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# GREEN'S POTENTIALS WITH PRESCRIBED BOUNDARY VALUES

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**1. Introduction.** Let $U$, $C$ denote the open unit disk and unit circumference, respectively and $G(z, w)$ be the Green's function on $U$. We say $v$ is the *Green's potential* of a mass distribution $\nu$ on $U$ if

$$v(z) = \int_U G(z, w) d\nu(w) \quad \text{and} \tag{1.1}$$
$$\int_U (1 - |z|) d\nu(z) < +\infty.$$

Littlewood [**3**, p. 391] showed that the radial limit of a Green's potential is zero at almost all points of $C$. Zygmund [**5**, pp. 644–645] pointed out that the nontangential limit of a Green's potential need not exist at any point on $C$. Several other authors, Tolsted [**6**], Arsove and Huber [**1**] have given conditions on the mass distribution $\nu$ sufficient for the almost everywhere existence of the nontangential limit of the Green's potential $v$. (Tolsted's variant of Zygmund's example [**5**, p. 646, (4.7)] violates the minimum principle for superharmonic functions.)

Our object is to study the existence of Green's potential $v$ with a preassigned radial limit on a certain subset of $C$ and nontangential limit almost everywhere on $C$; we give a simple application to Blaschke products. The following three theorems are proved. (Theorem 1 is an analogue of a theorem of Rudin [**4**, p. 808].)

THEOREM 1. *Suppose $E$ is a closed set of measure zero on $C$, $f$ is a nonnegative continuous function on $E$ and $\epsilon > 0$. Then there exists a continuous Green's potential $v$ such that*

(1) $\lim_{r \to 1} v(re^{i\phi}) = f(e^{i\phi})$

*uniformly for $e^{i\phi} \in E$, $u$ has boundary value zero on $C \backslash E$, and*

(2) $v(0) < \epsilon$.

THEOREM 2. *Suppose $E$ is a set of measure zero on $C$. Then there exists a Green's potential $v$ with non-tangential limit almost everywhere on $C \backslash E$, such that for $e^{i\phi} \in E$,*

$$\limsup_{r \to 1} v(re^{i\phi}) = +\infty.$$

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*Moreover, the mass distribution $\nu$ of $v$ can be given by a density function $\lambda(z)$ which is $O((1 - |z|)^{-2})$ as $|z| \to 1$. Here the exponent 2 can not be replaced by any smaller number.*

THEOREM 3. *Let $E$ be a set of measure zero on $C$. Then there exists a Blaschke product $B$ such that*

$$\liminf_{r \to 1} |B(re^{i\phi})| = 0$$

*whenever $e^{i\phi} \in E$.*

## 2. Lemmas.

LEMMA 1. *A bounded positive superharmonic function $v$ on $U$ is a Green's potential if and only if the radial limit of $v$ is zero almost everywhere on $C$.*

*Proof.* The necessary part is a result of Littlewood [**3**, p. 391]. To prove the sufficient part, we apply the Riesz decomposition theorem for superharmonic functions [**2**, p. 116] to $v$; $v$ is the sum of a Green's potential and a positive harmonic function $h$. Since $h$ is bounded harmonic with radial limit zero almost everywhere on $C$, $h \equiv 0$. Hence $v$ is a Green's potential.

LEMMA 2. *Suppose* $0 < a < 1$, $|\text{Arg } z| < 1 - a$ *and* $|z| > (1 + a)/2$. *Then*

$$|G(a, z)| > \frac{1}{100} \frac{1 - |z|}{1 - a}.$$

*Proof.* Write $z$ as $re^{i\theta}$, where $|\theta| < 1 - a$ and $(1 + a)/2 < r < 1$. Thus we have

$$\begin{aligned} \left| \frac{z - a}{1 - az} \right|^2 &\geqq \frac{[(1 + a)/2 - a]^2}{(1 - ar)^2 + 2ar(1 - \cos\theta)} \\ &\geqq \frac{(1 - a)^2/4}{[1 - a(1 + a)/2]^2 + \theta^2} > \frac{1}{13} > \frac{1}{25}. \end{aligned} \tag{2.1}$$

Using (2.1) and the mean value theorem, we proceed to find an lower bound for $G(a, z)$.

$$\begin{aligned} G(a, z) &= \frac{1}{2} \log \left| \frac{1 - az}{z - a} \right|^2 \\ &= \frac{1}{2} \frac{1}{c} \left( \left| \frac{1 - az}{z - a} \right|^2 - 1 \right) \quad \text{where } 1 < c < \left| \frac{1 - az}{z - a} \right|^2 \\ &> \frac{1}{50} \frac{(1 - a^2)(1 - r^2)}{(r - a)^2 + 2ra(1 - \cos\theta)} \\ &> \frac{1}{50} \frac{(1 - a)(1 - r)}{(1 - a)^2 + \theta^2} \\ &> \frac{1}{100} \frac{1 - r}{1 - a}. \end{aligned}$$

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We also need the following lemma, which was used by W. Rudin [**4**, p. 810] to prove a theorem similar to Theorem 1 for analytic functions.

LEMMA 3. *Suppose $E$ is a closed totally disconnected set on $C$ (for example, $E$ is a closed set of measure zero). If $f$ is a nonnegative continuous function on $E$, bounded above by $M$, then there exists a sequence $\{f_n\}$ of simple continuous functions on $E$ such that*

$$f(z) = \sum_{n=1}^{\infty} f_n(z) \quad \text{and} \quad 0 \leqq f_n(z) \leqq 2^{-n} M \quad \text{for } 1 \leqq n < \infty .$$

We quote a theorem by Arsove and Huber [**1**, p. 125], which will be used to prove Theorem 2.

THEOREM (Arsove and Huber). *Let $v$ be a Green's potential and suppose the mass distribution for $v$ is given by a density function $\lambda$. If $\lambda(z) = O((1 - |z|)^{-2})$ as $|z| \to 1$, then $v$ has nontangential limit zero at almost all points on $C$. The exponent 2 is the largest possible.*

**3. Proof of Theorem 1.** For each $a$ in $(0, 1)$ and each set $S$ on $C$, let $T_a(S) = \{cz : a \leqq c < 1, z \in S\}$. For the moment we fix $a$ and omit the subscript.

First we shall construct a continuous Green's potential with property (1) in Theorem 1.

In case $f$ is a simple continuous function with values $\alpha_i$ on closed sets $E_i$, $1 \leqq i \leqq k$, we introduce $w_i$ as follows. Each $w_i$ is a continuous function on $U$, harmonic on $U \backslash T(E_i)$ with value $\alpha_i$ on $T(E_i)$ and with boundary value 0 on $C \backslash E_i$. Because $U \backslash T(E_i)$ is a Dirichlet region, each $w_i$ is well-defined. We note that each $w_i$ is superharmonic on $U$ and $\sum_{i=1}^{k} w_i$ satisfies (1) of Theorem 1.

For an arbitrary continuous function $f$ on $E$, let $M$ be an upper bound for $f$ and let $\{f_n\}$ be a sequence of continuous functions with the properties in Lemma 3. To each $f_n$, following the argument in the last paragraph, we may find a continuous superharmonic function $w_n$ satisfying (1) of Theorem 1 with respect to the function $f_n$. Let $u$ be the continuous superharmonic function on $U$, harmonic on $U \backslash T(E)$ with value $M$ on $T(E)$ and boundary value 0 on $C \backslash E$. By the continuity of $w_n$, the function $v_n$ defined by $\min \{2^{-n}u, w_n\}$ is still continuous superharmonic on $U$ and satisfies (1) of Theorem 1 relative to $f_n$. Since $0 \leqq v_n \leqq 2^{-n}u \leqq 2^{-n}M$, $\sum_{n=1}^{\infty} v_n$ converges uniformly on $U$; we denote the sum by $v$. Thus $v$ is bounded continuous superharmonic and has the desired boundary limiting property (1). From Lemma 1, $v$ is indeed a Green's potential.

We note that $v$ and $u$ are dependent on $a$. For this reason we denote $v$, $u$ by $v_a$, $u_a$ respectively and observe that $v_a \leqq u_a$. Hence we may conclude Theorem 1 by showing that $u_a(0) < \epsilon$ if $a$ is chosen to be sufficiently small.

Because $E$ may be covered by an open set $S$ of arbitrarily small measure and $u_{1/2}$ converges to 0 uniformly on $C \backslash S$, there exists a number $b$ close to 1, $1/2 < b < 1$, such that the average of $u_{1/2}$ on $|z| = b$ is less than $\epsilon$. Choose $a$,

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$b < a < 1$. From the maximum principle we see that $u_{1/2} > u_a$ on $U$. Since $u_a$ is harmonic on $|z| < a$,

$$u_a(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u_a(be^{i\theta})d\theta.$$

Consequently,

$$u_a(0) < \frac{1}{2\pi} \int_{-\pi}^{\pi} u_{1/2}(be^{i\theta})d\theta < \epsilon.$$

This completes the proof of Theorem 1.

**4. Proof of Theorem 2.** Let $l$ be the Lebesgue measure on $C$. Let $\{V_m\}$ be a sequence of coverings of $E$ by disjoint open arcs such that

i) $V_{m+1}$ is a refinement of $V_m$;

ii) the total length of the open arcs in $V_{m+1}$ is less than half that of $V_m$; and

iii) if $S$ is an arc in $V_m$ then $(2 \cdot n!)^{-1} \leqq l(s) \leqq (n!)^{-1}$ for some $n \geqq 3$ and those arcs in $V_{m+1}$ which are contained in $S$ are of length at most $[(n+1)!]^{-1}$.

To each open arc $S$ in $V_m$, $1 \leqq m < \infty$, we use $n$ to denote the chosen integer satisfying $(2 \cdot n!)^{-1} \leqq l(S) \leqq (n!)^{-1}$ and use $B$ to denote the annular sector $\{re^{i\theta} : e^{i\theta} \in S \text{ and } 1 - (n!)^{-1} < r < 1 - (m \cdot n!)^{-1}\}$. We may regard $n$ and $B$ as functions of $S$ and observe that $n > m$. We shall sometimes identify $S$ with the corresponding segment on $[0, 2\pi)$.

From i) and iii) above, we see that to two different $S$'s the corresponding annular sectors $B$ are disjoint. Thus we may introduce the density function $\lambda$ by

$$\lambda(z) = \begin{cases} (1 - |z|)^{-2} & \text{if } z \in \bigcup_{m=1}^{\infty} \bigcup_{S \in V_m} B \\ 0 & \text{outside.} \end{cases}$$

The mass distribution $\lambda(z)dz$ satisfies (1.1). In fact,

$$\begin{aligned}
&\int_U (1 - |z|)\,\lambda(z)dz \\
&= \sum_{m=1}^{\infty} \sum_{S \in V_m} \int_B (1 - |z|)(1 - |z|)^{-2}dz \\
&= \sum_{m=1}^{\infty} \sum_{S \in V_m} \int_{1-(n!)^{-1}}^{1-(m \cdot n!)^{-1}} \int_S (1 - r)^{-1} r d\theta dr \\
&= \sum_{m=1}^{\infty} \sum_{S \in V_m} \int_{(m \cdot n!)^{-1}}^{(n!)^{-1}} l(S) x^{-1}(1 - x)dx \\
&\leqq \sum_{m=1}^{\infty} \log m \sum_{S \in V_m} l(S)
\end{aligned}$$

which is finite from ii) above. Therefore the Green's potential $v$ given by $\lambda(z)dz$ is well-defined.

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We now show $\limsup_{r\to 1} v(re^{i\phi}) = +\infty$ for each $e^{i\phi}$ in $E$. For each $m$, $1 \leqq m < \infty$, let $S_m$ be the arc in $V_m$ that contains $e^{i\phi}$. Assume $(2 \cdot n_m!)^{-1} \leqq l(S_m) \leqq (n_m!)^{-1}$, $1 \leqq m < \infty$. Let $r_m = 1 - 2(n_m!)^{-1}$ and $B_m$ be the annular sector corresponding to $S_m$, $1 \leqq m < \infty$. If $z$ is in $B_m$, we observe that $|\text{Arg}\,(ze^{-i\phi})| < l(S_m) < 1 - r_m$. With the aid of Lemma 2, we have

$$\begin{aligned} v(r_m e^{i\phi}) &\geqq \int_{B_m} G(r_m e^{i\phi}, z)(1 - |z|)^{-2} dz \\ &> 10^{-2} \int_{1-(n_m!)^{-1}}^{1-(m\cdot n_m!)^{-1}} \int_{S_m} \frac{1-r}{1-r_m} (1-r)^{-2} r d\theta dr \\ &= 10^{-2} \int_{1-(n_m!)^{-1}}^{1-(m\cdot n_m!)^{-1}} \frac{l(S_m)}{1-r_m} \frac{r}{1-r} dr \\ &> 10^{-3} \log m. \end{aligned}$$

Consequently, $\limsup_{r\to 1} v(re^{i\phi}) = +\infty$.

The nontangential limit of $v$ is zero almost everywhere on $C$ by the cited theorem of Arsove and Huber.

If $\alpha(z)$ is a density function defined by $(1 - |z|)^{\epsilon-2}$, $\epsilon > 0$, clearly $\int_U (1 - |z|)\alpha(z)dz < \infty$; let $u$ be the Green's potential of $\alpha(z)dz$. From Littlewood's theorem [**3**, p. 391], $u(z)$ has radial limit zero at almost all points on $C$. Since $u(z)$ is constant on each circle, $u$ can be continued up to $C$ and with value 0 on $C$. Thus the exponent 2 is the best possible.

The proof of Theorem 2 is complete.

**5. Proof of Theorem 3.** First we want to construct a point mass distribution $\upsilon$ such that the Green's potential $v$ given by $\upsilon$ has the property

$$\limsup_{r\to 1} v(re^{i\phi}) = +\infty$$

if $e^{i\phi} \in E$. We retain the definition for $\{V_m\}$ from Section 4. To each $S$ in $V_m$, $1 \leqq m < \infty$, we assign a point mass $\delta_S$ of weight $m$ at the midpoint $P_S$ of the arc $(1 - 2/n!)S$, where $(2 \cdot n!)^{-1} \leqq l(S) \leqq (n!)^{-1}$. The mass distribution $\upsilon$ is defined as $\sum_{m=1}^{\infty} \sum_{S\in V_m} \delta_S$. We have

$$\begin{aligned} &\int_U (1 - |z|)d\upsilon \\ &\quad = \sum_{m=1}^{\infty} \sum_{S\in V_m} \frac{2}{n!} \cdot m \\ &\quad \leqq \sum_{m=1}^{\infty} 4m \sum_{S\in V_m} l(S) < +\infty, \end{aligned}$$

from ii) of the definition of $\{V_m\}$.

Let $v$ be the Green's potential of $\upsilon$, and let $e^{i\phi} \in E$. For each $m$, $1 \leqq m < \infty$, let $S_m$ be the arc in $V_m$ that contains $e^{i\phi}$. Assume $(2 \cdot n_m!)^{-1} \leqq l(S_m) \leqq (n_m!)^{-1}$,

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$1 \leqq m < \infty$. Let $r_m$ be $1 - 2(n_m!)^{-1}$, and $P_m$ be the midpoint of the arc $r_m S_m$, $1 \leqq m < \infty$. We observe that $|P_m - r_m e^{i\phi}| \leqq (n_m!)^{-1}$. Therefore,

$$v(r_m e^{i\phi}) \geqq mG(P_m, r_m e^{i\phi})$$
$$= m \log \left| \frac{1 - P_m r_m e^{-i\phi}}{P_m - r_m e^{i\phi}} \right|$$
$$\geqq m \log \left| \frac{1 - r_m}{P_m - r_m e^{i\phi}} \right| = m \log 2.$$

Hence we proved $\limsup_{r \to 1} v(re^{i\phi}) = +\infty$.

Now if $B$ is the Blaschke product with zeros of multiplicity $m$ at $P_S$, $S \in V_m$, $1 \leqq m < \infty$, then $\log 1/|B| = v$. This $B$ is our example for Theorem 3.

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