J. Austral. Math. Soc. (Series A) **55** (1993), 311–324

A NATURAL REPRESENTATION OF PARTITIONS AS TERMS OF A UNIVERSAL ALGEBRA

HARRY LAKSER

(Received September 17 1990)

Communicated by T. E. Hall

Abstract

We consider a variety of algebras with two binary commutative and associative operations. For each integer $n \geq 0$, we represent the partitions on an n-element set as n-ary terms in the variety. We determine necessary and sufficient conditions on the variety ensuring that, for each n, these representing terms be all the essentially n-ary terms and moreover that distinct partitions yield distinct terms.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary: 08 A 40; secondary 08 B 20, 05 A 17.
Keywords and phrases: Bell bisemigroup, essentialy n-ary, partition, variety.

1. Introduction

Following the notation of [4], we denote the set of n-ary term operations on an algebra $\mathbf{A}$ by $\mathrm{Clo}_n\,\mathbf{A}$. We say that an n-ary term $f(x_0, \dots, x_{n-1})$ *does not depend on the variable x_i in* $\mathbf{A}$ if the identity

$$f(x_0, \dots, x_{i-1}, y, x_{i+1}, \dots, x_{n-1}) = f(x_0, \dots, x_{i-1}, z, x_{i+1}, \dots, x_{n-1})$$

is satisfied in the algebra $\mathbf{A}$. Otherwise, we say that f *depends on x_i*. Similarly, if $\mathscr{V}$ is a variety of algebras, we say that the term $f(x_0, \dots, x_{n-1})$ *does not*

This research was supported by the NSERC of Canada

depend on the variable x_i *in* $\mathscr{V}$ if $\mathscr{V}$ satisfies the identity

$$f(x_0, \dots, x_{i-1}, y, x_{i+1}, \dots, x_{n-1}) = f(x_0, \dots, x_{i-1}, z, x_{i+1}, \dots, x_{n-1})$$

We say that the n-ary term $f(x_0, \dots, x_{n-1})$ is *essentially* n*-ary* if f depends on all the variables $x_0, \dots, x_{n-1}$. Following G. Grätzer [2], we denote by $P_n(\mathbf{A})$ the subset of $\mathrm{Clo}_n\,\mathbf{A}$ consisting of all essentially n-ary term operations of $\mathbf{A}$.

We denote the set of equivalence relations on a set X by $\mathrm{Eqv}\,X$ and, without further ado, think of them interchangeably as either equivalence relations or as partitions of X. Note that $\mathrm{Eqv}\,\emptyset = \{\emptyset\}$. We denote by $\mathbb{N}_n$ the set $\{0, \dots, n-1\}$. Note that $\mathbb{N}_0 = \emptyset$. We denote by $\mathbb{N}$ the set $\{0, 1, \dots\}$ of all natural numbers.

By a *commutative bisemigroup* $\mathbf{A}$ we mean an algebra $\langle A, +, \cdot\rangle$ such that $+$ and $\cdot$ are binary commutative and associative operations. Let $\mathbf{A}$ be a commutative bisemigroup with a nullary term denoted 0—whether 0 is a nullary fundamental operation or the value of a constant term of arity > 0 will turn out to be irrelevant. For each integer $n \geq 0$ we define a mapping

$$\Phi_n : \mathrm{Eqv}\,\mathbb{N}_n \to \mathrm{Clo}_n\,\mathbf{A}$$

by setting

$$\Phi_0(\emptyset) = 0$$

and, for $n > 0$ and each $\alpha \in \mathrm{Eqv}\,\mathbb{N}_n$, by setting

$$\Phi_n(B) = \prod (x_i \mid i \in B)$$

for each block B of the partition α, and setting

$$\Phi_n(\alpha) = \sum \Phi_n(B),$$

the sum being taken over all the blocks B of α. For example, for the unique partition α of $\mathbb{N}_1$, we have

$$\Phi_1(\alpha) = x_0,$$

and, for the partition $\alpha = \{\{0, 2, 4\}, \{1\}, \{3\}\}$ of $\mathbb{N}_5$, we have

$$\Phi_5(\alpha) = x_0 x_2 x_4 + x_1 + x_3.$$

We say that a commutative bisemigroup $\mathbf{A}$ is a *Bell bisemigroup* if, for each $n \geq 0$, the representation Φ_n is a bijection between $\mathrm{Eqv}\,\mathbb{N}_n$ and $P_n(\mathbf{A})$, that is, if the essentially n-ary term operations are precisely those term operations representing partitions of $\mathbb{N}_n$ and distinct partitions yield distinct term operations. We choose this terminology because the cardinality of Eqv $\mathbb{N}_n$ is often called the *Bell number* $B(n)$—see [5, page 33]. Clearly, whether or not the commutative bisemigroup **A** is a Bell bisemigroup depends only on the variety $\mathscr{V}$ generated by **A**; consequently, we say that a variety of bisemigroups is a *Bell variety of bisemigroups* if it is generated by a Bell bisemigroup.

2. Bell varieties of bisemigroups

In this section and the next we characterize Bell varieties of bisemigroups. We consider the following identities, where 0 denotes a nullary,

$$x + y = y + x \tag{1}$$
$$xy = yx \tag{2}$$
$$(x + y) + z = x + (y + z) \tag{3}$$
$$(xy)z = x(yz) \tag{4}$$
$$0 + x = 0 \tag{5}$$
$$0x = 0 \tag{6}$$
$$x + x = 0 \tag{7}$$
$$xx = 0 \tag{8}$$
$$x + xy = 0 \tag{9}$$
$$xy + xz = 0, \tag{10}$$

and the identities

$$(x + y)z = 0, \tag{11}$$
$$(x + y)z = xyz. \tag{11'}$$

In this section we prove:

LEMMA 1. *If $\mathscr{V}$ is a Bell variety of bisemigroups, then $\mathscr{V}$ satisfies the identities* (1)–(10) *and either the identity* (11) *or* (11′).

To prove Lemma 1, let $\mathscr{V}$ be a Bell variety of bisemigroups. Then there is exactly one nullary term,

$$0,$$

https://doi.org/10.1017/S1446788700034066 Published online by Cambridge University Press

exactly one unary term $f(x)$,

$$x,$$

exactly two essentially binary terms $f(x, y)$,

$$x + y \quad \text{and} \quad xy,$$

and exactly five essentially ternary terms $f(x, y, z)$,

$$x + y + z, \quad xyz, \quad x + yz, \quad y + xz, \quad xy + z.$$

Identities (1)–(4) hold by the definition of Bell variety. The key to the proof is the following lemma:

LEMMA 2. *$\mathscr{V}$ satisfies one of the identities* (11) *or* (11′).

PROOF. If the term $f(x, y, z) = (x + y)z$ does not depend on the variable z, then

$$(x + y)z = (x + y)(u + v) = (u + v)(x + y) = (u + v)w,$$

that is, $(x + y)z$ is constant, and so we have (11).

If $(x + y)z$ does not depend on x, then, by the commutativity of $+$, it also does not depend on y. Consequently, if $(x + y)z$ is not constant, the identity $(x + y)z = z$ holds. But then we have the sequence of identities

$$x + y = (x + z)(x + y) = (x + y)(x + z) = x + z,$$

contradicting the fact that $x + y$ is essentially binary.

Consequently, either (11) holds and we are done, or $(x + y)z$ is essentially ternary. In this latter case, by the symmetry in x and y (and the distinctness of the five essentially ternary terms), $\mathscr{V}$ must satisfy one of the identities

$$(11') \qquad (x + y)z = xyz$$

$$(12) \qquad (x + y)z = x + y + z$$

$$(13) \qquad (x + y)z = xy + z.$$

Identity (12) yields

$$x + y + uv = (x + y)uv = ((x + y)u)v = (x + y + u)v = x + y + u + v,$$

that is, the contradiction

$$\Phi_4(\{\{0\}, \{1\}, \{2\}, \{3\}\}) = \Phi_4(\{\{0\}, \{1\}, \{2, 3\}\}).$$

https://doi.org/10.1017/S1446788700034066 Published online by Cambridge University Press

Similarly, identity (13) yields the contradiction

$$xy + uv = (x + y)uv = ((x + y)u)v = (xy + u)v = xyu + v.$$

Thus, if $(x + y)z$ is essentially ternary, then identity (11′) holds, concluding the proof of Lemma 2.

We now establish identities (5)–(10). We first consider the unary term $f(x) = xx$. Either (8) holds or we have the identity

$$xx = x.$$

But then we have the identity

$$x + y = (x + y)(x + y).$$

If (11) holds, then we get the contradiction

$$x + y = 0.$$

If (11′) holds, then we get the contradiction

$$x + y = xy(x + y) = (x + y)xy = xyxy = xxyy = xy.$$

Thus, identity (8) is established.

Similarly, if (7) does not hold, then we have the identity

$$x + x = x.$$

But then

$$xy = (x + x)y.$$

If (11) holds, we get the contradiction

$$xy = 0.$$

Similarly, if (11′) holds, then, since (8) was established above,

$$xy = xxy = 0y,$$

yielding the contradiction that the term xy does not depend on x. Thus (7) is established.

https://doi.org/10.1017/S1446788700034066 Published online by Cambridge University Press

We now establish (6). If (11) holds, then (6) follows immediately from (7):

$$0x = (x + x)x = 0.$$

On the other hand, if (11′) holds and (6) does not hold, then we must have the identity

$$0x = x.$$

But then, using (11′), we get the contradiction

$$x + y = 0(x + y) = (x + y)0 = xy0 = x(y0) = x(0y) = xy.$$

Thus we have established (6).

We now establish (5). If (5) does not hold, then we have the identity

$$0 + x = x.$$

Then

$$xy = (0 + x)y.$$

If (11) holds, we get the immediate contradiction

$$xy = 0.$$

If (11′) holds, then, by (6), we again get the contradiction

$$xy = x0y = 0.$$

Thus (5) holds.

We now establish (10). If the term $xy + xz$ does not depend on at least one of the variables x, y, z, then, substituting 0 for that variable and using (5) and (6), we get (10). Thus we need only show that the term $xy + xz$ is not essentially ternary. Assume, to the contradictory, that it is. Then, by the symmetry in y and z, $\mathscr{V}$ must satisfy one of the identities

(14) $$xy + xz = x + y + z,$$

(15) $$xy + xz = xyz,$$

(16) $$xy + xz = x + yz.$$

If (14) holds, we get the identity

$$x + y + z + u = x(y + z) + xu.$$

https://doi.org/10.1017/S1446788700034066 Published online by Cambridge University Press

If $\mathscr{V}$ satisfies (11), we then get the contradiction

$$x + y + z + u = 0 + xu = 0.$$

If $\mathscr{V}$ satisfies (11′), we get the contradiction

$$x + y + z + u = xyz + xu = x + yz + u,$$

using (14) once more to get the second equality. Thus (14) does not hold.

If (15) holds, then we get the contradiction

$$xyuv = (xy)u + (xy)v = x(yu) + x(yv) = xyuyv = 0xuv = 0.$$

If (16) holds, then we get the contradiction

$$\begin{aligned} xy + uv = (xy)u + (xy)v &= x(yu) + x(yv) \\ &= x + yuyv = x + 0uv = x + 0 = 0. \end{aligned}$$

Consequently, $xy + xz$ is not essentially ternary, and so must be constant, that is, (10) holds.

Finally, we establish (9). If $x + xy$ is not constant, then $\mathscr{V}$ must satisfy one of the identities

$$\begin{aligned} x + xy &= xy, \\ x + xy &= x + y, \\ x + xy &= x, \\ x + xy &= y. \end{aligned}$$

But then, substituting uv for x and using (10), we get the respective contradictions

$$\begin{aligned} 0 &= uvy, \\ 0 &= uv + y, \\ 0 &= uv, \\ 0 &= y. \end{aligned}$$

Thus (9) holds, concluding the proof of Lemma 1.

https://doi.org/10.1017/S1446788700034066 Published online by Cambridge University Press

3. The free Bell bisemigroups

In this section we prove our main result:

THEOREM. *There are precisely two Bell varieties of bisemigroups, the variety* $\mathscr{B}$ *given by the identities* (1)–(11), *and the variety* $\mathscr{B}'$ *given by the identities* (1)–(10), (11′).

In the process of the proof we shall give a natural representation of the free algebras on $\aleph_0$ generators in these varieties.

We first note the following two lemmas:

LEMMA 3. *Let* $\mathscr{V}$ *be one of the varieties* $\mathscr{B}$, $\mathscr{B}'$, *and let* $f(x_0, \dots, x_{n-1})$ *be a term in which the variable* x_i, $0 \leq i < n$, *appears. Then* $\mathscr{V}$ *satisfies the identity*

$$f(x_0, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{n-1}) = 0.$$

PROOF. The proof follows in a straight-forward manner from identities (5) and (6) by induction on the complexity of the term f.

LEMMA 4. *Let* $\mathscr{V}$ *be one of the varieties* $\mathscr{B}$, $\mathscr{B}'$, *and let* $f(x_0, \dots, x_{n-1})$ *be a term in which all of the variables* $x_0, \dots, x_{n-1}$ *appear. Then either* $\mathscr{V}$ *satisfies the identity*

$$f(x_0, \dots, x_{n-1}) = 0$$

or there is an $\alpha \in Eqv\, \mathbb{N}_n$ *such that* $\mathscr{V}$ *satisfies the identity*

$$f(x_0, \dots, x_{n-1}) = \Phi_n(\alpha).$$

PROOF. The proof follows in a straight-forward manner from each of the sets of identities (1)–(11) and (1)–(10), (11′) by induction on the complexity of the term f. More precisely, we prove the following inductively:

Let $n \geq 1$ *and let* $\varphi : \mathbb{N}_n \to \mathbb{N}$, *the set of natural numbers, be an injection. If* f *is a term involving all of the variables* $x_{\varphi(0)}, \dots, x_{\varphi(n-1)}$ *and no others, then either* $\mathscr{V}$ *satisfies the identity*

$$f = 0$$

or there is an $\alpha \in Eqv\, \mathbb{N}_n$ *such that* $\mathscr{V}$ *satisfies the identity*

$$f = g(x_{\varphi(0)}, \dots, x_{\varphi(n-1)}),$$

where $g = \Phi_n(\alpha)$.

The details are left to the reader. We remark only that in both $\mathscr{B}$ and $\mathscr{B}'$ we have the identities

$$f + g = fg = 0$$

whenever the terms f and g have a variable in common. Thus, in view of identities (5) and (6), if any two subterms of f have a variable in common, we get the identity

$$f = 0.$$

Now let $\mathscr{W}$ be a proper subvariety of $\mathscr{V} = \mathscr{B}$ or $\mathscr{B}'$, and let

$$f = g$$

be an identity satisfied in $\mathscr{W}$ but not in $\mathscr{V}$. Without loss of generality, we may assume that f is not constant in $\mathscr{V}$ and so that $f = f(x_0, \dots, x_{n-1})$ for $n \geq 1$, where all the variables $x_0, \dots, x_{n-1}$ occur in f. Applying Lemma 4 to f, we get an $\alpha \in \operatorname{Eqv} \mathbb{N}_n$ such that $\mathscr{W}$ satisfies the identity

$$\Phi_n(\alpha)(x_0, \dots, x_{n-1}) = g,$$

not satisfied by $\mathscr{V}$. If there is a variable occurring on one side of this identity and not the other, then, substituting 0 for that variable and applying Lemma 3, we see that $\Phi_n(\alpha)$ is constant in $\mathscr{W}$, that is, that $\mathscr{W}$ is not Bell. Otherwise, the variables $x_0, \dots, x_{n-1}$ are precisely the variables occurring in g, and, applying Lemma 4 to g, we have a $\beta \in \operatorname{Eqv} \mathbb{N}_n$ such that the identity

$$\Phi_n(\alpha) = \Phi_n(\beta)$$

holds in $\mathscr{W}$ but not in $\mathscr{V}$. Then $\alpha \neq \beta$, and so Φ_n is not injective in $\mathscr{W}$, that is, again, $\mathscr{W}$ is not Bell. Thus no proper subvariety of $\mathscr{B}$ or $\mathscr{B}'$ is Bell. In view of Lemma 1, the proof of the theorem will be complete if we exhibit algebras in $\mathscr{B}$ and $\mathscr{B}'$ in which, for each $n \geq 1$, the term operations in $\Phi_n(\operatorname{Eqv} \mathbb{N}_n)$ are all essentially n-ary and are all distinct. We proceed to this task.

Let the sets $F = F'$ denote the set of all ordered pairs

$$\langle \alpha, X \rangle$$

where X is a finite subset of $\mathbb{N}$ and $\alpha \in \operatorname{Eqv} X$. We define two algebraic structures $\mathbf{F} = \langle F, +, \cdot \rangle$ and $\mathbf{F}' = \langle F', +, \cdot \rangle$. In both $\mathbf{F}$ and $\mathbf{F}'$ we set

(17) $$\langle \emptyset, \emptyset \rangle + \langle \alpha, X \rangle = \langle \alpha, X \rangle + \langle \emptyset, \emptyset \rangle = \langle \emptyset, \emptyset \rangle$$

and

$$(18)\qquad \langle\emptyset,\emptyset\rangle\langle\alpha, X\rangle = \langle\alpha, X\rangle\langle\emptyset,\emptyset\rangle = \langle\emptyset,\emptyset\rangle$$

for all $\langle\alpha, X\rangle$.

For any other pair $\langle\alpha, X\rangle$, $\langle\beta, Y\rangle$, that is, when $X \neq \emptyset$, $Y \neq \emptyset$, the sum is defined the same way in both algebras:

$$(19)\qquad \langle\alpha, X\rangle + \langle\beta, Y\rangle = \begin{cases} \langle\alpha\cup\beta, X\cup Y\rangle & \text{if } X\cap Y = \emptyset, \\ \langle\emptyset,\emptyset\rangle & \text{if } X\cap Y \neq \emptyset, \end{cases}$$

where $\alpha\cup\beta$ is the ordinary set union, and has the same effect whether we regard α and β as equivalence relations, that is, subsets of X^2, Y^2, respectively, or as partitions, that is, sets of subsets of X, Y, respectively; since the sets X and Y are disjoint, $\alpha\cup\beta \in \operatorname{Eqv}(X\cup Y)$.

The product is defined differently for $\mathbf{F}$ and $\mathbf{F}'$. If $X \neq \emptyset$, $Y \neq \emptyset$, then in $\mathbf{F}$ we set

$$(20)\,\langle\alpha, X\rangle\langle\beta, Y\rangle = \begin{cases} \langle\iota_{X\cup Y}, X\cup Y & \text{if } X\cap Y = \emptyset \text{ and } \alpha = \iota_X, \beta = \iota_Y, \\ \langle\emptyset,\emptyset\rangle & \text{otherwise,} \end{cases}$$

where, for each set X, ι_X (or simply ι when the context is clear) denotes the equivalence relation X^2, equivalently, the partition with only one block.

For $\mathbf{F}'$ we set

$$(21)\qquad \langle\alpha, X\rangle\langle\beta, Y\rangle = \begin{cases} \langle\iota_{X\cup Y}, X\cup Y\rangle & \text{if } X\cap Y = \emptyset, \\ \langle\emptyset,\emptyset\rangle & \text{if } X\cap Y \neq \emptyset, \end{cases}$$

whenever $X \neq \emptyset$ and $Y \neq \emptyset$. Note that in both $\mathbf{F}$ and $\mathbf{F}'$ we have

$$\langle\emptyset,\emptyset\rangle = 0.$$

LEMMA 5. *$\mathbf{F} \in \mathscr{B}$ and $\mathbf{F}' \in \mathscr{B}'$.*

PROOF. We need only show that $\mathbf{F}$ satisfies the identities (1)–(11) and that $\mathbf{F}'$ satisfies the identities (1)–(10) and (11′). This is safely left to the reader. However, in order to preserve a modicum of honesty and at the dire risk of boring the reader, we present the verification of (4) for both $\mathbf{F}$ and $\mathbf{F}'$, along with the verification of (11) for $\mathbf{F}$ and (11′) for $\mathbf{F}'$.

We first verify (4). In both $\mathbf{F}$ and $\mathbf{F}'$ both sides are $0 = \langle\emptyset,\emptyset\rangle$ if at least one of x, y, or z is 0. Otherwise, we may set $x = \langle\alpha, X\rangle$, $y = \langle\beta, Y\rangle$, $z = \langle\gamma, Z\rangle$

https://doi.org/10.1017/S1446788700034066 Published online by Cambridge University Press

for nonempty X, Y, Z. Then, for both **F** and **F**′ the left hand side of (4) is 0 unless $X \cap Y \neq \emptyset$ and $(X \cup Y) \cap Z \neq \emptyset$, that is, unless

$$X \cap Y \neq \emptyset, X \cap Z \neq \emptyset, \text{ and } Y \cap Z \neq \emptyset.$$

From (20) it follows that in **F**

$$(xy)z = \begin{cases} \langle \iota_{X \cup Y \cup Z}, X \cup Y \cup Z \rangle & \text{if } X \cap Y = X \cap Z = Y \cap Z = \emptyset \\ & \text{and } \alpha = \iota_X, \beta = \iota_Y, \gamma = \iota_Z, \\ 0 & \text{otherwise,} \end{cases}$$

which is exactly the computation for $x(yz)$.

Similarly, using (21), we compute in **F**′ that

$$(xy)z = \begin{cases} \langle \iota_{X \cup Y \cup Z}, X \cup Y \cup Z \rangle & \text{if } X \cap Y = X \cap Z = Y \cap Z = \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

which is exactly the computation for $x(yz)$.

This verifies (4) in **F** and in **F**′.

We now verify (11) for **F**. Again, we set $x = \langle \alpha, X \rangle$, $y = \langle \beta, Y \rangle$, $z = \langle \gamma, Z \rangle$. If one of X, Y is $\emptyset$, then (11) follows immediately from (17) and (18). Otherwise, since $\alpha \cup \beta$ can then not be $\iota_{X \cup Y}$, we again get

$$(x + y)z = 0,$$

concluding the verification of (11) for **F**.

To verify (11′) for **F**′ we note both sides are $\langle \emptyset, \emptyset \rangle$ unless X, Y, Z are all nonempty and $X \cap Y = X \cap Z = Y \cap Z = \emptyset$, in which case

$$x + y = \langle \alpha \cup \beta, X \cup Y \rangle$$

and so

$$(x + y)z = \langle \iota_{X \cup Y \cup Z}, X \cup Y \cup Z \rangle,$$

which is precisely xyz, as derived above.

This concludes Lemma 5.

Now let X be an n-element subset of $\mathbb{N}$ and let $\varphi: \mathbb{N}_n \to X$ be a bijection. For each $\alpha \in \operatorname{Eqv} X$ we denote by $\varphi^*\alpha$ the corresponding partition on $\mathbb{N}_n$,

$$\varphi^*\alpha = \{ \langle x, y \rangle \mid \langle \varphi(x), \varphi(y) \rangle \in \alpha \}.$$

The following lemma is then immediate from the definitions:

LEMMA 6. *Let X be an n-element subset of $\mathbb{N}$, let $\varphi\colon \mathbb{N}_n \to X$ be a bijection, and let $\alpha \in Eqv\, X$. Then, in both the algebras* $\mathbf{F}$ *and* $\mathbf{F}'$,

$$\langle \alpha, X\rangle = \Phi_n(\varphi^*\alpha)(\langle \iota, \{\varphi(0)\}\rangle, \dots, \langle \iota, \{\varphi(n-1)\}\rangle).$$

We can now conclude the proof of the theorem. Indeed, it follows easily from Lemma 6 that $\mathbf{F}$ and $\mathbf{F}'$ are Bell bisemigroups. Let $n \geq 1$ and let $\alpha \neq \beta$ be partitions of $\mathbb{N}_n$. Then, applying Lemma 6 with φ the identity mapping, we get

$$\begin{aligned}\Phi_n(\alpha)(\langle \iota, \{0\}\rangle, \dots, \langle \iota, \{n-1\}\rangle) &= \langle \alpha, \mathbb{N}_n\rangle \\ &\neq \langle \beta, \mathbb{N}_n\rangle \\ &= \Phi_n(\beta)(\langle \iota, \{0\}\rangle, \dots, \langle \iota, \{n-1\}\rangle),\end{aligned}$$

that is, Φ_n is injective. The variables that occur in $\Phi_n(\alpha)$ are precisely $x_0, \dots, x_{n-1}$. If $\Phi_n(\alpha)$ did not depend on one of these variables, then, substituting $0 = \langle \emptyset, \emptyset\rangle$ for that variable and appealing to Lemma 3, we would derive the contradiction

$$\langle \emptyset, \emptyset\rangle = \langle \alpha, \mathbb{N}_n\rangle.$$

Thus the image of Φ_n is a subset of the essentially n-ary term functions of $\mathbf{F}$, respectively of $\mathbf{F}'$. That the image is precisely the set of essentially n-ary term functions follows immediately from Lemma 4. Consequently, $\mathbf{F}$ and $\mathbf{F}'$ are Bell bisemigroups, concluding the proof of the theorem.

For the proof of the Theorem it would suffice to state Lemma 6 with $X = \mathbb{N}_n$ and φ the identity map. We chose the more general statement because it then follows immediately that $\mathbf{F}$ and $\mathbf{F}'$ are the free algebras generated by the countable set

$$\{\langle \iota, \{n\}\rangle \mid n \in \mathbb{N}\}$$

in $\mathscr{B}$ and $\mathscr{B}'$, respectively.

4. Concluding remarks

In terms of the concept of p_n *sequence*, [2], we have presented a representation of the sequence with

$$p_n = B(n), \qquad n \neq 1,$$

and

$$p_1 = B(1) - 1.$$

https://doi.org/10.1017/S1446788700034066 Published online by Cambridge University Press

(The condition on p_1 is quite natural, since in any nontrivial algebra the term x_0 is always essentially unary—p_1 was defined, somewhat artificially, as 1 less than the number of essentially unary terms in order that, a priori, every sequence of natural numbers be possible.) What is new here is the *naturality* of the representation; since $p_0 \neq 0$, the general theory of p_n sequences trivially yields (nonnatural) representations of this sequence. Essentially, one introduces as many fundamental operations as one needs and then cuts down the number of essentially n-ary terms by identifying any extra terms to a constant. The variety $\mathscr{B}$ is very much in this spirit, but $\mathscr{B}'$ is not.

A more interesting problem, and indeed not yet solved in general, is that of representing sequences with $p_0 = p_1 = 0$, that is, characterizing the p_n sequences of idempotent algebras with no constants. In the spirit of this paper, since $B(1) = 1$, we can consider the problem of presenting the sequence

$$\langle 0, 0, 2, \ldots, B(n), \ldots \rangle,$$

that is, modifying our results so that we have no constants. However, general results in the literature show that this is impossible. Indeed, let us assume that the algebra $\mathbf{A}$ represents the above sequence. Since $B(2) = 2$, $\mathbf{A}$ has exactly two essentially binary term functions. If one of them is commutative, then so is the other, that is, $\mathbf{A}$ has two commutative binary term operations. But then, by a result of Dudek [1],

$$p_3(\mathbf{A}) \geq 9 > 5 = B(3).$$

Thus, $\mathbf{A}$ has no binary commutative term functions. Then, by Kisielewicz [3, Theorem 4.1], there are natural numbers a_1, a_2 with

$$p_4(\mathbf{A}) = a_1\binom{4}{1} + a_2\binom{4}{3};$$

consequently, p_4 is divisible by 4, and so cannot be $B(4) = 15$. Thus the sequence is not representable.

References

[1] J. Dudek, 'A characterization of distributive lattices', in: *Contributions to Lattice Theory (Szeged 1980)*, volume 33 of *Colloq. Math. Soc. János Bolyai* (North-Holland, Amsterdam, 1983) pp. 325–335.

[2] G. Grätzer, 'Composition of functions', in: *Proceedings of the Conference on Universal Algebra, October 1969* (ed. G. H. Wenzel), Queen's Papers in Pure and Applied Mathematics (Queen's University, Kingston, 1970) pp. 1–106.

[3] A. Kisielewicz, 'Characterization of p_n-sequences for nonidempotent algebras', *J. of Algebra* **108** (1987), 102–115.

[4] R. N. McKenzie, G. F. McNulty and W. F. Taylor, in: *Algebras, Lattices, Varieties Volume* I, The Wadsworth & Brooks/Cole Mathematics Series (Wadsworth & Brooks/Cole, Monterey, 1987).

[5] R. P. Stanley, in: *Enumerative Combinatorics Volume* I, The Wadsworth & Brooks/Cole Mathematics Series (Wadsworth & Brooks/Cole, Monterey, 1986).

Department of Mathematics and Astronomy
University of Manitoba
Winnipeg, Canada R3T 2N2