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SOLVABLE GROUPS OF UNIPOTENT ELEMENTS IN A RING

BY

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ABSTRACT. Let R be a ring with 1 whose nil subrings are nilpotent modulo the sum of nilpotent ideals. It is proved that if G is a locally solvable group of unipotent elements in R, then the subring generated by $\{g-1 \mid g \in G\}$ is nil. This result implies a result of Sizer showing that a solvable group of unipotent matrices over a skew field can be simultaneously triangularized.

An element x in a ring with 1 is said to be *unipotent* if $x-1$ is nilpotent. A well-known theorem of Kolchin [3] states that a multiplicative semigroup of $n \times n$ unipotent matrices over a field can be put in simultaneous triangular form. The corresponding question over a skew field (infinitely dimensional over its center) is still open. Kaplansky [1] suggests considering a more general question: Let S be a multiplicative semigroup of unipotent elements. Does the set $\{x-1 \mid x \in S\}$ generate a nil subring?

Sizer [6] has considered a group of unipotent matrices over a skew field and has proved that if such a group is solvable then it can be simultaneously triangularized. Following Sizer we consider Kaplansky's question in the case S is a solvable group and we prove that the answer is positive for rings whose nil subrings are nilpotent modulo the nil radical. In particular, the answer is positive for rings whose nil subrings are nilpotent modulo the sum of nilpotent ideals. For a large class of rings in which the last property holds, see Rowen [5, Theorem 6].

A ring R is said to be of *bounded index n* if any nilpotent element $x \in R$ satisfies $x^n = 0$. If the ring of $n \times n$ matrices $M_n(R)$ over a ring R with 1 is of bounded index n, then by [2, Theorem 10] its nil subrings are nilpotent, so our result applies to $M_n(R)$. In particular, it applies to $M_n(D)$, D a skew field, [4], and Sizer's result follows since a nilpotent subring of $M_n(D)$ can be put in simultaneous triangular form.

The following notations and remarks will be used throughout the paper.

Given a multiplicative group, G, of elements of a ring R, we denote the set $\{g-1 \mid g \in G\}$ by $G-1$. If H is a normal subgroup of G, then given $g \in G$,

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$h \in H$ there exists $h' \in H$ such that $hg = gh'$, so $(h-1)g = g(h'-1)$. It follows that if S is the semigroup generated by $H-1$, then $GS = \{gs \mid g \in G,\ s \in S\}$ is the semigroup generated by $G(H-1) = \{g(h-1) \mid g \in G,\ h \in H\}$. Moreover, if U is the subring generated by S, then $GU = \{\sum g_i s_i \mid g_i \in G,\ s_i \in S\}$ is the subring generated by GS. It also follows that if U is nilpotent modulo an ideal of R, then the same is true for GU.

If V, GV denote the subrings generated by $G-1, G(G-1)$ respectively, then the ring GU, which is clearly an ideal in the subring generated by G, is an ideal in the ring GV. Note that GV is in fact the additive subgroup generated by $G(G-1)$.

The nil radical of a ring R will be denoted by $\mathrm{Nil}(R)$. We shall not assume that the group G is solvable, but merely that it is *locally solvable*, which means that every finitely generated subgroup is solvable. Our main result is:

THEOREM. *Let R be a ring whose nil subrings are nilpotent modulo* $\mathrm{Nil}(R)$. *If G is a locally solvable group of unipotent elements of R, then $G(G-1)$ (and in particular $G-1$) generates a nil subring.*

Proof. We have to prove that any given element of the subring generated by $G(G-1)$ is nilpotent. In the expression of such an element, only a finite number of elements of G occur. These elements generate a solvable subgroup since G is locally solvable. This shows that the result of the theorem will follow if it can be proved in the case G is solvable. So let us assume G is solvable and we prove the result by induction on the solvable length l of G.

If $l = 1$, then G is commutative, so the elements of $G(G-1)$ are nilpotent and it follows that $G(G-1)$ generates a nil subring.

Let $G^{(l+1)} = \{1\}$, so by the induction hypothesis applied to $H = G'$ we have that $H-1$ generates a nil subring U, which is nilpotent modulo $\mathrm{Nil}(R)$ by our assumption on R. It follows, by one of the above remarks, that GU is nilpotent modulo $\mathrm{Nil}(R)$ so GU is nil. If as above V denotes the subring generated by $G-1$ then GU is an ideal in GV, so the result of the theorem will follow if we prove that GV/GU is nil.

We first prove that GV/GU is commutative. If $g_1, g_2 \in G$ then $g_1g_2 - g_2g_1 = g_2g_1(g_1^{-1}g_2^{-1}g_1g_2 - 1) \in GU$. This implies that the subring generated by G is commutative mod GU, so also GV is commutative mod GU. To prove that GV/GU is nil, we recall that GV is the additive subgroup generated by $G(G-1)$, so it suffices to show that if $g \in G$ and $t \in G-1$ then gt is nilpotent mod GU. But this follows by commutativity mod GU and by the assumption that the elements of G are unipotent. With this the theorem has been proved.

The result of the theorem together with [2, Theorem 10] yield:

COROLLARY 1. *Let R be a ring with 1 and let $M_n(R)$ be of bounded index n. If G is a solvable group of unipotent matrices of $M_n(R)$, then the subring generated by $G(G-1)$ (so also that generated by $G-1$) is nilpotent.*

Note that the above corollary can be obtained from a more general one, which states that if R is a ring of bounded index and G is a locally solvable group of unipotent elements of R, then $G(G-1)$ generates a ring which is nilpotent modulo the sum of nilpotent ideals. This more general result holds since the class of rings which is given in [5, Theorem 6] includes the rings of bounded index. It has been pointed out in [5] that if R is a ring with 1 and $M_n(R)$ is of bounded index n then $M_n(R)$ has no nonzero nilpotent ideals, so it follows that $G(G-1)$ generates a nilpotent subring.

The following corollary follows from the previous one and the fact that a nilpotent ring of matrices over a skew field can be simultaneously triangularized.

COROLLARY 2 (Sizer [6]). *Let D be a skew field and G a solvable group of unipotent matrices of $M_n(D)$. Then there exists an invertible matrix $P \in M_n(D)$ such that $P^{-1}AP$ is triangular for all $A \in G$.*

The above corollaries have been stated for G solvable rather than for G locally solvable since in the rings considered, if G is locally solvable then it is necessarily solvable. This result follows from the following:

PROPOSITION 1. *Let G be a group of unipotent elements of a ring R. If $G-1$ generates a nilpotent semigroup (subring) then G is solvable.*

Proof. Assume that for some $r \geq 1$ we have $t_1 \cdots t_r = 0$ for all $t_1, \ldots, t_r \in G-1$. We prove by induction on r that $G^{(r-1)} = \{1\}$.

If $r=1$ the result is trivial. If $r \geq 2$, let $H = G'$ and we prove that $H-1$ generates a nilpotent semigroup of index $\leq r-1$ and this will imply $H^{(r-2)} = \{1\}$ by the induction hypothesis, so $G^{(r-1)} = \{1\}$.

In fact we show that $t_1 \cdots t_{r-2}s = 0$ for all $t_1, \ldots, t_{r-2} \in G-1$ and $s \in H-1$. First we show that this holds for $s = h-1$, $h = g_1g_2g_1^{-1}g_2^{-1}$ a commutator of any two elements $g_1, g_2 \in G$. Let $t_{r-1} = g_1^{-1}$, $t_r = g_2 - 1$ then by our assumption on G we have

$$t_1 \cdots t_{r-2}t_{r-1}t_r = 0 = t_1 \cdots t_{r-2}t_rt_{r-1}.$$

So

$$t_1 \cdots t_{r-2}(g_1g_2 - g_2g_1) = t_1 \cdots t_{r-2}(t_{r-1}t_r - t_rt_{r-1}) = 0$$

and this implies that $t_1 \cdots t_{r-2}(g_1g_2g_1^{-1}g_2^{-1} - 1) = 0$. Now if $s \in H-1$ then $s = h_1 \cdots h_q - 1$ where $h_1, \ldots, h_q$ are commutators of elements of G. We may assume by induction that $t_1 \cdots t_{r-2}(h_1 \cdots h_{q-1} - 1) = 0$, so we obtain

$$t_1 \cdots t_{r-2}s = t_1 \cdots t_{r-2}((h_1 \cdots h_{q-1} - 1)h_q + h_q - 1) = 0.$$

The above proposition together with corollary 1 yield:

PROPOSITION 2. *Let R be a ring such that $M_n(R)$ has bounded index n. Let G*

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be a group of unipotent matrices of $M_n(R)$. Then the semigroup generated by $G-1$ is nilpotent if and only if G is solvable.

We now go back to the theorem and make a weaker assumption on the subrings of the ring R. We assume that they are locally nilpotent modulo Nil(R). A positive result can still be obtained if we make a stronger assumption on G. We only state the result. Its proof is clear by our previous arguments.

PROPOSITION 3. *Let R be a ring whose nil subrings are locally nil-potent modulo* Nil(R) *and let G be a group of unipotent elements of R. If G is locally finite and locally solvable then $G(G-1)$ generates a nil subring.*

We conclude with a result which follows from Proposition 1.

PROPOSITION 4. *Let G be a locally finite group of unipotent elements of a ring R. If $G-1$ generates a locally nilpotent semigroup (subring), then G is locally solvable.*

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