(\exp\,(-s(X_j - Y_j)) - vT_j\big),$$

where E denotes expectation and s and v are complex numbers chosen so $\phi(s, v)$ is defined. It is clear from the definitions that, prior to the time period T_N in which the content first becomes zero (other than at time t_0), the storage content coincides with the random variable z_j defined as follows:

$$z_0 = W_0$$

$$z_n = W_0 + \sum_{j=1}^{n} (X_j - Y_j) \qquad (n = 1, 2, \cdots).$$

We assume W_0 is a random variable which is independent of the triples (X_j, Y_j, T_j). We shall let N denote the least integer n such that $z_n \leqq 0$. Thus, time period T_N is the first time period in which the content becomes zero at a time other than $t_0 = 0$.

For a random event A we shall set $\{A\}$ equal to one if the event A occurs and zero if A does not occur. Using this notation we define the following generating functions, with t a complex number, $|t| > 1$:

$$Q(t, s, v|W_0) = \sum_{n=0}^{\infty} t^n q_n(s, v|W_0) \text{ where } q_n(s, v) = E(\{N > n\} \exp(-sz_n - vt_n)|W_0)$$

$$P(t, s, v|W_0) = \sum_{n=0}^{\infty} t^n p_n(s, v|W_0) \text{ where } p_n(s, v) = E(\{N = n\} \exp(-sz_n - vt_n)|W_0)$$

$$G(t, s, v|W_0) = \sum_{n=0}^{\infty} t^n g_n(s, v|W_0) \text{ where } g_n(s, v) = E\big(\exp\,(-sW_n - vt_n)|W_0\big).$$

The basic relationship between these generating functions is the following (see Kemperman (3)):

$$\text{(2.1)} \qquad G(t, s, v|W) = Q(t, s, v|W) + P(t, 0, v|W|)G(t, s, v|0)$$

or

$$G(t, s, v|W_0) = Q(t, s, v|W_0) + \frac{P(t, 0, v|W_0)Q(t, s, v|W_0 = 0)}{1 - P(t, 0, v|W_0 = 0)}.$$

If $W_0 = 0$ this reduces to $G(t, s, v|0) = \dfrac{Q(t, s, v|0)}{1 - P(t, 0, v|0)}$.

We see the generating function $G(t, s, v|W_0)$ may be found if the generating functions $Q(t, s, v|W_0)$ and $P(t, s, v|W_0)$ have been determined. By using a slight modification of a technique given in Kemperman (3) we can determine these generating functions from the easily derived relation

$$(2.2)\qquad [1-t\phi(s, v)]Q(t, s, v|W_0) = E(\exp(-sW_0))-P(t, s, v|W_0).$$

The technique is based on the Wiener-Hopf type decomposition

$$(2.3)\qquad [1-t\phi(s, v)] = \frac{M_-(t, s, v)}{M_+(t, s, v)}$$

for $\text{Re}(s) = 0$. Here, and subsequently, $M_+(t, s, v)$ is continuous on the closed half plane $\text{Re}(s) \geqq 0$ and analytic on the open half plane $\text{Re}(s) > 0$; $M_-(t, s, v)$ is continuous on the closed half plane $\text{Re}(s) \leqq 0$ and analytic on the open half plane $\text{Re}(s) < 0$. In addition, $\lim M_-(t, s, v) = 1$, as $\text{Re}(s) \to -\infty$, $M_+(t, s, v)$ and $M_-(t, s, v)$ are bounded and bounded away from zero on their respective half-planes (Kemperman [3], p. 72). Before using (2.3) to solve Equation (2.2) for the desired generating functions, we shall define a convenient notation.

Let $\phi(s)$ denote the Laplace transform of a regular, complex valued, finite Borel measure on the real line, say,

$$\phi(s) = \int_{-\infty}^{\infty} \exp(-sx)\mu(dx),$$

where $\phi(s)$ is defined for each complex number $s = \sigma+i\tau$, $(i = \sqrt{-1})$, for which $\phi(\sigma) < \infty$. Then we let

$$[\phi(s)]^+ = \int_{0_-}^{\infty} \exp(-sx)\mu(dx),$$

$$[\phi(c)]^- = \int_{-\infty}^{0_-} \exp(-sx)\mu(dx).$$

Considered as a function of s,

(i) $[\phi(s)]^+$ is continuous and bounded for $\text{Re}(s) \geqq 0$ and is analytic for $\text{Re}(s) > 0$.

(ii) $[\phi(s)]^-$ is continuous and bounded for $\text{Re}(s) \leqq 0$ and is analytic for $\text{Re}(s) < 0$, $\lim [\phi(s)]^- = 0$ and $\text{Re}(s) \to -\infty$.

(iii) the decomposition of $\phi(s)$ into the sum of two such functions is unique.

We are now ready to prove

THEOREM 1. *For $P(t, s, v|W_0)$ and $Q(t, s, v|W_0)$ as previously defined and t, v chosen so $|t\phi(0, \text{Re}(v))| < 1$, we have*

$$(2.4)\qquad P(t, s, v|W_0) = M_-(t, s, v)\left[\frac{E(\exp(-sW_0))}{M_-(t, s, v)}\right]^-, \quad \text{Re}(s) \leqq 0,$$

$$(2.5)\qquad Q(t, s, v|W_0) = M_-(t, s, v)\left[\frac{E(\exp(-sW_0))}{M_-(t, s, v)^+}\right]^+, \quad \text{Re}(s) \geqq 0.$$

PROOF. Using (2.3) and the + and − operators we write (2.2) as follows for $\mathrm{Re}(s) = 0$:

$$\frac{Q(t, s, v|W_0)}{M_+(t, s, v)} - \left[\frac{E(\exp(-sW_0))}{M_-(t, s, v)}\right]^+ = \left[\frac{E(\exp(-sW_0))}{M_-(t, s, v)}\right]^- - \frac{P(t, s, v|W_0)}{M_-(t, s, v)}.$$

The theorem follows easily from Liouville's Theorem. For the special case $W_0 = 0$ we have

$$Q(t, s, v|0) = M_+(t, s, v) \quad \text{and} \quad P(t, s, v|0) = 1 - M_-(t, s, v).$$

3. Infinite capacity storage systems, discrete time

Using Theorem 1 we may write the fundamental relation (2.1) in the following form:

$$G(t, s, v|W_0) = M_+(t, s, v)\left\{\left[\frac{E(\exp(-sW_0))}{M_-(t, s, v)}\right]^+ + \left[\frac{E(\exp(-sW_0))}{M_-(t, s, v)}\right]^-_{s=0}\right\}$$

For many particular cases, the various terms in this expression are easy to determine. To illustrate the methods, we shall give various examples.

Suppose that X_j, Y_j and T_j are independently distributed with $T_j = 1$, $Y_j = M$ (positive constant) and X_j distributed according to a type III distribution, say

$$dF(x) = \frac{\mu^p}{\Gamma(p)} x^{p-1} e^{-\mu x} dx, \qquad \mu > 0, p > 0, x \geqq 0.$$

Then

$$\phi(s, v) = \frac{\mu^p}{(\mu+s)^p} \exp(-v+sM)$$

so that

$$M_-(t, s, v) = \frac{(\mu+s)^p - t\mu^p \exp(-v+sM)}{\prod_{i=1}^{p}(s-s_i)}$$

$$M_+(t, s, v) = \frac{(\mu+s)^p}{\prod_{i=1}^{p}(s-s_i)}$$

where $s_i = s_i(t)$ $(i = 1, 2, \cdots, p)$ are the roots (by Rouchés theorem) in the half plane $\mathrm{Re}(s) \leqq 0$ of the equation

$$\frac{t\mu^p \exp(-v+sM)}{(s+\mu)^p} = 1.$$

For $W_0 = 0$ we have

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$$G(t, s, v|0) = \frac{(\mu+s)^p}{\prod_{i=1}^{p}(s-s_i)} \frac{\prod_{i=1}^{p}(-s_i)}{\mu^p - t\mu^p \exp(-v)}.$$

For the special case $p = 1$ we have

$$G(t, s, v|0) = \frac{(\mu+s)s_1}{\mu(s_1-s)(1-t\exp(-v))}.$$

4. Asymptotic results

The results of Spitzer [4] may be used to show that the storage level W_n will converge in distribution to a random variable W_∞ if and only if the series

$$\sum n^{-1} \text{Prob}(z_n > W_0)$$

converges; moreover, if a limiting distribution exists, it is independent of W_0. Assuming the existence of a limiting distribution, we can find this distribution from our previous results. From Abel's Theorem we have

$$\begin{aligned}\lim_{n\to\infty} E(\exp(-uW)) &= \lim_{t\to 1-}(1-t)G(t, u, 0|W_0 = 0) \\ &= \lim_{t\to 1-}(1-t)\frac{M_+(t, u, 0)}{M_-(t, 0, 0)} \\ &= \lim_{t\to 1-}\frac{M_+(t, u, 0)}{M_+(t, 0, 0)}.\end{aligned}$$

Thus, in our preceding examples, we have the following results:

(i) When X_j is exponentially distributed, with parameter μ,

$$E(\exp(-uW_\infty)) = \frac{\mu+u}{\mu}\frac{s_0}{s_0-u}$$

where s_0 is the positive root of $\mu\exp(sM) = s+\mu$. This yields results of Gani and Prabhu [5].

For a gamma-type distribution, say Y_j has the distribution $(\lambda t)^k \exp(-\lambda t)/k!$, and arbitrary X_j, say with transform denoted by $\varphi_1(u)$, we have

$$E((\exp(uW_\infty)) = \lim_{t\to 1-}\frac{(1-t)\lambda^k}{(\lambda-u)^k - t\lambda^k\varphi_1(u)}\prod_{i=1}^{k}\frac{s_i-u}{s_i}$$

where the s_i are the roots of $(\lambda-s)^k = t\lambda^k\varphi_1(s)$. As t tends to one, one of the roots tends to zero. Suppose s_1 is the root; then

$$\lim_{t\to 1-} \frac{1-t}{s_1} = \lim_{t\to 1-} \frac{1-\dfrac{\lambda-s_1}{\lambda\varphi_1(s_1)}}{s_1} = \frac{\lambda\varphi_1'(0)+1}{\lambda}.$$

Hence, for $k = 1$

$$E(\exp(-uW_\infty)) = \frac{[1+\lambda\varphi_1'(0)]u}{u-\lambda+\lambda\varphi_1(u)}.$$

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