



Semistable reduction for overconvergent F -isocrystals, II: A valuation-theoretic approach

Kiran S. Kedlaya

ABSTRACT

We introduce a valuation-theoretic approach to the problem of semistable reduction (i.e. existence of logarithmic extensions on suitable covers) of overconvergent isocrystals with Frobenius structure. The key tool is the quasicompactness of the Riemann–Zariski space associated to the function field of a variety. We also make some initial reductions, which allow attention to be focused on valuations of height 1 and transcendence degree 0.

1. Introduction

This paper is the second of a series starting with [Ked07]. The goal of the series is to prove a ‘semistable reduction’ theorem for overconvergent F -isocrystals, a class of p -adic analytic objects associated to schemes of finite type over a field of characteristic $p > 0$. Such a theorem is expected to have consequences for the theory of rigid cohomology, in which overconvergent F -isocrystals play the role of coefficient objects.

In [Ked07], it was shown that the problem of extending an overconvergent isocrystal on a variety X to a log-isocrystal on a larger variety \overline{X} is governed by the triviality of some sort of ‘local monodromy’ along components of the complement of X . In this paper, we give a valuation-theoretic interpretation of this concept, which suggests an approach to the semistable reduction problem to be pursued later in this series.

The context of this result (including a complex analogue) and a description of potential applications have already been given in the introduction of [Ked07], so we will not repeat them here. Instead, we devote the remainder of this introduction to an overview of the results specific to this paper, and a survey of the structure of the various sections of the paper.

1.1 Valuations and semistable reduction

Let $X \hookrightarrow \overline{X}$ be an open immersion of varieties over a field k of characteristic $p > 0$, with X smooth and \overline{X} proper, and let \mathcal{E} be an F -isocrystal (isocrystal with Frobenius structure) on X overconvergent along $Z = \overline{X} \setminus X$. The semistable reduction problem, as described in [Ked07, §7], is to show that \mathcal{E} admits a logarithmic extension with nilpotent residues after being pulled back along some generically finite cover of \overline{X} . When X is a curve, this can be deduced from the p -adic local monodromy theorem (p LMT) of André [And02], Mebkhout [Meb02], and the present author [Ked04a]. This derivation is carried out in [Ked03]; the main point is that one can work locally, constructing the logarithmic extension separately for each point of Z .

When X has dimension greater than 1, one can still apply the p LMT along codimension 1 components of Z , but one only obtains a result that holds after ignoring a proper closed subset

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of the component. This would be fine if one were always able to use a finite cover in the pullback (by the analogues of Zariski–Nagata purity derived in [Ked07]), but that is not always possible: the result may be forced not to be smooth, in which case some blowing up is required, producing additional components of codimension 1 along which it is not clear that any control on monodromy has been imposed.

To get around this, it is helpful to think of the application of the p LMT as being parametrized by divisorial valuations, i.e. certain points on the Riemann–Zariski space associated to the function field of X . One is then naturally led to propose a version of the semistable reduction problem which is local in Riemann–Zariski space. In this paper, we formulate the local semistable reduction problem and explain its equivalence to the original semistable reduction problem, using the quasicompactness of the Riemann–Zariski space. We also perform some simplifying reductions that allow us to focus on what we call *minimal* valuations. We defer a direct assault on the local semistable reduction problem to subsequent papers in this series.

1.2 Structure of the paper

We conclude this introduction with a summary of the structure of the paper.

In §2, we review some relevant facts from valuation theory, most notably the construction of Riemann–Zariski spaces.

In §3, we describe the valuation-theoretic setup in more detail, formulating a local semistable reduction problem and verifying that it is equivalent to the semistable reduction problem described in [Ked07, §7].

In §4, we show that the local semistable reduction problem can be somewhat simplified. Specifically, we show that it suffices to solve it when k is algebraically closed, and the center valuation is of height 1 and has residue field k .

2. Review of valuation theory

In this chapter, we review some relevant facts from valuation theory, notably the definition of the Riemann–Zariski space of a field. We use the summary by Vaquié [Vaq00] as our primary reference; in turn, Vaquié’s underlying primary reference is Zariski and Samuel [ZS75].

CONVENTION 2.0.1. For A a local ring, let \mathfrak{m}_A denote the maximal ideal of A , and let $\kappa_A = A/\mathfrak{m}_A$ denote the residue field of A .

CONVENTION 2.0.2. By a *variety* over a field k , we mean a reduced separated (but not necessarily irreducible) scheme of finite type over k . For X an irreducible variety over k , let $k(X)$ denote the function field of X over k . By a *smooth pair* over a field k , we mean a pair (X, Z) consisting of a smooth irreducible k -variety X and a strict normal crossings divisor Z on X ; we will conflate this pair with the log-scheme it determines.

2.1 Totally ordered groups

We start with some standard facts about totally ordered groups, which are used to discuss valuations.

DEFINITION 2.1.1. By a *totally ordered group*, we will mean an abelian group Γ written additively, equipped with a total ordering \leq with the property that, for $a, b, c \in \Gamma$, $a \leq b$ if and only if $a + c \leq b + c$. As usual, we write $a < b$ to mean that $a \leq b$ but $a \neq b$ (so that $b \not\leq a$); and we write $a \geq b$ and $a > b$ to mean $b \leq a$ and $b < a$, respectively. Write Γ_∞ for the monoid $\Gamma \cup \{\infty\}$ in which $x + \infty = \infty$ for all $x \in \Gamma_\infty$, and extend the total ordering to Γ_∞ by declaring that, for all $x \in \Gamma$, $x < \infty$.

DEFINITION 2.1.2. Let Γ be a totally ordered group. A proper subgroup Δ of Γ is called an *isolated subgroup* if, for any $\alpha \in \Delta$, $\beta \in \Gamma$ with $\alpha \geq \beta \geq 0$, it follows that $\beta \in \Delta$. It is easily shown that the isolated subgroups are totally ordered under inclusion; define the *rank* of Γ , denoted $\text{rank}(\Gamma)$, to be the cardinality of the set of isolated subgroups of Γ .

PROPOSITION 2.1.3. *Let Γ be a totally ordered group. Then the following are equivalent:*

- (a) Γ is isomorphic, as a totally ordered group, to a subgroup of \mathbb{R} with its usual ordering;
- (b) $\text{rank}(\Gamma) \leq 1$;
- (c) Γ is archimedean: that is, for any $a, b \in \Gamma$ with $b > 0$, there exists an integer n with $nb \geq a$.

Proof. See [ZS75, §VI.10, p. 45]. □

COROLLARY 2.1.4. *A totally ordered group Γ has rank at most n if and only if it is isomorphic to a subgroup of \mathbb{R}^n with its lexicographic ordering.*

DEFINITION 2.1.5. Let Γ be a totally ordered group. Define the *rational rank* of Γ , denoted $\text{ratrank}(\Gamma)$, to be the dimension of the \mathbb{Q} -vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$.

PROPOSITION 2.1.6. *Let Γ be a totally ordered group, let $\bar{\Gamma}$ be an isolated subgroup of Γ , and equip $\Gamma' = \Gamma/\bar{\Gamma}$ with the induced total ordering. Then*

$$\begin{aligned} \text{ratrank}(\Gamma) &= \text{ratrank}(\bar{\Gamma}) + \text{ratrank}(\Gamma'), \\ \text{rank}(\Gamma) &= \text{rank}(\bar{\Gamma}) + \text{rank}(\Gamma'), \\ \text{rank}(\Gamma) &\leq \text{rank}(\bar{\Gamma}) + \text{ratrank}(\Gamma'). \end{aligned}$$

In particular, $\text{rank}(\Gamma) \leq \text{ratrank}(\Gamma)$.

Proof. See [Vaq00, Proposition 3.5]. □

2.2 Valuations

DEFINITION 2.2.1. Let R be a ring and let Γ be a totally ordered group. A *valuation* (or *Krull valuation*) on R with values in Γ is a function $v : R \rightarrow \Gamma_{\infty}$ satisfying the following conditions:

- (a) for $x, y \in R$, $v(xy) = v(x) + v(y)$;
- (b) for $x, y \in R$, $v(x + y) \geq \min\{v(x), v(y)\}$;
- (c) we have $v(1) = 0$ and $v(0) = \infty$.

We write *real valuation* as shorthand for ‘Krull valuation with values in \mathbb{R} ’. We say that the valuations v_1, v_2 on R , with values in Γ_1, Γ_2 , are *equivalent* if, for all $x, y \in R$,

$$v_1(x) \geq v_1(y) \iff v_2(x) \geq v_2(y).$$

Define the *value group* Γ_v of v to be the image of v , as a totally ordered group; equivalent valuations have isomorphic value groups, and in fact every valuation is equivalent to a surjective valuation (in which $\Gamma = \Gamma_v$).

Remark 2.2.2. With notation as in Definition 2.2.1, the inverse image $\mathfrak{p} = v^{-1}(\infty)$ is a prime ideal of R , and the induced map $v : R/\mathfrak{p} \rightarrow \Gamma_{\infty}$ extends uniquely to a valuation on $\text{Frac}(R/\mathfrak{p})$; see [Vaq00, Proposition 2.2].

DEFINITION 2.2.3. For F a field and $v : F \rightarrow \Gamma_{\infty}$ a valuation on F , the subset $\mathfrak{o}_v = \{x \in F : v(x) \geq 0\}$ is a valuation ring of F (a subring of F maximal among local rings with fraction field F under domination; see [Vaq00, Théorème 1.1] for other characterizations). In this case, we write κ_v for the residue field $\kappa_{\mathfrak{o}_v}$. Conversely, every valuation ring of F arises from a unique equivalence class of valuations [Vaq00, Proposition 2.3]. Let $\mathfrak{m}_v = \{x \in F : v(x) > 0\}$ denote the maximal ideal of \mathfrak{o}_v .

DEFINITION 2.2.4. For k an integral domain, there is a valuation $v : k \rightarrow \{0, \infty\}$ with $v(x) = \infty$ if and only if $x = 0$. We call v the *trivial valuation* (or *improper valuation*) on k . More generally, if R is a k -algebra, a *valuation on R over k* is a valuation which restricts to the trivial valuation on k .

DEFINITION 2.2.5. Define the *height* (or *rank*, or *real rank*) and *rational rank* of v as the rank and rational rank, respectively, of the value group of v , and denote these by $\text{height}(v)$ (or $\text{rank}(v)$) and $\text{ratrank}(v)$, respectively. By Proposition 2.1.6, we have $\text{height}(v) \leq \text{ratrank}(v)$. Also, note that by Proposition 2.1.3, a valuation has height 1 if and only if it is equivalent to a real valuation.

Valuations of height greater than 1 can be written as ‘compositions’ of valuations of smaller height.

DEFINITION 2.2.6. Let F be a field, and let $v : F \rightarrow \Gamma_\infty$ be a valuation. For $\bar{\Gamma}$ a nontrivial isolated subgroup of $\Gamma_v = v(F)$, put $\Gamma' = \Gamma/\bar{\Gamma}$, and let $v' : F \rightarrow \Gamma'_\infty$ be the composition of v with the quotient map $\Gamma \rightarrow \Gamma'$; it is again a valuation. Let $\bar{v} : \kappa_{v'} \rightarrow \bar{\Gamma}_\infty$ be the map induced by v ; it too is a valuation. In this situation, we write $v = v' \circ \bar{v}$ and say that v is the *composition* of v' and \bar{v} ; note that [Vaq00, Proposition 3.5]

$$\begin{aligned} \text{height}(v) &= \text{height}(v') + \text{height}(\bar{v}), \\ \text{ratrank}(v) &= \text{ratrank}(v') + \text{ratrank}(\bar{v}). \end{aligned}$$

Remark 2.2.7. Note that the convention ‘overbars denote reduction’ here applies to the valuation rings, rather than to the value groups.

DEFINITION 2.2.8. Let E/F be an extension of fields. If w is a valuation on E , the restriction of w to F is a valuation on F ; if that valuation is v , we say that w is an *extension to E* of v . Note that this happens if and only if $\mathfrak{o}_w \cap F = \mathfrak{o}_v$; in particular, since every integral domain is contained in a valuation ring (by a Zornication), every valuation on F admits at least one extension to E .

DEFINITION 2.2.9. Let E/F be a finite extension of fields, let v be a valuation on F , and let w be an extension to E of v . Define the *ramification index*

$$e(w/v) = [\Gamma_w : \Gamma_v]$$

and the *residual degree*

$$f(w/v) = [\kappa_w : \kappa_v].$$

These numbers are both finite and satisfy the fundamental inequality

$$e(w/v)f(w/v) \leq [E : F];$$

see [Vaq00, Proposition 5.1]. In particular, $\text{rank}(w) = \text{rank}(v)$ and $\text{ratrank}(w) = \text{ratrank}(v)$.

2.3 Riemann–Zariski spaces

We now recall the definition of a Riemann–Zariski space, following [Vaq00, § 7].

DEFINITION 2.3.1. Let F be a field and let R be a subring of F . Define the *Riemann–Zariski space* $S_{F/R}$ as the set consisting of the equivalence classes of valuations on F which are nonnegative on R . This set carries two natural topologies, the coarser *Zariski topology* and the finer *patch topology* (or *Zariski–Hausdorff topology*), in which bases are given by sets of one of the respective forms

$$\begin{aligned} \text{Zariski: } & \{v \in S_{F/R} : v(x_1) \geq 0, \dots, v(x_m) \geq 0\}, \\ \text{patch: } & \{v \in S_{F/R} : v(x_1) \geq 0, \dots, v(x_m) \geq 0; v(y_1) > 0, \dots, v(y_n) > 0\}, \end{aligned}$$

for $x_1, \dots, x_m, y_1, \dots, y_n \in F$. The patch topology is Hausdorff, while the Zariski topology is only T_1 in general. Note that, if E/F is a field extension, then there is a natural surjection $S_{E/R} \rightarrow S_{F/R}$

obtained by restricting valuations from E to F (see Definition 2.2.8 for the surjectivity), which is continuous for either consistent choice of topologies.

DEFINITION 2.3.2. For F a field, let T_F denote the set of functions from F to $\{0, +, -\}$, equipped with the product topology associated to the discrete topology on $\{0, +, -\}$. Consider the map from $S_{F/R}$ to T_F that associates to $v \in S_{F/R}$ the function $f_v \in T_F$ taking $x \in F$ to $0, +, -$ depending on whether $v(x) = 0$, $v(x) > 0$, or $v(x) < 0$. This map $S_{F/R} \rightarrow T_F$ is injective because from f_v we can recover \mathfrak{o}_v as $\{x \in F : f_v \in \{0, +\}\}$, and hence can recover v . Moreover, the subspace topology induced on $S_{F/R}$ is visibly the same as the patch topology. We may similarly recover the Zariski topology by starting with the topology on $\{0, +, -\}$ with open sets $\emptyset, \{0, +\}, \{0, +, -\}$.

The fundamental property of Riemann–Zariski spaces is the following result (see the proofs of [Vaq00, Théorème 7.2] and [ZS75, Theorem VI.40]).

THEOREM 2.3.3. *Let F be a field and let R be a subring of F . Then $S_{F/R}$ with the patch topology is a closed subset of T_F . Consequently (by Tykhonov’s theorem) $S_{F/R}$ is compact under the patch topology and quasicompact under the Zariski topology.*

2.4 Centers of valuations

DEFINITION 2.4.1. Let F be a field, let $v : F \rightarrow \Gamma_\infty$ be a valuation on F , and let A be a subring of \mathfrak{o}_v . The *center* of v on A is the ideal $A \cap \mathfrak{m}_v$ in A ; it is prime because it is the contraction of a prime ideal of \mathfrak{o}_v . If A is a local ring and $F = \text{Frac}(A)$, we say that v is *centered in A* if the center of v on A equals \mathfrak{m}_A ; it is equivalent to say that the valuation ring of v dominates A .

The concept of the center of a valuation also extends to schemes.

DEFINITION 2.4.2. Let X be an integral scheme, and let v be a valuation on the function field of X . Then the set of points x whose local rings $\mathcal{O}_{X,x}$ are contained in the valuation ring \mathfrak{o}_v is an irreducible or empty closed subset of X (see [Vaq00, Proposition 6.2]); we call this set (or the corresponding reduced closed subscheme) the *center* of v on X . Note that the generic point of the center of v is the unique point x such that v is centered in $\mathcal{O}_{X,x}$. If the center of v on X is nonempty, we say that v is *centered on X* . If X is proper over a field, then v is always centered on X ; see [Vaq00, Proposition 6.3]. (See also Lemma 2.4.5 below.)

PROPOSITION 2.4.3. *Let X be a proper irreducible variety over a field k , and let v be a valuation on $k(X)$ over k . Then the dimension of the center of v on X is at most $\text{trdeg}(\kappa_v/k)$. Moreover, there exists a blowup Y of X such that the center of v on Y has dimension equal to $\text{trdeg}(\kappa_v/k)$, and the same is true of any further blowup Y' of Y .*

Proof. The equality occurs whenever the local ring on Y of the generic point of the center of v contains elements lifting a transcendence basis of κ_v over k ; this clearly occurs for some Y and continues to occur after further blowing up. Compare [Vaq00, Proposition 6.4]. \square

Note the following relationship to the Riemann–Zariski space [Vaq00, Proposition 7.4].

PROPOSITION 2.4.4. *Let X be an integral noetherian scheme over a field k . Then the set of valuations $v \in S_{k(X)/k}$ with nonempty center on X is an open subset $U(X)$ of $S_{k(X)/k}$ for the Zariski topology, and the map $U(X) \rightarrow X$ carrying $v \in U(X)$ to the generic point of its center is continuous for the Zariski topology on $S_{k(X)/k}$ and the usual (Zariski) topology on X .*

LEMMA 2.4.5. *Let $f : X' \rightarrow X$ be a proper morphism between irreducible varieties over k . Let w be a valuation on $k(X')$ over k , and let v be the restriction of w to $k(X)$. Let Y be the center of v on X , and let Y' be the center of w on X' . Then the generic point of Y' maps to the generic point of Y .*

Proof. The proof is a relative version of [Vaq00, Proposition 6.3]. In the diagram

$$\begin{array}{ccc}
 \mathrm{Spec} k(X') & \longrightarrow & X' \\
 \downarrow & \nearrow & \downarrow f \\
 \mathrm{Spec} \mathfrak{o}_w & \longrightarrow & X
 \end{array}$$

the lower horizontal arrow factors as the surjection $\mathrm{Spec} \mathfrak{o}_w \rightarrow \mathrm{Spec} \mathfrak{o}_v$ followed by the map $\mathrm{Spec} \mathfrak{o}_v \rightarrow X$ whose image of the closed point of $\mathrm{Spec} \mathfrak{o}_w$ is the generic point of Y . By the valuative criterion for properness, into the diagram we can insert a map $\mathrm{Spec} \mathfrak{o}_w \rightarrow X'$, whose image of the closed point of $\mathrm{Spec} \mathfrak{o}_w$ is the generic point of Y' . This proves the claim. \square

2.5 Abhyankar’s inequality

We recall a fundamental result of Abhyankar [Vaq00, Théorème 9.2].

DEFINITION 2.5.1. We say that a valuation v is *discrete* if its value group is isomorphic to \mathbb{Z}^d under the lexicographic ordering, for some nonnegative integer d . Note that this is more inclusive than the layman’s definition of a ‘discrete valuation’; that concept corresponds in standard valuation-theoretic terminology to a *divisorial valuation*. See Definition 2.5.3.

THEOREM 2.5.2 (Abhyankar). *Let A be a noetherian local ring, and put $F = \mathrm{Frac}(A)$. Let $v : F \rightarrow \Gamma_\infty$ be a valuation on F centered in A .*

(a) *The following inequality holds:*

$$\mathrm{ratrank}(v) + \mathrm{trdeg}(\kappa_v/\kappa_A) \leq \dim(A). \tag{2.5.2.1}$$

(b) *If equality holds in (a), then $\Gamma \cong \mathbb{Z}^{\mathrm{ratrank}(v)}$ (as an abstract group) and κ_v is a finitely generated field extension of κ_A .*

(c) *If $\mathrm{rank}(v) + \mathrm{trdeg}(\kappa_v/\kappa_A) = \dim(A)$, then v is discrete.*

DEFINITION 2.5.3. Let A be a noetherian local ring, and put $F = \mathrm{Frac}(A)$. Let $v : F \rightarrow \Gamma_\infty$ be a valuation on F whose valuation ring contains A . We say that v is an *Abhyankar valuation* if equality holds in (2.5.2.1). We say a valuation v is *divisorial* if

$$\mathrm{rank}(v) = 1, \quad \mathrm{trdeg}(\kappa_v/\kappa_A) = \dim(A) - 1.$$

We say that v is *monomial* if

$$\mathrm{rank}(v) = 1, \quad \mathrm{ratrank}(v) = \dim(A), \quad \kappa_v = \kappa_A;$$

beware that some authors may prefer not to include the rank restriction.

Remark 2.5.4. For k a field, any valuation v over k on a finitely generated field K over k is subject to Abhyankar’s inequality. That is because we may choose a proper variety X over k with function field K , on which v will be centered (see Definition 2.4.2). In particular, v is centered on some affine chart of X , whose coordinate ring is noetherian, as is its localization at the center of v .

3. The local approach to semistable reduction

In this chapter, we recall the statement of the semistable reduction problem, then demonstrate its equivalence with a formally weaker form which is in some sense local at a valuation.

CONVENTION 3.0.1. Throughout this section, let k be a field of characteristic $p > 0$, and fix a power q of the prime p . Let K be a discretely valued field of characteristic 0 with residue field k . Assume that there exists a continuous endomorphism $\sigma_K : K \rightarrow K$ lifting the q -power Frobenius morphism on k , and fix a choice of σ_K .

Remark 3.0.2. The restriction to K discretely valued is necessitated in part by that restriction in the work of Shiho [Shi00, Shi02] invoked in [Ked07], and in part by that restriction in the construction of slope filtrations for Frobenius modules [Ked04a], which will intervene at more than one point in this series.

We retain notation and terminology as set in [Ked07]; for convenience, we recall some of the less standard notation.

DEFINITION 3.0.3. Let Γ^* be the divisible closure of $|K^*|$. We say that an interval $I \subseteq [0, +\infty)$ is *aligned* if each endpoint at which I is closed is either zero or an element of Γ^* . For I an aligned interval, let $A_K^m(I)$ denote the admissible subspace of the rigid analytic m -plane with coordinates t_1, \dots, t_m , consisting of points where $|t_i| \in I$ for $i = 1, \dots, m$. We drop the parentheses around I when it is written out explicitly; for instance, we write $A_K^m[0, 1)$ for the open unit polydisc. If we need to specify the name t of the family of dummy variables, we add it to the notation as a subscript, e.g. $A_{K,t}^m(I)$.

3.1 Alterations

Before proceeding further, we recall the fundamental concept of alterations, from [DJ96, § 2.20].

DEFINITION 3.1.1. For X a noetherian integral scheme, an *alteration* of X is a proper dominant morphism $f : X_1 \rightarrow X$ with X_1 irreducible and f generically finite, i.e. there exists $U \subseteq X$ open dense such that $f^{-1}(U) \rightarrow U$ is finite.

DEFINITION 3.1.2. Let X be an irreducible k -variety, and let Z be a proper closed subset of X . A *quasiresolution* of the pair (X, Z) consists of:

- (i) an alteration $f : X_1 \rightarrow X$ over k , which is required to be generically étale if k is perfect; and
- (ii) an open immersion $j : X_1 \hookrightarrow \overline{X_1}$ over k , with $\overline{X_1}$ projective over k ;

such that $(\overline{X_1}, j(f^{-1}(Z)) \cup (\overline{X_1} \setminus j(X_1)))$ form a smooth pair. By a quasiresolution of X , we mean a quasiresolution of the pair (X, \emptyset) .

In terms of this definition, de Jong's alterations theorem is as follows [DJ96, Theorem 4.1].

THEOREM 3.1.3. *Let X be an irreducible k -variety, and let Z be a proper closed subset of X . Then the pair (X, Z) admits a quasiresolution.*

3.2 The semistable reduction problem

We now formulate the semistable reduction problem, following Shiho [Shi02, Conjecture 3.1.8]. (This was done once already in [Ked07, Conjecture 7.1.2].)

DEFINITION 3.2.1. Let (X, Z) be a smooth pair, and let \mathcal{E} be a convergent isocrystal on $U = X \setminus Z$. We say that \mathcal{E} is *log-extendable (on X)* if \mathcal{E} extends to a convergent log-isocrystal \mathcal{F} with nilpotent residues on the log-scheme (X, Z) , in the sense of Shiho [Shi00, Shi02].

The following alternate formulation is [Ked07, Theorem 6.4.5] (the main result of [Ked07]).

THEOREM 3.2.2. *Let (X, Z) be a smooth pair, and let \mathcal{E} be an isocrystal on $U = X \setminus Z$ overconvergent along Z . Then \mathcal{E} has unipotent monodromy along Z in the sense of [Ked07, Definition 4.4.2] if and only if \mathcal{E} is log-extendable on X . Moreover, the restriction functor, from convergent log-isocrystals with nilpotent residues on (X, Z) to isocrystals on U overconvergent along Z , is fully faithful; in particular, a log-extension with nilpotent residues is unique if it exists, and any Frobenius on \mathcal{E} also acts on such an extension.*

Remark 3.2.3. Let (X, Z) and (X', Z') be smooth pairs, and let $f : X' \rightarrow X$ be a morphism with $f(X' \setminus Z') \subseteq X \setminus Z$. If \mathcal{E} is a convergent isocrystal on $X \setminus Z$ which is log-extendable on X , then $f^*\mathcal{E}$ is log-extendable on X' ; this follows from the existence of pullbacks in Shiho’s category of convergent log-isocrystals.

DEFINITION 3.2.4. Let X be a smooth irreducible k -variety, and let \mathcal{E} be an overconvergent F -isocrystal on X . We say that \mathcal{E} admits semistable reduction if there exists a quasiresolution $(f : X_1 \rightarrow X, j : X_1 \hookrightarrow \overline{X_1})$ of X such that $f^*\mathcal{E}$ is log-extendable on $\overline{X_1}$.

We now recall, in the present terminology, the formulation of Shiho’s conjecture [Shi02, Conjecture 3.1.8] given earlier in this series [Ked07, Conjecture 7.1.2].

CONJECTURE 3.2.5. Let X be a smooth irreducible k -variety and let \mathcal{E} be an overconvergent F -isocrystal on X . Then \mathcal{E} admits semistable reduction.

Actually, Shiho’s original conjecture only required k perfect; however, the distinction between this and the general case is illusory. In fact, one may even reduce to considering algebraically closed base fields, as follows.

PROPOSITION 3.2.6. *Let X be a smooth irreducible k -variety, and let \mathcal{E} be an overconvergent F -isocrystal on X . Let K^{unr} be the maximal unramified extension of K , and let K' be the completion of the direct limit $K^{\text{unr}} \xrightarrow{\sigma_K} K^{\text{unr}} \xrightarrow{\sigma_K} \dots$. Let \mathcal{E}' be the overconvergent F -isocrystal on $X' = X \times_k k^{\text{alg}}$, with coefficients in K' , obtained by base extension from \mathcal{E} . If \mathcal{E}' admits semistable reduction, then so does \mathcal{E} .*

Proof. If \mathcal{E}' admits semistable reduction, then there exists a quasiresolution $(f' : X'_1 \rightarrow X', j : X'_1 \hookrightarrow \overline{X'_1})$ such that $(f')^*\mathcal{E}'$ is log-extendable on $\overline{X'_1}$. Since specifying the data of this quasiresolution only involves a finite number of elements of k^{alg} , we can realize it over some finite extension k' of k . This means that we can produce an alteration $f_1 : X_1 \rightarrow X$ such that $X_1 \times_k k'$ is a disjoint union of copies of X'_1 . Unfortunately, X_1 need not be smooth over k ; however, if we construct a quasiresolution $(f_2 : X_2 \rightarrow X_1, j_2 : X_2 \hookrightarrow \overline{X_2})$, then the base extension of $(f_2 \circ f_1)^*\mathcal{E}'$ to k^{alg} is log-extendable on $\overline{X_2} \times_k k^{\text{alg}}$. Since local unipotency can be checked after a field extension [Ked07, Remark 3.4.4], we may apply Theorem 3.2.2 to deduce that $(f_2 \circ f_1)^*\mathcal{E}$ is also log-extendable on $\overline{X_2}$. Hence \mathcal{E} admits semistable reduction. □

3.3 Local semistable reduction

We next formulate a local version of semistable reduction, then relate it to global semistable reduction via the quasicompactness of Riemann–Zariski spaces.

DEFINITION 3.3.1. Let X be a smooth irreducible k -variety, and let \mathcal{E} be an overconvergent F -isocrystal on X . For v a valuation on the function field $k(X)$ over k , we say that \mathcal{E} admits local semistable reduction at v if there exists a quasiresolution $(f : X_1 \rightarrow X, j : X_1 \hookrightarrow \overline{X_1})$ of X such that $f^*\mathcal{E}$ is log-extendable on some open subscheme of $\overline{X_1}$ on which each extension of v to $k(X_1)$ is centered.

Remark 3.3.2. If $g : Y \rightarrow X$ is any alteration, and $g^*\mathcal{E}$ admits local semistable reduction at every extension of v to $k(Y)$, then \mathcal{E} admits local semistable reduction at v .

LEMMA 3.3.3. *Let X be a smooth irreducible k -variety, let $(f : X_1 \rightarrow X, j : X_1 \hookrightarrow \overline{X_1})$ be a quasiresolution of X , and let U be an open subscheme of $\overline{X_1}$. Then the set of $v \in S_{k(X)/k}$, all of whose extensions to $k(X_1)$ are centered on U , is an open subset of $S_{k(X)/k}$ for the patch topology.*

Proof. Let A be the set of valuations on $k(X_1)$ centered on U ; then A is open in $S_{k(X_1)/k}$. Put $B = S_{k(X_1)/k} \setminus A$, which is thus closed for the patch topology; since $S_{k(X_1)/k}$ is compact by Theorem 2.3.3, then so is B . Let C be the image of B under the restriction map $S_{k(X_1)/k} \rightarrow S_{k(X)/k}$; then C is quasicompact since it is the image of a quasicompact topological space under a continuous map. Since $S_{k(X)/k}$ is Hausdorff under the patch topology, C is Hausdorff, hence compact, hence closed. The set we are looking for is the complement of C , so we are done. \square

PROPOSITION 3.3.4. *Let X be a smooth irreducible k -variety, and let \mathcal{E} be an overconvergent F -isocrystal on X . Suppose that \mathcal{E} admits local semistable reduction at every valuation on $k(X)$ over k . Then \mathcal{E} admits semistable reduction.*

Proof. Consider the Riemann–Zariski space $S_{k(X)/k}$ equipped with the patch topology. By hypothesis, for each $v_i \in S_{k(X)/k}$, we may choose a quasiresolution $(f_i : X_i \rightarrow X, j_i : X_i \hookrightarrow \overline{X}_i)$ of X such that $f_i^* \mathcal{E}$ is log-extendable on some open subscheme U_i of \overline{X}_i containing the center of each extension of v_i to $k(X_i)$. Let B_i be the set of valuations $w \in S_{k(X)/k}$ each of whose extensions to $k(X_i)$ is centered in U_i ; by Lemma 3.3.3, B_i is an open neighborhood of v_i in $S_{k(X)/k}$.

By Theorem 2.3.3, $S_{k(X)/k}$ is compact, so there exist finitely many valuations $v_1, \dots, v_n \in S_{k(X)/k}$ such that $B_1 \cup \dots \cup B_n = S_{k(X)/k}$. Apply Theorem 3.1.3 (to the closure in $\overline{X}_1 \times_k \dots \times_k \overline{X}_n$ of an irreducible component of $X_1 \times_X \dots \times_X X_n$) to produce a smooth pair (\overline{Y}, E) with \overline{Y} projective, admitting maps $g_i : (\overline{Y}, E) \rightarrow (\overline{X}_i, \overline{X}_i \setminus X_i)$ for $i = 1, \dots, n$. Let Y be the inverse image of $X_1 \times_X \dots \times_X X_n$ in \overline{Y} , so that Y is open dense in \overline{Y} and the g_i induce a projective map $g : Y \rightarrow X$. Then by Remark 3.2.3, $g^* \mathcal{E} = (g_i \circ f_i)^* \mathcal{E}$ is log-extendable on $g_i^{-1}(U_i)$ for each i .

Let H be a component of E ; then H corresponds to a divisorial valuation on $k(Y)$, whose restriction to $k(X)$ must lie in one of the B_i . For any such i , $g_i^{-1}(U_i)$ meets H , so by the easy direction of Theorem 3.2.2 plus [Ked07, Proposition 4.4.4], $g^* \mathcal{E}$ has unipotent local monodromy along H . Since this is true for each H , we may apply the other direction of Theorem 3.2.2 to deduce that $g^* \mathcal{E}$ is log-extendable on \overline{Y} . Hence \mathcal{E} admits semistable reduction, as desired. \square

Since we are now using a local strategy, it is sensible to refer to unipotent local monodromy in terms of valuations rather than divisors.

DEFINITION 3.3.5. Let \mathcal{E} be an overconvergent F -isocrystal on a smooth irreducible k -variety X , and let v be a divisorial valuation on $k(X)$ over k . We say that \mathcal{E} has unipotent local monodromy along v if there exists a birational morphism $f : X' \rightarrow X$ such that v is centered on a smooth divisor Z of X' , and $f^* \mathcal{E}$ has unipotent local monodromy along Z . By [Ked07, Proposition 4.4.1], the same will then be true for any other choice of f . Similarly, it is well defined to say that \mathcal{E} acquires unipotent local monodromy along v over a finite separable extension of $k(X)$.

3.4 Partial compactifications

In some applications, it may be helpful to have some sort of semistable reduction even for isocrystals which are only partially overconvergent. Here is the correct formulation of the global and local problems.

DEFINITION 3.4.1. Let X be a smooth irreducible k -variety, and let \overline{X} be a partial compactification of X (i.e. \overline{X} is a k -variety equipped with an open immersion $X \hookrightarrow \overline{X}$). Let \mathcal{E} be an F -isocrystal on X overconvergent along $\overline{X} \setminus X$. We say that \mathcal{E} admits semistable reduction if there exists a quasiresolution $(f : X_1 \rightarrow \overline{X}, j : X_1 \rightarrow \overline{X}_1)$ of the pair $(\overline{X}, \overline{X} \setminus X)$ such that $f^* \mathcal{E}$ is log-extendable to X_1 . As in Proposition 3.2.6, it is sufficient to check semistable reduction after extending scalars from k to its algebraic closure.

CONJECTURE 3.4.2. Let X be a smooth irreducible k -variety, let \overline{X} be a partial compactification of X , and let \mathcal{E} be an F -isocrystal on X overconvergent along $\overline{X} \setminus X$. Then \mathcal{E} admits semistable reduction.

DEFINITION 3.4.3. Let X be a smooth irreducible k -variety, let \overline{X} be a partial compactification of X , and let v be a valuation on $k(X)$ centered on \overline{X} . Let \mathcal{E} be an F -isocrystal on X overconvergent along $\overline{X} \setminus X$. We say that \mathcal{E} admits local semistable reduction at v if there exists a quasiresolution $(f : X_1 \rightarrow \overline{X}, j : X_1 \rightarrow \overline{X}_1)$ of the pair $(\overline{X}, \overline{X} \setminus X)$ such that $f^*\mathcal{E}$ is log-extendable to some open subset of X_1 on which v is centered.

Remark 3.4.4. Note that, if \overline{X} is proper, then \mathcal{E} is just an overconvergent F -isocrystal, and the two possible interpretations of semistable reduction (Definitions 3.2.4 and 3.4.1) are consistent; similarly, the two possible interpretations of local semistable reduction (Definitions 3.3.1 and 3.4.3) are consistent. More generally, equivalent partial compactifications in the sense of [Ked07, Definition 4.1.2] give rise to equivalent categories of F -isocrystals, and to equivalent notions of global and local semistable reduction.

PROPOSITION 3.4.5. Let X be a smooth irreducible k -variety, and let \overline{X} be a partial compactification of X . Let \mathcal{E} be an F -isocrystal on X overconvergent along $\overline{X} \setminus X$. Suppose that \mathcal{E} admits local semistable reduction at every valuation on $k(X)$ over k centered on \overline{X} . Then \mathcal{E} admits semistable reduction.

Proof. As in Proposition 3.3.4. □

4. Simplification of the local problem

In this section, we demonstrate that the local semistable reduction problem need only be considered around valuations of height 1, by an inductive argument. We also show that valuations whose residue fields have positive transcendence degree over the base field need not be treated separately, by comparison between an isocrystal and its ‘generic fibre’.

4.1 Étale covers of affine spaces

Besides de Jong’s alterations theorem, it will also be useful to have a method for pushing forward isocrystals onto simple spaces. The following result [Ked05, Theorem 2] (based on a technique of Abhyankar for constructing finite étale morphisms in positive characteristic) will be of use in this regard.

PROPOSITION 4.1.1. Let X be an irreducible k -variety of dimension n , let $x \in X$ be a smooth point (whose existence forces X to be geometrically reduced), and let D_1, \dots, D_m be smooth irreducible divisors in X meeting transversely at x . Then there exists an open neighborhood U of x in X and a finite étale morphism $f : U \rightarrow \mathbb{A}_k^n$ such that D_1, \dots, D_m map to coordinate hyperplanes.

The relevance of Proposition 4.1.1 to our study comes from the following observation.

LEMMA 4.1.2. Let (X, Z) be a smooth pair over k , put $U = X \setminus Z$, let $f : Y \rightarrow X$ be a finite étale morphism of k -varieties, and let \mathcal{E} be an isocrystal on $f^{-1}(U)$ overconvergent along $f^{-1}(Z)$. Then \mathcal{E} is log-extendable to Y if and only if $f_*\mathcal{E}$ is log-extendable to X .

Proof. By [Ked07, Theorem 6.4.5], we may check log-extendability by checking unipotence along each component of the boundary divisor. On the one hand, if $f_*\mathcal{E}$ has unipotent local monodromy, then so does $f^*f_*\mathcal{E}$; however, \mathcal{E} injects into $f^*f_*\mathcal{E}$ by adjunction (see [Ked07, Definition 2.6.8]), so by [Ked07, Proposition 3.2.20] it too has unipotent local monodromy. On the other hand, suppose

\mathcal{E} has unipotent local monodromy; we can then push forward a log-extension of \mathcal{E} to obtain a log-extension of $f_*\mathcal{E}$. □

4.2 Composite valuations

We next show that the semistable reduction problem can be reduced to the restricted local semistable reduction problem which is only centered at valuations of height 1.

We first formulate an extension of the full faithfulness theorem for overconvergent-to-convergent restriction [Ked04b, Theorem 1.1]. Although we only need the case of a smooth pair, for future reference we formulate the general theorem and prove it using a descent argument.

THEOREM 4.2.1. *Let $U \hookrightarrow X$ be an open immersion of k -varieties with dense image, with U smooth. Then the restriction functor from $F\text{-Isoc}^\dagger(U, X/K)$ to $F\text{-Isoc}(U/K)$ is fully faithful.*

Proof. Let $\text{Hom}(\mathcal{O}_U, \mathcal{E}; U, X/K)$ and $\text{Hom}(\mathcal{O}_U, \mathcal{E}; U/K)$ be the morphisms from \mathcal{O}_U to \mathcal{E} in the categories $F\text{-Isoc}^\dagger(U, X/K)$ and $F\text{-Isoc}(U/K)$. (These morphisms can be identified with F -invariant horizontal sections on appropriate realizations of \mathcal{E} .) For any $\mathcal{E} \in F\text{-Isoc}^\dagger(U, X/K)$, restriction induces an injection $\text{Hom}(\mathcal{O}_U, \mathcal{E}; U, X/K) \rightarrow \text{Hom}(\mathcal{O}_U, \mathcal{E}; U/K)$, and the desired result is that this arrow is always surjective.

First suppose that $(X, X \setminus U)$ is a smooth pair (this is the only case that will be used in this paper). Then this statement follows from [Tsu02, Proposition 6.2.1], under the assumption of [Tsu02, Conjecture 2.3.2]. However, the latter conjecture is verified by [Ked04b, Theorem 5.1], so we may unconditionally deduce the desired result.

In the general case, we may assume that X is irreducible. Choose a quasiresolution $(f : X_1 \rightarrow X, j : X_1 \hookrightarrow \overline{X_1})$ of the pair $(X, X \setminus U)$, and put $U_1 = f^{-1}(U)$. Then (U_1, X_1) is a smooth pair; hence given $\mathcal{E} \in F\text{-Isoc}^\dagger(U, X/K)$, we may apply the previous paragraph to show that the map $\text{Hom}(\mathcal{O}_{U_1}, f^*\mathcal{E}; U_1, X_1/K) \rightarrow \text{Hom}(\mathcal{O}_{U_1}, f^*\mathcal{E}; U_1/K)$ is bijective.

Suppose we are given $\mathbf{v} \in \text{Hom}(\mathcal{O}_U, \mathcal{E}; U/K)$; we can pull \mathbf{v} back to $\text{Hom}(\mathcal{O}_{U_1}, f^*\mathcal{E}; U_1/K)$. By the previous paragraph, this element lifts to $\text{Hom}(\mathcal{O}_{U_1}, f^*\mathcal{E}; U_1, X_1/K)$. Let U' be the subscheme of U over which f is finite étale, and put $U'_1 = f^{-1}(U')$. We can restrict \mathbf{v} to $\text{Hom}(\mathcal{O}_{U'_1}, f^*\mathcal{E}; U'_1, X_1/K)$, which by adjunction for finite étale morphisms (see [Ked07, Definition 2.6.8] and [Tsu02, § 5.1]) maps to $\text{Hom}(\mathcal{O}_{U'}, \mathcal{E}; U', X/K)$. (Note that we are overloading notation slightly, by using \mathcal{E} and $f^*\mathcal{E}$ to refer also to the restrictions to $F\text{-Isoc}^\dagger(U', X/K)$ and $F\text{-Isoc}^\dagger(U'_1, X_1/K)$, respectively.)

By [Ked07, Theorem 5.2.1], \mathbf{v} lifts to a morphism in $\text{Isoc}^\dagger(U, X/K)$. Since we can check compatibility with Frobenius over U' , we have $\mathbf{v} \in F\text{-Isoc}^\dagger(U, X/K)$, as desired. (A proof using Tsuzuki's cohomological descent theorem [Tsu03, Theorem 2.1.3] is also possible.) □

We next verify a particular geometric instance of the general statement we are after; ultimately we will reduce back to this case.

LEMMA 4.2.2. *Put $\mathbb{A}_k^m = \text{Spec } k[t_1, \dots, t_m]$ and put $D = \text{Spec } V(t_1 \cdots t_m) \subset \mathbb{A}_k^m$. Put $\mathbb{A}_k^n = \text{Spec } k[u_1, \dots, u_n]$ and put $E = \text{Spec } V(u_1 \cdots u_n) \subset \mathbb{A}_k^n$. Let \mathcal{E} be an F -isocrystal on $(\mathbb{A}_k^m \setminus D) \times (\mathbb{A}_k^n \setminus E)$ overconvergent along $D \times (\mathbb{A}_k^n \setminus E) \cup (\mathbb{A}_k^m \setminus D) \times E$, with unipotent monodromy along each component of $D \times (\mathbb{A}_k^n \setminus E)$. Apply Theorem 3.2.2 to extend \mathcal{E} to a convergent log-isocrystal on $(\mathbb{A}_k^m \times (\mathbb{A}_k^n \setminus E), D \times (\mathbb{A}_k^n \setminus E))$, then restrict to $\{0\} \times (\mathbb{A}_k^n \setminus E)$. Let \mathcal{F} be the resulting convergent F -isocrystal, and suppose that \mathcal{F} is log-extendable to $\{0\} \times \mathbb{A}_k^n$. Then, for any sufficiently large integer N , \mathcal{E} is log-extendable to*

$$\text{Spec } k[t_1(u_1 \cdots u_n)^{-N}, \dots, t_m(u_1 \cdots u_n)^{-N}, u_1, \dots, u_n].$$

Proof. To check log-extendability of \mathcal{E} to

$$\text{Spec } k[t_1(u_1 \cdots u_n)^{-N}, \dots, t_m(u_1 \cdots u_n)^{-N}, u_1, \dots, u_n],$$

by Theorem 3.2.2 it suffices to check unipotence along $V(u_j)$ for $j = 1, \dots, n$. By generization in the sense of [Ked07, Proposition 3.4.3] (or more precisely, from [Ked07, Proposition 4.4.1]), we may reduce to the case $n = 1$.

We may realize \mathcal{E} as a ∇ -module on a space of the form $A_{K,t}^m[\epsilon, 1] \times A_{K,u}^1[\delta, 1]$ for some $\delta, \epsilon \in (0, 1) \cap \Gamma^*$. By [Ked07, Lemma 5.1.1(b)], for suitable δ , we can extend \mathcal{E} to a log- ∇ -module \mathcal{E}' with nilpotent residues on $A_{K,t}^m[0, 1] \times A_{K,u}^1[\delta, 1]$.

The restriction of \mathcal{E}' to $\{0\} \times A_{K,u}^1[1, 1]$ is isomorphic to \mathcal{F} , which we assumed admits a log-extension \mathcal{G} . By Theorem 4.2.1, again for suitable δ the restriction of \mathcal{E}' to $\{0\} \times A_{K,u}^1[\delta, 1]$ is isomorphic to a corresponding restriction of \mathcal{G} . (Here we are using that the restriction of \mathcal{E}' is overconvergent with respect to u_1 ; this follows from the same fact on \mathcal{E}' itself. By [Ked07, Lemma 3.1.6], we may check this after restriction to the subspace on which $|t_1| = \dots = |t_n| = 1$, where we are given that \mathcal{E} is overconvergent.)

Pick any $\eta \in (\epsilon, 1) \cap \Gamma^*$. By [Ked07, Proposition 3.5.3], there exists $\delta \in (0, 1) \cap \Gamma^*$ such that \mathcal{E} is unipotent on $A_{K,t}^m[\eta, \eta^{1/q}] \times A_{K,u}^1[\delta, 1]$. In other words, on that space, \mathcal{E} is isomorphic to a successive extension of ∇ -modules pulled back from $A_{K,u}^1[\delta, 1]$. By the hypothesis on the log-extendability of \mathcal{F} , plus [Ked07, Proposition 3.6.9], we can choose δ so that the resulting ∇ -modules on $A_{K,u}^1[\delta, 1]$ all become unipotent on $A_{K,u}^1[\delta, 1]$. Hence \mathcal{E} admits a filtration with trivial successive quotients on

$$\{(t_1, \dots, t_m, u_1) \in \mathbb{A}_K^{m+1} : \delta \leq |u_1| < 1; \eta \leq |t_i| \leq \eta^{1/q} \ (i = 1, \dots, m)\}.$$

In particular, for any N with $\eta^{1/N} \geq \delta$ (which holds for N sufficiently large), this space contains

$$\{(t_1, \dots, t_m, u_1) \in \mathbb{A}_K^{m+1} : \eta^{1/N} \leq |u_1| \leq \eta^{1/(qN)}; |t_i/u_1^N| = 1 \ (i = 1, \dots, m)\},$$

so \mathcal{E} is unipotent on the latter. By applying Frobenius repeatedly, we see that, for each nonnegative integer h , \mathcal{E} is unipotent on

$$\{(t_1, \dots, t_m, u_1) \in \mathbb{A}_K^{m+1} : \eta^{1/(q^h N)} \leq |u_1| \leq \eta^{1/(q^{h+1} N)}; |t_i/u_1^N| = 1 \ (i = 1, \dots, m)\}.$$

By glueing, \mathcal{E} is unipotent on

$$\{(t_1, \dots, t_m, u_1) \in \mathbb{A}_K^{m+1} : \eta^{1/N} \leq |u_1| < 1; |t_i/u_1^N| = 1 \ (i = 1, \dots, m)\}.$$

Hence \mathcal{E} has unipotent monodromy along the subspace $V(u_1)$ in $\text{Spec } k[t_1/u_1^N, \dots, t_m/u_1^N, u_1]$, so Theorem 3.2.2 yields the desired result. \square

We now state a partially restricted version of the local semistable reduction problem; we will restrict even further in Conjecture 4.3.3.

CONJECTURE 4.2.3. Let X be a smooth irreducible k -variety, let \overline{X} be a partial compactification of X , and let \mathcal{E} be an F -isocrystal on X overconvergent along $\overline{X} \setminus X$. Then \mathcal{E} admits local semistable reduction at any valuation v on $k(X)$ over k of height 1 centered on \overline{X} .

PROPOSITION 4.2.4. Suppose that Conjecture 3.4.2 holds for all varieties of dimension less than n , and that Conjecture 4.2.3 holds for all varieties of dimension n . Then Conjecture 3.4.2 also holds for varieties of dimension n . In particular, Conjecture 4.2.3 (for a given k and K) implies Conjecture 3.4.2 (for the same k and K).

Proof. Let X be a smooth irreducible k -variety of dimension n , let \overline{X} be a partial compactification of X , and let \mathcal{E} be an F -isocrystal on X overconvergent along $\overline{X} \setminus X$. By Proposition 3.4.5, it suffices to show that, for any valuation v on $k(X)$ of height greater than 1 centered on \overline{X} , \mathcal{E} admits local semistable reduction at v . As in Definition 2.2.6, write v as a composition $v' \circ \overline{v}$, where v' is a valuation on X of height 1 centered on \overline{X} .

We establish a series of reductions of this statement to more restrictive versions. To begin with, we may assume by Proposition 3.2.6 that:

(a) The field k is algebraically closed.

Note that, at any point, we may pull back along an alteration and replace v by each of its extensions in turn; Lemma 2.4.5 guarantees that these stay centered in the right places. By Theorem 3.1.3, we may thus assume that:

(b) There exists a smooth pair $(\overline{Y}, \overline{D})$, such that $X = Y \setminus D$ for Y the complement of a union of components of \overline{D} and $D = Y \cap \overline{D}$, such that v' is centered on Y and v is centered on \overline{Y} .

Note also that the condition that \mathcal{E} is log-extendable to an open subset of Y on which v is centered is local on Y , thanks to the full faithfulness aspect of Theorem 3.2.2.

By the hypothesis that Conjecture 4.2.3 holds for all varieties of dimension n , we know that \mathcal{E} admits local semistable reduction at v' . Hence by passing up a suitable quasiresolution and shrinking, we may thus assume that:

(c) The isocrystal \mathcal{E} is log-extendable to Y .

By shrinking Y and enlarging D , we can ensure that:

(d) The intersection E of all of the components of D is nonempty and irreducible, and the center of v' on Y is equal to E .

By shrinking X and \overline{Y} , then applying Proposition 4.1.1 and Lemma 4.1.2, we can ensure that:

(e) We have $Y = \mathbb{A}_k^m \times X'$ and $\overline{Y} = \mathbb{A}_k^m \times Y'$, and writing $\mathbb{A}_k^m = \text{Spec } k[t_1, \dots, t_m]$, we have $D = V(t_1 \cdots t_m)$.

By Theorem 3.1.3 again (applied this time to $(Y', Y' \setminus X')$), we may assume that:

(f) There exists a smooth pair (Y', D') with $X' = Y' \setminus D'$, such that \overline{v} is centered on Y' .

By applying Proposition 4.1.1 and Lemma 4.1.2, we can ensure that:

(g) We have $Y' = \mathbb{A}^{n-m}$, and writing $\mathbb{A}_k^{n-m} = \text{Spec } k[u_1, \dots, u_{n-m}]$, we have $D' = V(u_1 \cdots u_{n-m})$.

By [Ked07, Lemma 5.1.1] (applied on affine subspaces of $\mathbb{P}_k^m \times \mathbb{P}_k^n$), we may realize the log-extension of \mathcal{E} to Y as a log- ∇ -module with nilpotent residues on $A_{K,t}^m[0, 1] \times A_{K,u}^{n-m}[\delta, 1]$ for some $\delta \in (0, 1) \cap \Gamma^*$, which is convergent with respect to the parameters $t_1, \dots, t_m, u_1, \dots, u_{n-m}$. The restriction of this log- ∇ -module to $\{0\} \times A_{K,u}^{n-m}[\delta, 1]$ represents an F -isocrystal on X' overconvergent along D' . Let \mathcal{F} denote the underlying convergent F -isocrystal on X' ; since \overline{v} is a well-defined valuation on $k(X')$ and $\dim(X') < n$, we may invoke the induction hypothesis in order to ensure that:

(h) The isocrystal \mathcal{F} is log-extendable to Y' .

This disturbs restriction (g), but we may apply Proposition 4.1.1 and Lemma 4.1.2 to re-establish it without losing any of the other restrictions. The desired result in this case follows from Lemma 4.2.2. □

4.3 Positive transcendence degree

We now give an argument to eliminate the need for separately treating valuations whose residue fields are not algebraic over k . This again amounts to generization; as in the previous section, we calculate in a simple geometric setting and then reduce the general case back to the simple one.

LEMMA 4.3.1. *Let v be a valuation on $k(t_1, \dots, t_m, u_1, \dots, u_n)$ over k , with center on $\mathbb{A}_k^{m+n} = \mathbb{A}_k^m \times \mathbb{A}_k^n$ equal to $\{0\} \times \mathbb{A}_k^n$; note that this implies that v is trivial on $\ell = k(u_1, \dots, u_n)$. Put $D = V(t_1 \cdots t_m) \subset \mathbb{A}_k^m$ and $D_\ell = V(t_1 \cdots t_m) \subset \mathbb{A}_\ell^m$. Let \mathcal{E} be an F -isocrystal on $(\mathbb{A}_k^m \setminus D) \times \mathbb{A}_k^n$ overconvergent along $D \times \mathbb{A}_k^n$. Let L be the p -adic completion of $K(u_1, \dots, u_n)$, and let \mathcal{F} be the*

induced isocrystal on $\mathbb{A}_\ell^m \setminus D_\ell$ overconvergent along D_ℓ , with coefficient field L (as in [Ked07, Proposition 3.4.3]). If \mathcal{F} admits local semistable reduction at v , then so does \mathcal{E} .

Proof. Choose a quasiresolution $(f_1 : X_1 \rightarrow (\mathbb{A}_\ell^m \setminus D_\ell), j_1 : X_1 \hookrightarrow \overline{X_1})$ of $\mathbb{A}_\ell^m \setminus D_\ell$, such that $f_1^* \mathcal{F}$ is log-extendable on an open subset on which each extension of v to $\ell(X_1)$ is centered. Note that each such valuation has center equal to a closed point; by Theorem 3.2.2, $f_1^* \mathcal{F}$ has unipotent local monodromy along any divisor passing through that point.

Choose a quasiresolution $(f_2 : X_2 \rightarrow \mathbb{A}_k^{m+n}, j_2 : X_2 \hookrightarrow \overline{X_2})$ of the pair $(\mathbb{A}_k^{m+n}, D \times \mathbb{A}_k^n)$, such that $k(X_2)$ contains the maximal separable subextension of the normal closure of $\ell(X_1)$ over $\ell(t_1, \dots, t_m) = k(t_1, \dots, t_m, u_1, \dots, u_n)$. Put $U = f_2^{-1}((\mathbb{A}_k^m \setminus D) \times \mathbb{A}_k^n)$, so that $(\overline{X_2}, \overline{X_2} \setminus U)$ is a smooth pair and $f_2^* \mathcal{E}$ is an F -isocrystal on U overconvergent along $X_2 \setminus U$.

Let w be any extension of v to $k(X_2)$; we may view w also as an extension of v to $\ell(X_1)$ over ℓ . Let Y be the center of w on $\overline{X_2}$; by Lemma 2.4.5, $f_2(Y) = \{0\} \times \mathbb{A}_k^n$. Let E be a component of $\overline{X_2} \setminus U$ containing Y , and let w_E be the corresponding divisorial valuation on $k(X_2)$. Let v_E be the restriction of w_E to $k(t_1, \dots, t_m, u_1, \dots, u_n)$; then the center of v_E on \mathbb{A}_k^{m+n} contains $f_2(Y) = \{0\} \times \mathbb{A}_k^n$.

We deduce that v_E is trivial on ℓ ; we may thus view w_E as a divisorial valuation on $\ell(X_1)$ over ℓ , whose center contains the center of w . As noted above, this means that $f_1^* \mathcal{F}$ has unipotent local monodromy along the center of w_E . By [Ked07, Proposition 3.4.3], $f_2^* \mathcal{E}$ has unipotent local monodromy along E .

By Theorem 3.2.2, we may conclude that $f_2^* \mathcal{E}$ is log-extendable to a subscheme of $\overline{X_2}$ on which each extension of v to $k(X_2)$ is centered. This implies that \mathcal{E} admits local semistable reduction at v , as desired. □

DEFINITION 4.3.2. Let X be an irreducible variety over k . By a *minimal valuation* on X , we mean a valuation v on the function field $k(X)$ over k such that $\text{height}(v) = 1$ and $\text{trdeg}(\kappa_v/k) = 0$.

We now give our most refined version of the local semistable reduction problem.

CONJECTURE 4.3.3. Let X be a smooth irreducible k -variety, let \overline{X} be a partial compactification of X , and let \mathcal{E} be an F -isocrystal on X overconvergent along $\overline{X} \setminus X$. Then \mathcal{E} admits local semistable reduction at any minimal valuation v on $k(X)$ centered on \overline{X} .

THEOREM 4.3.4. *Suppose that for some integer n , Conjecture 4.3.3 holds for varieties of dimension at most n for all algebraically closed k . Then Conjecture 3.4.2 holds for varieties of dimension at most n for all k .*

Proof. We proceed by induction on n ; we may thus assume Conjecture 3.4.2 for all varieties of dimension less than n . Let X be a smooth irreducible k -variety of dimension n , let \overline{X} be a partial compactification of X , and let \mathcal{E} be an F -isocrystal on X overconvergent along $\overline{X} \setminus X$. By Proposition 4.2.4, it suffices to show that, for any valuation v on $k(X)$ of height 1 centered on \overline{X} , \mathcal{E} admits local semistable reduction at v . This follows from the assumption of Conjecture 4.3.3 in the case $\text{trdeg}(\kappa_v/k) = 0$, so hereafter we assume instead that $\text{trdeg}(\kappa_v/k) = d > 0$.

As in the proof of Proposition 4.2.4, we make a sequence of reductions, again starting by applying Proposition 3.2.6 to reduce to the case where:

- (a) The field k is algebraically closed.

By Theorem 3.1.3, we may assume that:

- (b) There exists a smooth pair (Y, D) with $X = Y \setminus D$, such that v is centered on Y .

By Proposition 2.4.3 (plus Theorem 3.1.3 again), we can blow up X and Y to ensure that:

(c) The dimension of the center of v on Y is equal to d .

By shrinking X and Y , we may assume that:

(d) D consists of $n - d$ components whose intersection E is the center of v on Y .

By Proposition 4.1.1 and Lemma 4.1.2, we may assume that:

(e) $Y = \mathbb{A}_k^n = \text{Spec } k[t_1, \dots, t_n]$ and $D = V(t_1 \cdots t_{n-d})$.

The desired result now follows from Lemma 4.3.1. \square

Remark 4.3.5. Theorem 4.3.4 and the p -adic local monodromy theorem imply that local semistable reduction holds at any divisorial valuation. One way to interpret Theorem 3.2.2 is that local semistable reduction at a general valuation v is equivalent to *uniform* local semistable reduction at all divisorial valuations in some neighborhood of v .

Remark 4.3.6. One can deduce refinements of Theorem 4.3.4 by inspecting its proof and the proof of Proposition 4.2.4. For instance, local semistable reduction for all Abhyankar valuations follows from local semistable reduction for all monomial valuations.

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Kiran S. Kedlaya kedlaya@mit.edu
Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue,
Cambridge, MA 02139, USA