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# JEŚMANOWICZ' CONJECTURE REVISITED

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**Abstract**

Let $a, b, c$ be relatively prime positive integers such that $a^2 + b^2 = c^2$. In 1956, Jeśmanowicz conjectured that for any positive integer $n$, the only solution of $(an)^x + (bn)^y = (cn)^z$ in positive integers is $(x, y, z) = (2, 2, 2)$. In this paper, we consider Jeśmanowicz' conjecture for Pythagorean triples $(a, b, c)$ if $a = c - 2$ and $c$ is a Fermat prime. For example, we show that Jeśmanowicz' conjecture is true for $(a, b, c) = (3, 4, 5), (15, 8, 17), (255, 32, 257), (65535, 512, 65537)$.

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## 1. Introduction

Let $a, b, c$ be relatively prime positive integers such that $a^2 + b^2 = c^2$ with $2 \mid b$. Clearly, the Diophantine equation

$$(na)^x + (nb)^y = (nc)^z \tag{1.1}$$

has the solution $(x, y, z) = (2, 2, 2)$. In 1956, Sierpiński [7] showed that there is no other solution when $n = 1$ and $(a, b, c) = (3, 4, 5)$; and Jeśmanowicz [2] proved that when $n = 1$ and $(a, b, c) = (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61)$, then the only solution of (1.1) is $(x, y, z) = (2, 2, 2)$. Moreover, he conjectured that for any positive integer $n$, (1.1) has no solution other than $(x, y, z) = (2, 2, 2)$. In [1], Deng and Cohen showed that Jeśmanowicz' conjecture is true for $(a, b, c) = (3, 4, 5)$. In [8], the authors of this paper proved that Jeśmanowicz' conjecture is true for $(a, b, c) = (15, 8, 17)$. For related problems, see [5, 6].

In this paper, we obtain the following results.

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THEOREM 1.1. *Let $k$ be a positive integer. If $F_k = 2^{2^k} + 1$ is a Fermat prime, then for any positive integer $n$, the Diophantine equation*

$$((F_k - 2)n)^x + (2^{2^{k-1}+1}n)^y = (F_k n)^z \tag{1.2}$$

*has no solution $(x, y, z)$ satisfying $z < \min\{x, y\}$.*

THEOREM 1.2. *Let $k \le 4$ be a positive integer and $F_k = 2^{2^k} + 1$. Then, for any positive integer $n$, (1.2) has no solution other than $(x, y, z) = (2, 2, 2)$.*

## 2. Proofs

LEMMA 2.1 [4]. *The only solution of the Diophantine equation $(4m^2 - 1)^x + (4m)^y = (4m^2 + 1)^z$ is $(x, y, z) = (2, 2, 2)$.*

LEMMA 2.2 [1, Lemma 2]. *If $z \ge \max\{x, y\}$, then the Diophantine equation $a^x + b^y = c^z$, where $a, b$ and $c$ are any positive integers (not necessarily relatively prime) such that $a^2 + b^2 = c^2$, has no solution other than $(x, y, z) = (2, 2, 2)$.*

LEMMA 2.3 [3]. *If the Diophantine equation $(na)^x + (nb)^y = (nc)^z$ (with $a^2 + b^2 = c^2$) has a solution $(x, y, z) \ne (2, 2, 2)$, then $x, y, z$ must be distinct.*

PROOF OF THEOREM 1.1. By Lemma 2.1, we may suppose that $n \ge 2$ and that (1.2) has one solution $(x, y, z)$ with $z < \min\{x, y\}$. By Lemma 2.3, it is sufficient to consider the following two cases.

*Case 1.* $x < y$. By (1.2),

$$n^{x-z}((F_k - 2)^x + 2^{(2^{k-1}+1)y} n^{y-x}) = F_k^z. \tag{2.1}$$

If $\gcd(n, F_k) = 1$, then by (2.1) and $n \ge 2$ we have $x = z$, a contradiction. If $\gcd(n, F_k) = F_k$, then write $n = F_k^r n_1$, where $r \ge 1$ and $\gcd(F_k, n_1) = 1$. By (2.1),

$$n_1^{x-z} F_k^{r(x-z)}((F_k - 2)^x + 2^{(2^{k-1}+1)y} n_1^{y-x} F_k^{r(y-x)}) = F_k^z.$$

Noting that

$$\gcd(2^{(2^{k-1}+1)y} n_1^{y-x} F_k^{r(y-x)} + (F_k - 2)^x, F_k) = 1,$$

we have $2^{(2^{k-1}+1)y} n_1^{y-x} F_k^{r(y-x)} + (F_k - 2)^x = 1$, which is also impossible.

*Case 2.* $x > y$. By (1.2),

$$n^{y-z}(2^{(2^{k-1}+1)y} + (F_k - 2)^x n^{x-y}) = F_k^z. \tag{2.2}$$

If $\gcd(n, F_k) = 1$, then by (2.2) and $n \ge 2$ we have $y = z$, a contradiction. If $\gcd(n, F_k) = F_k$, then write $n = F_k^r n_1$, where $r \ge 1$ and $\gcd(F_k, n_1) = 1$. By (2.2),

$$n_1^{y-z} F_k^{r(y-z)}((F_k - 2)^x F_k^{r(x-y)} n_1^{x-y} + 2^{(2^{k-1}+1)y}) = F_k^z.$$

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Noting that $(F_k-2)^x F_k^{r(x-y)} n_1^{x-y} + 2^{(2^{k-1}+1)y} > 1$ and

$$\gcd((F_k-2)^x F_k^{r(x-y)} n_1^{x-y} + 2^{(2^{k-1}+1)y}, F_k) = 1,$$

we have another contradiction.

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We know that $F_0 = 3$ and $F_k$ $(1 \le k \le 4)$ are Fermat primes, so by Theorem 1.1 and Lemmas 2.2 and 2.3, it is sufficient to prove that if $k \le 4$ then (1.2) has no solution $(x, y, z)$ satisfying $y < z < x$ or $x < z < y$.

(i) By Lemma 2.1, we may suppose that $n \ge 2$ and (1.2) has one solution $(x, y, z)$ with $y < z < x$. By (1.2),

$$2^{(2^{k-1}+1)y} = n^{z-y}(F_k^z - (F_k-2)^x n^{x-z}). \tag{2.3}$$

If $\gcd(n, 2) = 1$, then by (2.3) and $n \ge 2$ we have $y = z < x$, a contradiction.

If $\gcd(n, 2) = 2$, then write $n = 2^r n_1$, where $r \ge 1$ and $\gcd(2, n_1) = 1$. By (2.3),

$$2^{(2^{k-1}+1)y} = n_1^{z-y} 2^{r(z-y)}(F_k^z - (F_k-2)^x 2^{r(x-z)} n_1^{x-z}).$$

Then $(2^{k-1}+1)y = r(z-y)$ and $n_1 = 1$. Thus,

$$F_k^z - (F_k-2)^x 2^{r(x-z)} = 1. \tag{2.4}$$

We have $F_k^z \equiv 1 \pmod 3$, $z \equiv 0 \pmod 2$. Write $z = 2z_1$; by (2.4),

$$\Big(\prod_{i=0}^{k-1} F_i\Big)^x 2^{r(x-z)} = (F_k-2)^x 2^{r(x-z)} = (F_k^{z_1}-1)(F_k^{z_1}+1).$$

Noting that $\gcd(F_k^{z_1}-1, F_k^{z_1}+1) = 2$, and $F_{k-1}$ is a Fermat prime, we have $F_{k-1}^x \mid F_k^{z_1}+1$ or $F_{k-1}^x \mid F_k^{z_1}-1$. Moreover,

$$F_{k-1}^x = (2^{2^{k-1}}+1)^x > (2^{2^{k-1}}+1)^{2z_1} > F_k^{z_1}+1,$$

a contradiction.

(ii) By Lemma 2.1, we may suppose that $n \ge 2$ and (1.2) has one solution $(x, y, z)$ with $x < z < y$. By (1.2),

$$\Big(\prod_{i=0}^{k-1} F_i\Big)^x = n^{z-x}(F_k^z - 2^{(2^{k-1}+1)y} n^{y-z}). \tag{2.5}$$

If $\gcd(n, \prod_{i=0}^{k-1} F_i) = 1$, then by (2.5) and $n \ge 2$ we have $x = z$, a contradiction.

If $\gcd(n, \prod_{i=0}^{k-1} F_i) > 1$, then the form of $n$ must be one of the following cases.

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*Case 1.* $\gcd(n, \prod_{i=0}^{k-1} F_i) = F_\lambda$, where $\lambda \in \{0, \ldots, k-1\}$.

For fixed $\lambda \in \{0, \ldots, k-1\}$, let $n = F_\lambda^\alpha n_1$, where $\alpha \geq 1$ and $\gcd(\prod_{i=0}^{k-1} F_i, n_1) = 1$. Let

$$T_1 = \prod_{\substack{i=0 \\ i \neq \lambda}}^{k-1} F_i.$$

By (2.5),

$$T_1^x = F_k^z - 2^{(2^{k-1}+1)y} F_\lambda^{\alpha(y-z)}. \tag{2.6}$$

*Subcase 1.1.* $k = 1$. We have $T_1 = 1$ and $F_\lambda = 3$, so $2^z \equiv 1 \pmod 3$, $z \equiv 0 \pmod 2$.

*Subcase 1.2.* $k = 2, 3, 4$. We have $T_1 \equiv 3, 5$ or $7 \pmod 8$. By (2.6), $T_1^x \equiv 1 \pmod 8$, so $x \equiv 0 \pmod 2$. Moreover, $T_1^x \equiv 2^z \pmod{F_\lambda}$. Noting that $x \equiv 0 \pmod 2$, by calculation, we have $z \equiv 0 \pmod 2$.

Write $z = 2z_1$, $x = 2x_1$ (so, in particular, $x_1 = 0$ if $k = 1$). By (2.6),

$$2^{(2^{k-1}+1)y} F_\lambda^{\alpha(y-z)} = (F_k^{z_1} - T_1^{x_1})(F_k^{z_1} + T_1^{x_1}).$$

Noting that

$$\gcd(F_k^{z_1} - T_1^{x_1}, F_k^{z_1} + T_1^{x_1}) = 2,$$

we have

$$2^{(2^{k-1}+1)y-1} \mid F_k^{z_1} - T_1^{x_1}, \quad 2 \mid F_k^{z_1} + T_1^{x_1},$$

or

$$2 \mid F_k^{z_1} + T_1^{x_1}, \quad 2^{(2^{k-1}+1)y-1} \mid F_k^{z_1} - T_1^{x_1}.$$

Then

$$2^{(2^{k-1}+1)y-1} > 2^{(2^{k-1}+1)2z_1} > (F_k + F_k - 2)^{z_1} > F_k^{z_1} + T_1^{x_1},$$

a contradiction.

*Case 2.* $\gcd(n, \prod_{i=0}^{k-1} F_i) = F_\lambda F_\mu$, where $\lambda, \mu \in \{0, \ldots, k-1\}$ and $\lambda < \mu$.

In this case, $k = 2, 3, 4$. For fixed $\lambda, \mu \in \{0, \ldots, k-1\}$, let $n = F_\lambda^\alpha F_\mu^\beta n_1$, where $\alpha, \beta \geq 1$ and $\gcd(\prod_{i=0}^{k-1} F_i, n_1) = 1$. Let

$$T_2 = \prod_{\substack{i=0 \\ i \neq \lambda,\mu}}^{k-1} F_i.$$

By (2.5),

$$T_2^x = F_k^z - 2^{(2^{k-1}+1)y} F_\lambda^{\alpha(y-z)} F_\mu^{\beta(y-z)}. \tag{2.7}$$

*Subcase 2.1.* $k = 2$. We have $T_2 = 1$, so $2^z \equiv 1 \pmod 3$, $z \equiv 0 \pmod 2$.

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*Subcase 2.2.* $k = 3, 4$. If $T_2 \equiv 3, 5$ or $7 \pmod 8$, then by $T_2^x \equiv 1 \pmod 8$, we have $x \equiv 0 \pmod 2$. If $T_2 \equiv 17 \pmod{32}$, then by $T_2^x \equiv 1 \pmod{32}$, we have $x \equiv 0 \pmod 2$. Moreover, $T_2^x \equiv 2^z \pmod{F_\lambda}$. Noting that $x \equiv 0 \pmod 2$, by calculation, we have $z \equiv 0 \pmod 2$.

Write $z = 2z_1$, $x = 2x_1$ (so, in particular, $x_1 = 0$ if $k = 2$). By (2.7),

$$2^{(2^{k-1}+1)y} F_\lambda^{\alpha(y-z)} F_\mu^{\beta(y-z)} = (F_k^{z_1} - T_2^{x_1})(F_k^{z_1} + T_2^{x_1}). \tag{2.8}$$

As in the proof of Case 1, we know that (2.8) cannot hold.

*Case 3.* $\gcd(n, \prod_{i=0}^{k-1} F_i) = F_\lambda F_\mu F_\nu$, where $\lambda, \mu, \nu \in \{0, \ldots, k-1\}$ and $\lambda < \mu < \nu$.

In this case, $k = 3, 4$. For fixed $\lambda, \mu, \nu \in \{0, \ldots, k-1\}$, let $n = F_\lambda^\alpha F_\mu^\beta F_\nu^\gamma n_1$, where $\alpha, \beta, \gamma \geq 1$ and $\gcd(\prod_{i=0}^{k-1} F_i, n_1) = 1$. Let

$$T_3 = \prod_{\substack{i=0 \\ i \neq \lambda, \mu, \nu}}^{k-1} F_i.$$

By (2.5),

$$T_3^x = F_k^z - 2^{(2^{k-1}+1)y} F_\lambda^{\alpha(y-z)} F_\mu^{\beta(y-z)} F_\nu^{\gamma(y-z)}. \tag{2.9}$$

*Subcase 3.1.* $k = 3$. We have $T_3 = 1$, so $2^z \equiv 1 \pmod 3$, $z \equiv 0 \pmod 2$.

*Subcase 3.2.* $k = 4$. If $T_3 = 3$ or $5$, then by $T_3^x \equiv 1 \pmod 8$, we have $x \equiv 0 \pmod 2$. If $T_3 = 17$, then by $T_3^x \equiv 1 \pmod{32}$, we have $x \equiv 0 \pmod 2$. If $T_3 = 257$, then by $T_3^x \equiv 1 \pmod{512}$, we have $x \equiv 0 \pmod 2$. Moreover, $T_3^x \equiv 2^z \pmod{F_\lambda}$. Noting that $x \equiv 0 \pmod 2$, by calculation, we have $z \equiv 0 \pmod 2$.

Write $z = 2z_1$, $x = 2x_1$ (so, in particular, $x_1 = 0$ if $k = 3$). By (2.9),

$$2^{(2^{k-1}+1)y} F_\lambda^{\alpha(y-z)} F_\mu^{\beta(y-z)} F_\nu^{\gamma(y-z)} = (F_k^{z_1} - T_3^{x_1})(F_k^{z_1} + T_3^{x_1}). \tag{2.10}$$

As in the proof of Case 1, we know that (2.10) cannot hold.

*Case 4.* $\gcd(n, \prod_{i=0}^{k-1} F_i) = F_\lambda F_\mu F_\nu F_\omega$, where $\lambda, \mu, \nu, \omega \in \{0, \ldots, k-1\}$ and $\lambda < \mu < \nu < \omega$. In this case, $k = 4$. Let $n = F_\lambda^\alpha F_\mu^\beta F_\nu^\gamma F_\omega^\delta n_1$, where $\alpha, \beta, \gamma, \delta \geq 1$ and $\gcd(\prod_{i=0}^{k-1} F_i, n_1) = 1$. By (2.5),

$$F_k^z - 2^{(2^{k-1}+1)y} F_\lambda^{\alpha(y-z)} F_\mu^{\beta(y-z)} F_\nu^{\gamma(y-z)} F_\omega^{\delta(y-z)} = 1.$$

Thus, $2^z \equiv 1 \pmod 3$, $z \equiv 0 \pmod 2$. With $z = 2z_1$,

$$2^{(2^{k-1}+1)y} F_\lambda^{\alpha(y-z)} F_\mu^{\beta(y-z)} F_\nu^{\gamma(y-z)} F_\omega^{\delta(y-z)} = (F_k^{z_1} - 1)(F_k^{z_1} + 1). \tag{2.11}$$

As in the proof of Case 1, we know that (2.11) cannot hold.

This completes the proof of Theorem 1.2. □

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