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Rigidity of Hamiltonian Actions

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Abstract. This paper studies the following question: Given an ω'-symplectic action of a Lie group on a manifold M which coincides, as a smooth action, with a Hamiltonian ω-action, when is this action a Hamiltonian ω'-action? Using a result of Morse-Bott theory presented in Section 2, we show in Section 3 of this paper that such an action is in fact a Hamiltonian ω'-action, provided that M is compact and that the Lie group is compact and connected. This result was first proved by Lalonde-McDuff-Polterovich in 1999 as a consequence of a more general theory that made use of hard geometric analysis. In this paper, we prove it using classical methods only.

1 Introduction

In [3], Lalonde-McDuff gave a classical proof that an ω'-symplectic action of a compact Lie group on M which coincides, as a smooth action, with a Hamiltonian ω-action, must be Hamiltonian too, provided that ω' is sufficiently close to ω. They used that proximity condition to ensure that the 1-form corresponding to the ω'-action is non-degenerate (and has same indices as the 1-form corresponding to the ω-action). In this note, we get rid of that condition. Indeed, we will show the following simple proposition: if S^1 acts in a Hamiltonian way on (M,ω), then $H_1(M,C_0)$ vanishes, where C_0 is the (connected) submanifold of M which consists of all the points of Morse index 0 for the Morse-Bott function H corresponding to that action. This follows from the fact that the gradient flow of H collapses to C_0 any loop in M, after a small perturbation that disjoins it from all stable manifolds of codimension greater then one. Indeed, this holds because all critical submanifolds have even indices and therefore the stable manifolds corresponding to non-minimal critical submanifolds must have codimension at least two. This implies that $H^1_{DR}(M,C_0)$ vanishes. But any S^1-ω'-symplectic action that has the same dual vector field as the one corresponding to the ω-Hamiltonian action must have the same critical set and therefore must vanish on C_0. Because $H^1_{DR}(M,C_0)$ vanishes, it must be exact too. We can extend this from S^1 to a compact connected Lie group by elementary methods, and refine the argument so that the result be also true under the hypothesis that the two actions are only homotopic (instead of being the same). Note that this theorem implies that Hamiltonian actions are stable under small deformations of the symplectic form. See [3] for the proof. Here is the main theorem of this note:

Theorem 1.1 *Let $\Psi_t\colon G \to \mathrm{Diff}(M)$, $t \in [0,1]$, be a smooth family of actions of a compact connected Lie group G on a compact manifold M such that Ψ_0 is a Hamiltonian*

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action on (M, ω) *and* Ψ_1 *is a symplectic action on* (M, ω')*, where* ω *and* ω' *are two symplectic forms on* M*. Then,* Ψ_1 *is also a Hamiltonian action on* (M, ω')*.*

I would like to thank François Lalonde, my Master's adviser, who suggested that problem to me and gave me possible approaches and ideas how to start.

2 Morse-Bott Functions

For the sake of completeness, we will first quickly review the basic elements of Morse-Bott theory. In the following section, M will denote a connected compact Riemannian manifold.

Definition 2.1 A smooth function $f \colon M \to \mathbb{R}$ is said to be *Morse-Bott* if its set of critical points $\operatorname{Crit}(f) = \{x \in M \mid df(x) = 0\}$ is a submanifold of M such that:

$$T_x \operatorname{Crit}(f) = \ker \nabla^2 f(x) \quad \forall x \in \operatorname{Crit}(f) \tag{1}$$

where the linear map $\nabla^2 f \colon T_xM \to T_xM$ is obtained from the Hessian

$$\operatorname{Hess}(f)_x \colon T_xM \times T_xM \to \mathbb{R}$$

via some metric g on M.

In particular, any Morse function is also a Morse-Bott function. In fact, when f is a Morse-Bott function, $\operatorname{Crit}(f)$ splits into a finite number of connected submanifolds which we call *critical manifolds*. Using the fact that$\nabla^2 f(x)$ is self-adjoint with respect to the metric g, it is possible to define on each critical manifold an index which generalizes the Morse index.

Definition 2.2 Let $C \subset \operatorname{Crit}(f)$ be a (connected) critical submanifold of M for some Morse-Bott function f. The *index* of C, noted $\operatorname{Ind}(C)$, is the dimension of the greatest subspace of T_xM on which $\nabla^2 f(x)$ is negative definite, where $x \in C$. The condition (1) ensures that this definition does not depend on the choice of x.

We can now generalize the Morse Lemma (Lemma 2.2 in [5]) in the following way. Although it is a well-known result, the proof of it does not seem to appear in the literature. Therefore, to be complete, we will give a proof which is basically the proof in [5] for the Morse case with slight modifications to cover the Morse-Bott case.

Lemma 2.3 *If* C *is a connected critical manifold of a Morse-Bott function* f*, then, for each* $x \in C$*, there exists a coordinate system* $(y_1, \dots, y_n)$ *in a neighborhood* $\mathcal{U}$ *of* x *such that:*

- $y_i(x) = 0 \, \forall i \in \{1, \dots, n\}$
- $p \in \mathcal{U} \cap C \Leftrightarrow y_i(p) = 0 \, \forall i \leq k$
- $f(y_1, \dots, y_n) = f(x) - y_1^2 - \dots - y_\lambda^2 + y_{\lambda+1}^2 + \dots + y_k^2$ *in* $\mathcal{U}$

where $k = \operatorname{codim}(C)$*,* $\lambda = \operatorname{Ind}(C)$ *and* $n = \dim(M)$*.*

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Proof It is clear that if such a coordinate system exists, we must have $\lambda = \operatorname{Ind}(C)$ and $k = \operatorname{codim}(C)$. To show the existence, first note that we can suppose without loss of generality that $f(x) = 0$. Moreover, by assumption, C is a submanifold of M, so there exists a coordinate system $(u_1, \dots, u_n)$ in a neighborhood $\mathcal{U}$ of x such that:

- $u_i(x) = 0 \, \forall i \in \{1, \dots, n\}$
- $p \in \mathcal{U} \cap C \Leftrightarrow u_i(p) = 0 \, \forall i \leq k$.

To obtain the third property, we write f as:

$$f(u_1, \dots, u_n) = \sum_{i=1}^{k} u_i g_i(u_1, \dots, u_n) \quad \text{where } g_i \in C^\infty(\mathcal{U}) \quad \forall i \leq k.$$

For instance, we can take $g_i = \int_0^1 \frac{\partial f}{\partial u_i}(tu_1, \dots, tu_k, u_{k+1}, \dots, u_n)\, dt$ since:

$$\begin{aligned} f(u_1, \dots, u_n) &= \int_0^1 \frac{df}{dt}(tu_1, \dots, tu_k, u_{k+1}, \dots, u_n)\, dt \quad \text{since } f|_{\mathcal{U} \cap C} = 0 \\ &= \sum_{i=1}^{k} u_i \int_0^1 \frac{\partial f}{\partial u_i}(tu_1, \dots, tu_k, u_{k+1}, \dots, u_n)\, dt. \end{aligned}$$

Note that $g_i|_{\mathcal{U}\cap C} = \frac{\partial f}{\partial u_i}|_{\mathcal{U}\cap C} = 0$, so we can use the same trick with the functions g_i, that is, we can write them as:

$$g_i(u_1, \dots, u_n) = \sum_{j=1}^{k} u_j h_{ji}(u_1, \dots, u_n) \quad \text{where } h_{ji} \in C^\infty(\mathcal{U}) \quad \forall i, j \leq k.$$

So that we can finally write f as:

$$f(u_1, \dots, u_n) = \sum_{i,j=1}^{k} u_i u_j h_{ij}(u_1, \dots, u_n).$$

We can assume without loss of generality that $h_{ij} = h_{ji}$, because if it is not the case, we can take $h'_{ij} = \frac{h_{ij} + h_{ji}}{2}$ so that $f = \sum_{i,j=1}^{k} u_i u_j h'_{ij}$. We then apply a standard argument for the diagonalization of quadratic forms. We proceed by induction. We suppose that there exists a coordinate system $(v_1, \dots, v_n)$ in a neighborhood $\mathcal{U}$ of x with $v_i = u_i$ for all $i > k$ and such that:

$$f = \pm v_1^2 \pm \cdots \pm v_{r-1}^2 + \sum_{i,j=r}^{k} v_i v_j H_{ij}(v_1, \dots, v_n) \quad \text{where } H_{ij} = H_{ji} \in C^\infty(M) \quad \forall i, j.$$

The condition (1) ensures us that the matrix $\|H_{ij}(x)\|$ is not identically zero. Thus, after a linear transformation of the coordinates $v_r, \dots, v_k$, we can always assume that

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$H_{rr}(x) \neq 0$. Set $g = \sqrt{|H_{rr}|}$. This is a smooth non-zero function in $\mathcal{U}$, taking $\mathcal{U}$ smaller if necessary. This allows us to introduce new coordinates in $\mathcal{U}$:

$$w_i = v_i \quad \text{if } i \neq r$$

$$w_r = g\Big(v_r + \sum_{i=r+1}^{k} \frac{v_i H_{ir}}{H_{rr}} \Big).$$

At x, the absolute value of the determinant of the Jacobian associated to this change of coordinates is clearly $g(x) \neq 0$, hence, by the inverse functions theorem, taking $\mathcal{U}$ smaller if necessary, we know that $(w_1, \ldots, w_n)$ is a coordinate system in $\mathcal{U}$. But in this coordinate system:

$$f = \sum_{i=1}^{r} \pm w_i^2 - \frac{1}{H_{rr}} \Big(\sum_{i=r+1}^{k} w_i H_{ir} \Big)^2 + \sum_{i,j=r+1}^{k} w_i w_j H_{ij} \quad \text{taking } \pm w_r^2 = \frac{H_{rr} w_r^2}{|H_{rr}|}$$

$$= \sum_{i=1}^{r} \pm w_i^2 + \sum_{i,j=r+1}^{k} w_i w_j H'_{ij} \quad \text{where } H'_{ij} = -\frac{H_{ir} H_{jr}}{H_{rr}} + H_{ij}.$$

This concludes the inductive step. Applying this step at most k times to the coordinate system$(u_1, \ldots, u_n)$, we obtain the required coordinate system. ■

Definition 2.4 The *negative gradient flow* $\phi_t \colon M \to M$ of a Morse-Bott function f is defined by the equation:

$$\frac{d\phi_t}{dt} = -\nabla f \circ \phi_t, \quad \phi_0 = \text{id}\,. \tag{2}$$

This equation has a unique solution for all $t \in \mathbb{R}$ since M is compact. Of course, this definition of the negative gradient flow depends on the choice of a metric g on M.

The negative gradient flow allows us to define stable and unstable manifolds as follows.

Definition 2.5 If C is a critical manifold of a Morse-Bott function f, then its associate *stable manifold* $W^s(C)$ and *unstable manifold* $W^u(C)$ are:

$$W^s(C) = \{x \in M \mid \lim_{t\to+\infty} \phi_t(x) \in C\}$$

$$W^u(C) = \{x \in M \mid \lim_{t\to-\infty} \phi_t(x) \in C\}.$$

An important result about $W^s(C)$ and $W^u(C)$ is the following:

Lemma 2.6 *$W^s(C)$ and $W^u(C)$ are submanifolds of M.*

Proof By standard results on local integration of flows (see [2]), for any $x \in C$, where C is some critical manifold, there exists a neighborhood $\mathcal{U}$ of x such that $W^s(C) \cap \mathcal{U}$ and $W^u(C) \cap \mathcal{U}$ are submanifolds. Using the negative gradient flow ϕ_t, we can conclude that $W^s(C)$ and $W^u(C)$ are submanifolds. ■

An easy consequence of Definition 2.5 is the following:

Lemma 2.7 *For $x \in C$, $T_xM = T_xC \oplus E_x^+ \oplus E_x^-$, $T_xW^s(C) = T_xC \oplus E_x^+$ and $T_xW^u(C) = T_xC \oplus E_x^-$, where E_x^+ and E_x^- are respectively the positive and negative eigenspaces of $\nabla^2 f(x)$. In particular,* $\operatorname{codim}\big(W^s(C)\big) = \dim(E_x^-) = \operatorname{Ind}(C)$.

Remark 2.8 In the case where $\operatorname{Ind}(C) = 0$, $W^s(C)$ is an open set. Indeed, for $x \in W^s(C)$ sufficiently close to C, it is clear using Lemma 2.7 that there exists an open neighborhood $\mathcal{U}$ of x which is contained in $W^s(C)$. For an arbitrary $x \in W^s(C)$, we take $t > 0$ such that $\phi_t(x)$ is sufficiently close to $W^s(C)$, that is, such that there exists an open neighborhood $\mathcal{U}$ of $\phi_t(x)$ which is contained in $W^s(C)$. Then, $\phi_{-t}(\mathcal{U})$ is an open neighborhood of x which is contained in $W^s(C)$. Thus, for any x in $W^s(C)$, there exists open neighborhood $\mathcal{U}$ of x which is contained in $W^s(C)$, that is, $W^s(C)$ is open.

It is a well-known result that:

Lemma 2.9 *For all $x \in M$, $\phi_t(x)$ must converge to some critical manifold C as t tends to $\pm\infty$.*

What Lemma 2.9 says is in fact that:

$$M = \bigcup_C W^s(C) = \bigcup_C W^u(C).$$

When the Morse-Bott function f has no critical manifold of index 1, the topology of M has some interesting properties. We state them in the next proposition which is the main result of this section.

Proposition 2.10 *Let M be a compact connected Riemannian manifold and f a Morse-Bott function with no critical manifold of index 1. Then, there is only one connected critical manifold C_0 of index 0 and $H_1(M, C_0) = 0$.*

Proof By Lemma 2.7, first note that for any critical manifold C,

$$\operatorname{codim} W^s(C) = \operatorname{Ind}(C).$$

Since M is compact, the function f must have a minimum, so there exists at least one critical manifold of index 0. Hence, if C_0 denotes the union of all critical manifolds of f of index 0, we see by Lemma 2.9 that the complement of $W^s(C_0)$ is a finite union of

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submanifolds $W^s(C)$ with $\mathrm{Ind}(C) \neq 0$ of codimension greater than one. Therefore, $W^s(C_0)$ must be connected since M is connected. Now, if

$$C_0 = C_0^1 \cup C_0^2 \quad \text{with} \quad C_0^1 \cap C_0^2 = \varnothing, \quad C_0^1 \neq \varnothing \quad \text{and} \quad C_0^2 \neq \varnothing$$

where C_0^1 and C_0^2 are disconnected critical manifolds, then we have

$$W^s(C_0) = W^s(C_0^1) \cup W^s(C_0^2) \quad \text{with} \quad W^s(C_0^1) \cap W^s(C_0^2) = \varnothing,$$

contradicting the fact that $W^s(C_0)$ is connected. Consequently, C_0 must be connected.

For the second statement, consider a closed continuous path $\beta\colon S^1 \to M$. By transversality, there exists a path β' homotopic to β which is transversal to $\bigcup_{C\neq C_0} W^s(C)$. But since $\mathrm{codim}\big(W^s(C)\big) = \mathrm{Ind}(C) > 1$ for $C \neq C_0$, this means that β' is outside $\bigcup_{C\neq C_0} W^s(C)$. Thus, by Lemma 2.9, this means that β' lies in $W^s(C_0)$. Hence:

$$\forall s \in S^1, \quad \lim_{t\to\infty} \phi_t\big(\beta'(s)\big) \in C_0.$$

Using this fact, we can naturally construct a homotopy $\tau_t\colon S^1 \to M$ with $t \in [0,1]$ defined by:

$$\tau_t(s) = \begin{cases} \phi_{\tan(t\frac{\pi}{2})} \circ \beta'(s) & \text{if } t \in [0,1) \\ \lim_{\theta\to 1^-} \phi_{\tan(\theta\frac{\pi}{2})} \circ \beta'(s) & \text{if } t = 1. \end{cases}$$

One can show using Remark 2.8 that τ_t is well-defined, that is, $(t,s) \mapsto \tau_t(s)$ is continuous. By construction, τ_0 is β' and τ_1 is a continuous path homotopic to β' which lies in C_0. Hence, $H_1(M, C_0) = 0$. ■

It is also possible to prove that $H_1(M, C_0) = 0$ without using Lemma 2.6:

Second Proof We suppose by contradiction that $H_1(M, C_0) \neq 0$. Then, there exists a continuous path $\gamma\colon S^1 \to M$ such that $[\gamma] \neq 0$ in $H_1(M, C_0)$. Without loss of generality, we can assume $\gamma(0) \in C_0$ and $f|_{C_0} \equiv 0$, where S^1 is considered as $[0,1]$ with 0 and 1 identified. We let Γ denote the homotopy class of γ and we define:

$$\alpha = \inf\{\sup_{t\in S^1} f \circ \beta(t) \mid \beta \in \Gamma\}.$$

Clearly, $\alpha \geq 0$, since the minimum of f is 0. In fact, $\alpha > 0$. To see this, we consider a small open neighborhood $\mathcal{U}$ of C_0 such that $\overline{\mathcal{U}}$ is homotopic to C_0 and each point of $\overline{\mathcal{U}}$ is contained in a neighborhood of the type described in Lemma 2.3. Hence, every path of Γ must go out of $\overline{\mathcal{U}}$, so at least one of its points belongs to $\partial\mathcal{U}$. Now, by construction, f is strictly positive on $\partial\mathcal{U}$ and since $\partial\mathcal{U}$ is clearly compact, the infimum m of f on $\partial\mathcal{U}$ is strictly positive. In particular, we must have $0 < m \leq \alpha$, thus α is strictly positive as we were claiming.

By definition of α, there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of continuous paths in Γ such that:

$$\lim_{n\to\infty} \sup_{t\in S^1}\big(f \circ \gamma_n(t)\big) = \alpha.$$

Let us see that we can assume without loss of generality that all these paths lie outside an open neighborhood Ω of:

$$\bigcup_{C \neq C_0} C$$

where C ranges over the critical manifolds of f. To prove this, take $\beta \in \Gamma$ arbitrary and fix a critical manifold C, $C \neq C_0$. For each point $x \in C$, there exists a neighborhood $\mathcal{V}_x$ of x with coordinates $(y_1, \ldots, y_n)$ as in Lemma 2.3. Moreover, taking $\mathcal{V}_x$ small enough, we can assume that $\mathcal{V}_x \cap C' = \varnothing$ for any critical manifold C' other than C. With respect to the coordinates $(y_1, \ldots, y_n)$, we can define two neighborhoods $\mathcal{W}_x$ and $\mathcal{W}'_x$ of x with $\mathcal{W}_x \subset \mathcal{W}'_x \subset \mathcal{V}_x$ in the following way:

$$\mathcal{W}_x = \{(y_1, \ldots, y_n) \in \mathcal{V}_x \mid y_1^2 + \cdots + y_\lambda^2 < r_x^2 \text{ and } y_{\lambda+1}^2 + \cdots + y_n^2 < l_x^2\}$$
$$\mathcal{W}'_x = \{(y_1, \ldots, y_n) \in \mathcal{V}_x \mid y_1^2 + \cdots + y_\lambda^2 < r_x^2 \text{ and } y_{\lambda+1}^2 + \cdots + y_n^2 < (l'_x)^2\}$$

where $\frac{r_x}{2} > l'_x > l_x > 0$ must be chosen small enough. Then,

$$\mathcal{W} = \bigcup_{x \in C} \mathcal{W}_x$$

is an open covering of C. Since C is clearly compact, we can extract a finite subcover, say $\{\mathcal{W}_1, \ldots, \mathcal{W}_m\}$.

Now, if the closed path β passes through $\mathcal{W}'_i$, in terms of the coordinates, we have:

$$\beta(t) = \big(y_1(t), \ldots, y_n(t)\big)$$

for $t \in S^1$ such that $\beta(t) \in \mathcal{W}'_i$. Clearly, we can modify homotopically this path in $\mathcal{W}'_i$ such that it does not pass through

$$\mathcal{V}_0 = \{(y_1, \ldots, y_n) \in \mathcal{W}'_i \mid y_1 = \cdots = y_\lambda = 0\}$$

only using the coordinates $y_1, \ldots, y_\lambda$, thus, in such a way that we do not increase the value of f along the path. This is possible since $\lambda = \operatorname{Ind}(C) > 1$. Then, we can pull β out of $\mathcal{W}_i$ using the homotopy $\tau_t \colon \overline{\mathcal{W}'_i} \setminus \mathcal{V}_0 \to \overline{\mathcal{W}'_i} \setminus \mathcal{V}_0$ defined for $t \in [0, 1]$ by:

$$\tau_t(y) = \tau_t(y_u, y_s)$$
$$= \begin{cases} \Big(\frac{r_i y_u}{(1-t)r_i + t\|y_u\|}, y_s\Big) & \text{if } y \in W_i \\ \Big(\frac{y_u}{(1-t) + t\frac{(l'_i - l_i)\|y_u\|}{r_i(l'_i - \|y_s\|)}}, y_s\Big) & \text{if } l'_i > \|y_s\| \geq l_i \text{ and } \|y_u\| \leq \frac{r_i(l'_i - \|y_s\|)}{l'_i - l_i} \\ (y_u, y_s) & \text{otherwise} \end{cases}$$

where $y_u = (y_1, \ldots, y_\lambda)$ and $y_s = (y_{\lambda+1}, \ldots, y_n)$. Obviously, τ_0 is the identity and the image of τ_1 lies outside $\mathcal{W}_i$. Morever, it is not hard to see that we can extend this homotopy to all $M \setminus \mathcal{V}_0$ by setting τ_t to be the identity outside $\mathcal{W}'_i$. Then $\tau_1 \circ \beta$ is a continuous path homotopic to β which lies outside $\mathcal{W}_i$. Moreover, we can convince ourselves:

$$\forall t \in S^1, \quad f \circ \tau_1 \circ \beta(t) \le f \circ \beta(t).$$

Doing this modification successively for $i = 1, \dots, m$, we finally obtain a continuous path β' homotopic to β, but this new path is not necessarily outside

$$\bigcup_{i=1}^{m} \mathcal{W}_i.$$

However, the condition $r_i > 2l_i'$ ensures us that β' lies outside the open neighborhood

$$\Big\{x \in M \mid f(x) > f(x_c) - \frac{3r^2}{4}\Big\} \cap \Big(\bigcup_{i=1}^{m} \mathcal{W}_i\Big)$$

of C, where $x_C \in C$ and $r = \min\{r_1, \dots, r_m\}$. The main point is that:

$$\forall t \in S^1, \quad f \circ \beta'(t) \le f \circ \beta(t).$$

Now, if C' is another critical manifold, using the same technique, we can find in $M \setminus (\mathcal{W}_1 \cup \cdots \cup \mathcal{W}_m)$ an open neighborhood $\mathcal{V}$ of C' such that we can pull β out of $\mathcal{V}$ without increasing the value of f on β. We can continue this process again and again until we have been through all the critical manifolds of f other than C_0. At the end, we finally obtain that there exists an open neighborhood Ω of all the critical manifolds other than C_0 with the property that we can deform β homotopically to a path β'' which lies outside Ω and in such a way that:

$$\forall t \in S^1, \quad f \circ \beta''(t) \le f \circ \beta(t).$$

Since $\beta \in \Gamma$ is arbitrary, this shows that we can assume without loss of generality that all the paths γ_n lie outside an open neighborhood Ω of $\bigcup_{C \neq C_0} C$. Considering again the homotopic neighborhood $\mathcal{U}$ of C_0 defined above, we then see that each path reaches its supremum in the closed subset $\mathcal{W} = M \setminus (\Omega \cup \mathcal{U})$. We can let the negative gradient flow ϕ_t acts on M, so that we can define the following continuous function on M:

$$\forall x \in M, h(x) = f(x) - f \circ \phi_1(x).$$

By construction, the function h is strictly positive in $\mathcal{W}$ since there is no critical manifold in $\mathcal{W}$. Hence, since $\mathcal{W}$ is compact, there exists $\epsilon > 0$ such that $h(w) \ge \epsilon$ for all $w \in \mathcal{W}$. Since ϕ_1 is diffeotopic to the identity diffeomorphism ϕ_0, $\{\phi_1 \circ \gamma_n\}_{n \in \mathbb{N}}$ will be a sequence of paths in Γ with the property that:

$$\sup_{t \in S^1}(f \circ \phi_1 \circ \gamma_n) \le \sup_{t \in S^1}(f \circ \gamma_n) - \epsilon \quad \forall n \in \mathbb{N}.$$

However, by definition of the limit, there exists $n_0 \in \mathbb{N}$ such that:

$$\sup_{t \in S^1}(f \circ \gamma_{n_0}) < \alpha + \frac{\epsilon}{2}.$$

Hence,

$$\sup_{t\in S^1}\big(f\circ\phi_1\circ\gamma_{n_0}(t)\big)\leq\sup_{t\in S^1}\big(f\circ\gamma_{n_0}(t)\big)-\epsilon<\alpha-\frac{\epsilon}{2}<\alpha.$$

This contradicts the defintion of α. To avoid a contradiction, we must admit that $H_1(M, C_0) = 0$. ■

Remark 2.11 The proof of Proposition 2.10 establishes actually a stronger result, namely that $\pi_1(M/C_0)$ vanishes.

3 Rigidity of Hamiltonian Actions

Let G be a compact connected Lie group acting on a symplectic manifold (M,ω) by symplectomorphisms. Precisely, we consider a symplectic action:

$$\Psi\colon G\to\mathrm{Symp}(M,\omega)$$
$$g\mapsto\psi_g$$

such that $\psi_g\circ\psi_h=\psi_{gh}$ for all $g,h\in G$ and $\psi_e=\mathrm{id}$. If $\mathfrak{g}$ denote the Lie algebra of G, then:

$$d\Psi\colon\mathfrak{g}\to\mathfrak{X}(M)$$
$$\xi\mapsto X_\xi$$

is the differential of Ψ. Precisely, for $\xi\in\mathfrak{g}$, X_ξ is given by:

$$X_\xi=\frac{d}{dt}\psi_{\exp(t\xi)}\Big|_{t=0}.$$

Since the action of G on (M,ω) is symplectic, $\mathcal{L}_{X_\xi}\omega=0$ for all $\xi\in\mathfrak{g}$. As a consequence, if we define the pairing:

$$\Phi\colon\mathfrak{X}(M)\to\Omega^1(M)$$
$$X\mapsto\iota(X)\omega$$

then $\Phi(X_\xi)$ must be a closed form for all $\xi\in\mathfrak{g}$, since:

$$\begin{aligned}d\Phi(X_\xi)&=d\iota(X_\xi)\omega\\&=\mathcal{L}_{X_\xi}\omega-\iota(X_\xi)d\omega\quad\text{since }\mathcal{L}_{X_\xi}=d\iota(X_\xi)+\iota(X_\xi)d\\&=0\quad\text{since the action is symplectic and }\omega\text{ is closed.}\end{aligned}$$

Definition 3.1 We cannot expect in general that $\Phi(X_\xi)$ is exact for all $\xi\in\mathfrak{g}$. However, if it is the case, we say that the action is *Hamiltonian*.

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Thus, when the action Ψ is Hamiltonian, we can find for each $\xi \in \mathfrak{g}$ a smooth function $H_\xi\colon M \to \mathbb{R}$ such that $dH_\xi = \Phi(X_\xi)$. The functions H_ξ are determined up to a constant.

Definition 3.2 A Hamiltonian action Ψ is *strongly Hamiltonian* if it is possible to choose the functions H_ξ in a way that the map:

$$\Theta\colon \mathfrak{g} \to C^\infty(M)$$
$$\xi \mapsto H_\xi$$

is a homomorphism of Lie algebras, where $C^\infty(M)$ is considered as a Lie algebra with the Poisson bracket. In other words, Θ is a Lie algebras homomorphism if:

$$\forall \xi, \eta \in \mathfrak{g}, \quad \Theta([\xi, \eta]) = \{H_\xi, H_\eta\} = \omega(X_\xi, X_\eta).$$

In general, a Hamiltonian action is not strongly Hamiltonian. However, in the case where G is compact, it is possible using an averaging argument to show that any Hamiltonian action is strongly Hamiltonian.

Now, suppose $\Psi\colon G \to \mathrm{Symp}(M, \omega)$ is a Hamiltonian action of a compact connected Lie group G on a compact symplectic manifold (M, ω). Furthermore, assume ω' is another symplectic form on M such that Ψ considered as an action on (M, ω') is symplectic, that is, $\mathcal{L}_{X_\xi}\omega' = 0$ for all $\xi \in \mathfrak{g}$. Does this implies that Ψ, considered as an action on (M, ω'), is also Hamiltonian? In terms of differential forms, the question is whether we have the implication:

$$(3) \qquad \forall \xi \in \mathfrak{g}, \quad d\iota(X_\xi)\omega' = 0 \quad \Rightarrow \quad \forall \xi \in \mathfrak{g}, \quad \iota(X_\xi)\omega' \quad \text{is exact.}$$

As we will see, this implication is true, that is to say: whenever Ψ considered as an action on (M, ω') is symplectic, it must be Hamiltonian. The first step is to reduce the proof to the simpler case where $G = S^1$. This is the task of the next lemma.

Lemma 3.3 *Let $\Psi\colon G \to \mathrm{Symp}(M, \omega)$ be a symplectic action of a compact connected Lie group G on a symplectic manifold (M, ω) such that for any subgroup H of G with $H \cong S^1$,*

$$\Psi|_H\colon H \to \mathrm{Symp}(M, \omega)$$

is a Hamiltonian action. Then, Ψ is a Hamiltonian action.

Proof Take $\xi \in \mathfrak{g}$ arbitrary. We need to show that $\iota(X_\xi)\omega$ is exact. Consider the connected closed abelian subgroup

$$H = \overline{\{\exp(t\xi) \mid t \in \mathbb{R}\}}.$$

The fact that H is a closed subgroup implies that H is a Lie group (see [6]). Moreover, using Theorem 3.6 in [1], which says that a connected abelian Lie group is the

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product of a torus and a vector space, we deduce that $H \cong T^k$ for some $k \in \mathbb{N}$, since H is compact. Now, choose a basis $\xi_1, \dots, \xi_k$ of $\mathfrak{h}$, the Lie algebra of H, such that:

$$\forall i \in \{1, \dots, k\}, \quad \{\exp(t\xi_i) \mid t \in \mathbb{R}\} \cong S^1.$$

By hypothesis, $\iota(X_{\xi_i})\omega$ must be exact for all $i \in \{1, \dots, k\}$. Hence, since $\xi \in \mathfrak{h}$, we can write ξ in a unique way as:

$$\xi = \sum_{i=1}^{k} a_i \xi_i$$

where $a_i \in \mathbb{R}$ for all $i \in \{1, \dots, k\}$. Consequently, choosing smooth functions $F_1, \dots, F_k$ such that $\iota(X_{\xi_i})\omega = dF_i$ for all $i \in \{1, \dots, k\}$, we obtain:

$$\iota(X_\xi)\omega = \sum_{i=1}^{k} a_i \iota(X_{\xi_i})\omega = \sum_{i=1}^{k} a_i dF_i = d\Big(\sum_{i=1}^{k} a_i F_i\Big)$$

which shows that $\iota(X_\xi)\omega$ is exact. ■

Since the Lie algebra of S^1 is ismorphic to $\mathbb{R}$, a Hamiltonian action of S^1 is completely specified once we know the Hamiltonian function H associated to the element of the Lie algebra of S^1 corresponding to 1 in $\mathbb{R}$. The following lemma is a well-known result about the Hamiltonian function H (see [4] for a proof).

Lemma 3.4 *Let $\Psi\colon S^1 \to \mathrm{Symp}(M, \omega)$ be a Hamiltonian action on a symplectic manifold (M, ω). Then, the Hamiltonian function H of this action is a Morse-Bott function with even dimensional critical manifolds of even indices.*

We are now in position to prove the rigidity of Hamiltonian actions.

Theorem 3.5 *Let $\Psi\colon G \to \mathrm{Symp}(M, \omega)$ be a Hamiltonian action of a compact connected Lie group G on a symplectic manifold (M, ω) and suppose ω' is another symplectic form on M such that Ψ, considered as an action on (M, ω') is symplectic. Then Ψ, considered as an action on (M, ω'), is Hamiltonian.*

Proof By Lemma 3.3, we only need to prove the result in the case where $G = S^1$. Therefore, let $\Psi\colon S^1 \to \mathrm{Symp}(M, \omega)$ be a Hamiltonian action with Hamiltonian function H. What we have to show is that $\iota(X)\omega'$ is not only closed, but exact, where $X \in \mathfrak{X}(M)$ is such that $\iota(X)\omega = dH$. Since ω' and ω are nondegenerate 2-forms, for all $x \in M$, we have:

$$\begin{aligned} \iota(X)\omega'_x = 0 &\iff X(x) = 0 \\ &\iff \iota(X)\omega_x = 0 \\ &\iff dH_x = 0. \end{aligned}$$

Thus, the 1-form $\iota(X)\omega'$ vanishes on the set $\mathrm{Crit}(H)$ of critical points of H. Now, by Lemma 3.4, H is a Morse-Bott function with even-dimensional critical manifolds

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of even indices. Therefore, we can apply Proposition 2.10, which says that there is a unique connected critical manifold C_0 of index 0 and that $H_1(M, C_0)$ vanishes. Let $\gamma\colon S^1 \to M$ be any closed smooth path. Then there exists a closed smooth path $\gamma'\colon S^1 \to C_0 \subset M$ homologous to γ. Hence,

$$\int_\gamma \iota(X)\omega' = \int_{\gamma'} \iota(X)\omega' = 0$$

because $\iota(X)\omega'$ vanishes on C_0. We conclude that $\iota(X)\omega'$ is exact. ■

As it was said in the introduction, it is possible to refine the argument so that the result is also true under the hypothesis that the two actions are only homotopic instead of being the same. To prove this, we need the following lemma.

Lemma 3.6 *Let $\Psi\colon S^1 \to \mathrm{Symp}(M,\omega)$ be a Hamiltonian action on (M,ω) with Hamiltonian function H. Then, for any smooth closed path $\gamma\colon S^1 \to M$, the map*

$$\beta_\gamma\colon \quad \begin{array}{ccc} T^2 & \to & M \\ (\theta_1,\theta_2) & \mapsto & \Psi(\theta_1)\big(\gamma(\theta_2)\big) \end{array} \quad t \in [0,1]$$

is such that $[\beta_\gamma] = 0$ when β_γ is considered as an element of $H_2(M)$.

Proof By Lemma 3.4, we know that H is a Morse-Bott function with even dimensional critical manifolds of even indices. By Proposition 2.10, H has a unique critical manifold C_0 of index 0 and $H_1(M, C_0) = 0$. Hence, for an arbitrary smooth closed path $\gamma\colon S^1 \to M$, there exists a homologous smooth path

$$\gamma'\colon S^1 \to C_0$$

which lies in C_0. Since the map $\gamma \to \beta_\gamma$ leads to a homomorphism of groups:

$$\Theta\colon \quad \begin{array}{ccc} H_1(M) & \to & H_2(M) \\ [\gamma] & \mapsto & [\beta_\gamma] \end{array}$$

we deduce that $\beta_{\gamma'}$ and β_γ are homologous. But since any point of C_0 is fixed by the action of Ψ, the map $\beta_{\gamma'}$ is simply:

$$\beta_{\gamma'}\colon \quad \begin{array}{ccc} T^2 & \to & M \\ (\theta_1,\theta_2) & \mapsto & \gamma'(\theta_2) \end{array} \quad t \in [0,1]$$

which is obviously equal to zero considered as an element of $H_2(M)$. Therefore, $[\beta_\gamma] = [\beta_{\gamma'}] = 0$ in $H_2(M)$. ■

Theorem 3.7 *Let $\Psi_t\colon G \to \mathrm{Diff}(M)$, $t \in [0,1]$, be a smooth family of actions of a compact connected Lie group G on a compact manifold M such that Ψ_0 is a Hamiltonian action on (M,ω) and Ψ_1 is a symplectic action on (M,ω'), where ω and ω' are two symplectic forms on M. Then, Ψ_1 is also a Hamiltonian action on (M,ω').*

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Proof By Lemma 3.3, it is sufficient to consider the case where $G = S^1$. Therefore, let $\Psi_t \colon S^1 \to \mathrm{Diff}(M)$ be a smooth family of actions such that Ψ_0 is a Hamiltonian action on (M, ω) with Hamiltonian function H and Ψ_1 is a symplectic action on (M, ω'), where ω and ω' are two symplectic forms on M. To conclude that $\iota(X)\omega$ is exact, we have to show that for any smooth closed path $\gamma \colon S^1 \to M$,

$$\int_\gamma \iota(X)\omega' = 0 \quad \text{where} \quad X = \frac{d\Psi_1(\theta)}{d\theta}\bigg|_{\theta=0}.$$

Hence, take an arbitrary smooth closed path $\gamma \colon S^1 \to M$. First note that for any $s \in S^1$:

$$\begin{aligned}
d\Psi_1(s)X &= d\Psi_1(s)\frac{d\Psi_1(\theta)}{d\theta}\bigg|_{\theta=0} = \frac{d}{d\theta}\big(\Psi_1(s)\circ\Psi_1(\theta)\big)\bigg|_{\theta=0} \\
&= \frac{d}{d\theta}\Psi_1(s+\theta)\bigg|_{\theta=0} = \frac{d}{d\theta}\Psi_1(\theta+s)\bigg|_{\theta=0} \\
&= \frac{d}{d\theta}\big(\Psi_1(\theta)\circ\Psi_1(s)\big)\bigg|_{\theta=0} = \frac{d}{d\theta}\Psi_1(\theta)\bigg|_{\theta=0}\circ\Psi_1(s) \\
&= X\circ\Psi_1(s).
\end{aligned} \tag{4}$$

Hence, for all $s \in S^1$,

$$\begin{aligned}
\int_\gamma \iota(X)\omega' &= \int_{S^1}\gamma^*\big(\iota(X)\omega'\big) = \int_0^1 \iota(X)\omega'\big(\dot\gamma(t)\big)\,dt \\
&= \int_0^1 \omega'\big(X\circ\gamma(t),\dot\gamma(t)\big)\,dt \\
&= \int_0^1 \Psi_1(s)^*\omega'\big(X\circ\gamma(t),\dot\gamma(t)\big)\,dt \quad \text{since } \Psi_1(s)^*\omega' = \omega' \\
&= \int_0^1 \omega'\big(d\Psi_1(s)X\circ\gamma(t), d\Psi_1(s)\dot\gamma(t)\big)\,dt \\
&= \int_0^1 \omega'\Big(X\circ\Psi_1(s)\circ\gamma(t), \frac{d}{dt}\big(\Psi_1(s)\circ\gamma(t)\big)\Big)\,dt \quad \text{by (4)} \\
&= \int_0^1 \omega'\big(X\circ\gamma_s(t),\dot\gamma_s(t)\big)\,dt \quad \text{where } \gamma_s = \Psi_1(s)\circ\gamma.
\end{aligned} \tag{5}$$

Moreover, the map

$$\beta_1 \colon \begin{array}{ccc} T^2 & \to & M \\ (\theta_1,\theta_2) & \mapsto & \Psi_1(\theta_1)\big(\gamma(\theta_2)\big) \end{array} \qquad t \in [0,1]$$

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is such that:

$$\frac{\partial \beta_1}{\partial \theta_1} = \frac{\partial}{\partial \theta_1}\big(\Psi_1(\theta_1)\big) \circ \gamma(\theta_2) = X\big(\Psi_1(\theta_1) \circ \gamma(\theta_2)\big) = X\big(\gamma_{\theta_1}(\theta_2)\big)$$

$$\frac{\partial \beta_1}{\partial \theta_2} = d\Psi_1(\theta_1)\dot{\gamma}(\theta_2) = \dot{\gamma}_{\theta_1}(\theta_2).$$

Therefore,

$$\begin{aligned}
\int_{\beta_1} \omega' = \int_{T^2} \beta_1^* \omega' &= \int_0^1 \int_0^1 \omega' \left(\frac{\partial \beta_1}{\partial \theta_1}, \frac{\partial \beta_1}{\partial \theta_2} \right) d\theta_1 \, d\theta_2 \\
&= \int_0^1 \int_0^1 \omega'\big(X \circ \gamma_{\theta_1}(\theta_2), \dot{\gamma}_{\theta_1}(\theta_2)\big) \, d\theta_1 \, d\theta_2 \\
&= \int_0^1 \left(\int_\gamma \iota(X)\omega' \right) d\theta_1 \quad \text{by (5)} \\
&= \int_\gamma \iota(X)\omega'.
\end{aligned}$$

Now,

$$\begin{array}{rccc} \beta_t : & T^2 & \to & M \\ & (\theta_1, \theta_2) & \mapsto & \Psi_t(\theta_1)\big(\gamma(\theta_2)\big) \end{array} \qquad t \in [0,1]$$

is a homotopy between β_0 and β_1 and by the previous lemma, $[\beta_0] = 0$ in $H_2(M)$ since Ψ_0 is a Hamiltonian action on (M, ω'). Thus, $[\beta_1] = 0$ in $H_2(M)$ as well and we obtain finally that:

$$\int_\gamma \iota(X)\omega' = \int_{\beta_1} \omega' = 0. \qquad \blacksquare$$

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