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Note on real and imaginary parts of harmonic quasiregular mappings

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Abstract. If $f = u + iv$ is analytic in the unit disk $\mathbb{D}$, it is known that the integral means $M_p(r,u)$ and $M_p(r,v)$ have the same order of growth. This is false if f is a (complex-valued) harmonic function. However, we prove that the same principle holds if we assume, in addition, that f is K-quasiregular in $\mathbb{D}$. The case $0 < p < 1$ is particularly interesting, and is an extension of the recent Riesz-type theorems for harmonic quasiregular mappings by several authors. Further, we proceed to show that the real and imaginary parts of a harmonic quasiregular mapping have the same degree of smoothness on the boundary.

1 Introduction and background

1.1 Notations and preliminaries

Let $\mathbb{D}$ denote the open unit disk in the complex plane and $\mathbb{T}$ be the unit circle. For a function f analytic in $\mathbb{D}$, the *integral means* are defined as

$$M_p(r,f) := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} \quad \text{if} \quad 0 < p < \infty,$$

and

$$M_\infty(r,f) := \sup_{|z|=r} |f(z)|.$$

It is said that f is in the *Hardy space* $H^p (0 < p \le \infty)$ if

$$\|f\|_p := \sup_{0\le r<1} M_p(r,f) < \infty.$$

A function $f \in H^p$ has the radial limit

$$f(e^{i\theta}) := \lim_{r\to 1^-} f(re^{i\theta})$$

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in almost every direction, and $f(e^{i\theta}) \in L^p(\mathbb{T})$. Detailed surveys on Hardy spaces and integral means can be found in, for example, the book of Duren [6]. Throughout this article, we follow notations from [6].

A complex-valued function $f = u + iv$ is harmonic in $\mathbb{D}$ if u and v are real-valued harmonic functions in $\mathbb{D}$. Every such function has a unique representation $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ with $g(0) = 0$. Analogous to the H^p spaces, the *harmonic Hardy spaces* h^p are the class of harmonic functions f for which $M_p(r, f)$ is bounded.

1.2 Growth of conjugate functions

Given a real-valued harmonic function u in $\mathbb{D}$, let v be its harmonic conjugate with $v(0) = 0$. It is a natural question that if u has a certain property, whether so does v. In the context of boundary behavior, this is answered by a celebrated theorem of M. Riesz.

Theorem A [6, Theorem 4.1] *If $u \in h^p$ for some p, $1 < p < \infty$, then its harmonic conjugate v is also of class h^p. Furthermore, there is a constant A_p, depending only on p, such that*

$$M_p(r, v) \le A_p M_p(r, u),$$

for all $u \in h^p$.

Curiously, the theorem fails for $p = 1$ and $p = \infty$, examples can be found in [6, p. 56]. Although the harmonic conjugate of an h^1-function need not be in h^1, Kolmogorov proved that it does belong to h^p for all $p < 1$. Later, Zygmund established that the condition $|u| \log^+ |u| \in L^1(\mathbb{T})$ is the "minimal" growth restriction on u which implies $v \in h^1$. We refer to the paper of Pichorides [14] for the optimal constants in the Riesz, Kolmogorov, and Zygmund theorems.

In [7], Hardy and Littlewood showed that in the case $0 < p < 1$, Riesz's theorem is false in a much more comprehensive sense. Kolmogorov's result might suggest that if $u \in h^p$, then v, while not necessarily in h^p, should belong to h^q for $0 < q < p$. But this is false, and in fact, v need not belong to h^q for any $q > 0$.

Nevertheless, they proved that the symmetry is restored in these latter cases if instead of the boundedness of the means, one considers their order of growth.

Theorem B [7, Theorem 4] *Let $0 < p \le \infty$ and $\beta > 0$. Suppose $f = u + iv$ is analytic in $\mathbb{D}$, and*

$$M_p(r, u) = O\left(\frac{1}{(1 - r)^\beta}\right).$$

Then,

$$M_p(r, v) = O\left(\frac{1}{(1 - r)^\beta}\right).$$

The proof of this theorem is based on an extremely complicated (as remarked by the authors themselves) result, which can be stated as follows.

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Theorem C [7, Theorems 2 and 3] *If $f = u + iv$ is analytic in $\mathbb{D}$, and*

$$M_p(r,u) = O\left(\frac{1}{(1-r)^{\beta}}\right), \quad 0 < p \le \infty, \quad \beta \ge 0,$$

then

$$M_p(r,f') = O\left(\frac{1}{(1-r)^{\beta+1}}\right).$$

Further, the converse is true for all $\beta > 0$.

Let us note that the functions $|u|^p$ and $|v|^p$ are subharmonic when $p \ge 1$, but not when $p < 1$, and therefore, $M_p(r,u)$ and $M_p(r,v)$ are not necessarily monotonic for $p < 1$. This is the principal difficulty in dealing with the case $0 < p < 1$ for harmonic functions.

1.3 Riesz theorem for harmonic quasiregular mappings

For $K \ge 1$, a sense-preserving harmonic function $f = h + \overline{g}$ is said to be K-*quasiregular* if its *complex dilatation* $\omega = g'/h'$ satisfies the inequality

$$|\omega(z)| \le k < 1 \quad (z \in \mathbb{D}),$$

where

$$k := \frac{K-1}{K+1}. \tag{1.1}$$

The function f is K-*quasiconformal* if it is K-quasiregular as well as homeomorphic in $\mathbb{D}$. One can find the H^p-theory for quasiconformal mappings in, for example, the paper of Astala and Koskela [1]. It is worth mentioning that harmonic quasiconformal mappings have generated considerable interest in recent times, perhaps from a novel point of view. In [16], Wang et al. constructed independent extremal functions for harmonic quasiconformal mappings, which were then further explored by Li and Ponnusamy in [12]. Recently, in [3], Baernstein-type extremal results were obtained on the analytic and co-analytic parts of functions in the harmonic quasiconformal Hardy space.

Suppose $f = u + iv$ is a harmonic function in $\mathbb{D}$, and $u \in h^p$ for some $p > 1$. Then, the imaginary part v does not necessarily belong to h^p, i.e., the Riesz theorem is not true for harmonic functions. One naturally asks under which additional condition(s) a harmonic analog of the Riesz theorem would hold. Recently, Liu and Zhu [13] showed that such a condition is the quasiregularity of f.

Theorem D [13] *Let $f = u + iv$ be a harmonic K-quasiregular mapping in $\mathbb{D}$ such that $u \ge 0$ and $v(0) = 0$. If $u \in h^p$ for some $p \in (1,2]$, then also v is in h^p. Furthermore, there is a constant $C(K,p)$, depending only on K and p, such that*

$$M_p(r,v) \le C(K,p) M_p(r,u).$$

Moreover, if $K = 1$, i.e., f is analytic, then $C(1,p)$ coincides with the optimal constant in the Riesz theorem.

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The condition $u \geq 0$ was subsequently removed by Chen et al. [2], who remarkably extended the result for all $p \in (1, \infty)$. Later in [10], Kalaj produced a couple of Kolmogorov type theorems for harmonic quasiregular mappings. Very recently, a quasiregular analog of Zygmund's theorem has been obtained by Kalaj [11], and also independently by Das, Huang, and Rasila [4].

The purpose of this article is to show that the real and imaginary parts of a harmonic quasiregular mapping have the same order of growth for all $p > 0$. This extends Theorem D to the cases $0 < p < 1$ and $p = \infty$. The main results and their proofs are presented in the next section.

2 Main results and proofs

In what follows, we always assume that K and k are related by (1.1).

Theorem 1 *Suppose $0 < p \leq \infty$ and $\beta > 0$, and let $f = u + iv$ be a harmonic K-quasiregular mapping in $\mathbb{D}$. If*

$$M_p(r, u) = O\left(\frac{1}{(1-r)^\beta}\right),$$

then

$$M_p(r, v) = O\left(\frac{1}{(1-r)^\beta}\right).$$

Proof For $1 < p < \infty$, we could apply the result of Chen et al. from [2], but here we shall give a simple proof which makes no appeal to this deeper result.

Let us write $f = h + \overline{g}$, and let $F = h + g$. Then,

$$\operatorname{Re} F = \operatorname{Re} f = u.$$

If $M_p(r, u)$ has the given order of growth, it follows from Theorem C that

$$M_p(r, F') = O\left(\frac{1}{(1-r)^{\beta+1}}\right).$$

Now, we observe

$$F' = h' + g' = (1 + \omega)h',$$

so that

$$|F'| \geq (1 - |\omega|)|h'| \geq (1 - k)|h'|,$$

as $|\omega| \leq k$. This readily implies

$$M_p(r, h') \leq \frac{1}{1-k} M_p(r, F') = O\left(\frac{1}{(1-r)^{\beta+1}}\right).$$

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Since $|g'| \leq k|h'|$, we also have

$$M_p(r, g') = O\left(\frac{1}{(1-r)^{\beta+1}}\right).$$

Therefore, the converse part of Theorem C shows that

$$M_p(r, h) = O\left(\frac{1}{(1-r)^{\beta}}\right) = M_p(r, g).$$

For $1 \leq p \leq \infty$, Minkowski's inequality gives

$$M_p(r, f) \leq M_p(r, h) + M_p(r, g),$$

while for $0 < p < 1$, we have

$$M_p^p(r, f) \leq M_p^p(r, h) + M_p^p(r, g).$$

In either case, we find that

$$M_p(r, f) = O\left(\frac{1}{(1-r)^{\beta}}\right),$$

which, in turn, implies

$$M_p(r, v) = O\left(\frac{1}{(1-r)^{\beta}}\right).$$

This completes the proof. ∎

The next theorem deals with the case $\beta = 0$. If $f = u + iv$ is harmonic K-quasiregular and $u \in h^p$ for some $p < 1$, then of course, v need not be in any h^q, as discussed before. Nevertheless, it is still possible to give an estimate on $M_p(r, v)$, as we show in Theorem 2. The proof is somewhat similar to that of Theorem 1, and relies on the following lemma from [5].

Lemma A [5] *Let $0 < p < 1$. Suppose $f = h + \overline{g}$ is a locally univalent, sense-preserving harmonic function in $\mathbb{D}$ with $f(0) = 0$. Then,*

$$\|f\|_p^p \leq C \int_0^1 (1-r)^{p-1} M_p^p(r, h')\, dr,$$

where $C > 0$ is a constant independent of f.

Theorem 2 *Suppose $f = u + iv$ is a harmonic K-quasiregular mapping in $\mathbb{D}$, and $u \in h^p$ for some $p \in (0, 1)$. Then,*

$$M_p(r, v) = O\left(\left(\log \frac{1}{1-r}\right)^{1/p}\right).$$

Proof As before, we write $f = h + \overline{g}$ and $F = h + g$. Since $M_p(r, u)$ is bounded, an appeal to Theorem C, for $\beta = 0$, shows that

$$M_p(r, F') = O\left(\frac{1}{1-r}\right).$$

The quasiregularity of f, like in the previous proof, then implies

$$M_p(r, h') = O\left(\frac{1}{1-r}\right).$$

Without any loss of generality, we assume that $f(0) = 0$. For $0 < r < 1$, let $f_r(z) = f(rz)$. Applying Lemma A for the function f_r, we find

$$\begin{aligned} M_p^p(r, f) &\le C \int_0^1 (1-t)^{p-1} M_p^p(rt, h')\,dt \le C \int_0^1 \frac{(1-t)^{p-1}}{(1-rt)^p}\,dt \\ &= C\left[\int_0^r \frac{(1-t)^{p-1}}{(1-rt)^p}\,dt + \int_r^1 \frac{(1-t)^{p-1}}{(1-rt)^p}\,dt\right] \\ &\le C\left[\int_0^r \frac{1}{1-t}\,dt + \frac{1}{(1-r)^p}\int_r^1 (1-t)^{p-1}\,dt\right] \\ &= O\left(\log\frac{1}{1-r}\right). \end{aligned}$$

Therefore, it follows that

$$M_p(r, v) \le M_p(r, f) = O\left(\left(\log\frac{1}{1-r}\right)^{1/p}\right).$$

The proof is thus complete. ■

Generally speaking, Theorem 1 suggests that the real and imaginary parts of a harmonic quasiregular mapping have the same "order of infinity." We now wish to show that they also have the same degree of smoothness on the boundary (see Theorem 3).

Let $\Lambda_\alpha (\alpha > 0)$ be the class of functions $\varphi : \mathbb{R} \to \mathbb{C}$ satisfying a Hölder condition of order α, i.e.,

$$|\varphi(x) - \varphi(y)| \le A|x-y|^\alpha,$$

for some constant $A > 0$. If $\alpha > 1$, Λ_α is the class of constant functions, hence, we restrict attention to the case $0 < \alpha \le 1$. Clearly, $\Lambda_\beta \subset \Lambda_\alpha$ for $\alpha < \beta$.

The following principle of Hardy and Littlewood says that an analytic function f is Hölder continuous on the boundary if f' has a "slow" rate of growth, and conversely.

Theorem E [8, Theorem 40] *Let f be an analytic function in $\mathbb{D}$. Then, f is continuous in the closed disk $\overline{\mathbb{D}}$ and $f(e^{i\theta}) \in \Lambda_\alpha (0 < \alpha \le 1)$, if and only if*

$$|f'(re^{i\theta})| = O\left(\frac{1}{(1-r)^{1-\alpha}}\right).$$

We are now prepared to discuss the final result of this article.

Theorem 3 *Let $f = u + iv$ be a harmonic K-quasiregular mapping in $\mathbb{D}$, and suppose u is continuous in $\overline{\mathbb{D}}$. If $u(e^{i\theta}) \in \Lambda_\alpha$, $0 < \alpha < 1$, then v is continuous in $\overline{\mathbb{D}}$ and $v(e^{i\theta}) \in \Lambda_\alpha$.*

Proof First, we note that if v is continuous on $\mathbb{T}$, then $v(re^{i\theta})$ is the Poisson integral of $v(e^{i\theta})$. Hence, the continuity of $v(e^{i\theta})$ would imply the continuity of v in $\overline{\mathbb{D}}$. Therefore, it is enough to show that $v(e^{i\theta}) \in \Lambda_\alpha$.

Now, suppose $u(e^{i\theta}) \in \Lambda_\alpha$ and $f = h + \overline{g}$. As before, we write $F = h + g$ so that

$$\operatorname{Re} F = \operatorname{Re} f = u.$$

Since u is continuous in $\overline{\mathbb{D}}$, we can represent F by the Poisson integral formula

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, u(e^{it})\, dt + i\operatorname{Im} F(0).$$

This implies

$$\begin{aligned} F'(z) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial z}\left(\frac{e^{it}+z}{e^{it}-z}\right) u(e^{it})\, dt \\ &= \frac{1}{\pi}\int_0^{2\pi} \frac{e^{it}}{(e^{it}-z)^2}\, u(e^{it})\, dt. \end{aligned}$$

Therefore, for $z = re^{i\theta}$, we have

$$F'(re^{i\theta}) = \frac{1}{\pi}\int_0^{2\pi} \frac{e^{it}}{(e^{it}-re^{i\theta})^2}\, u(e^{it})\, dt. \tag{2.1}$$

Also, from the Cauchy integral formula, it is easy to see

$$0 = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{d\zeta}{(\zeta - z)^2} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}}{(e^{it}-re^{i\theta})^2}\, dt,$$

so that

$$0 = \frac{1}{\pi}\int_0^{2\pi} \frac{e^{it}}{(e^{it}-re^{i\theta})^2}\, u(e^{i\theta})\, dt. \tag{2.2}$$

Subtracting (2.2) from (2.1), and taking absolute value, we find

$$|F'(re^{i\theta})| \le \frac{1}{\pi}\int_0^{2\pi} \frac{\left|u(e^{i(\theta+t)}) - u(e^{i\theta})\right|}{1 - 2r\cos t + r^2}\, dt. \tag{2.3}$$

Since $u(e^{i\theta}) \in \Lambda_\alpha$, we have

$$\left|u(e^{i(\theta+t)}) - u(e^{i\theta})\right| \le A|t|^\alpha,$$

for some constant $A > 0$. Therefore, it follows from (2.3) that

$$|F'(re^{i\theta})| \le \frac{A}{\pi}\int_0^{2\pi} \frac{|t|^\alpha}{1-2r\cos t + r^2}\, dt = \frac{2A}{\pi}\int_0^{\pi} \frac{t^\alpha}{1-2r\cos t + r^2}\, dt.$$

For $0 \le t \le \pi$, we can estimate the denominator as

$$1 - 2r\cos t + r^2 = (1-r)^2 + 4r\sin^2\frac{t}{2} \ge (1-r)^2 + \frac{4r}{\pi^2}t^2,$$

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which implies

$$|F'(re^{i\theta})| \le \frac{2A}{\pi}\int_0^{\pi} \frac{t^{\alpha}}{(1-r)^2 + (4r/\pi^2)t^2}\,dt.$$

Now, we substitute $u = t/(1-r)$ to obtain

$$\begin{aligned}|F'(re^{i\theta})| &\le \frac{2A}{\pi}\frac{1}{(1-r)^{1-\alpha}}\int_0^{\pi/(1-r)} \frac{u^{\alpha}}{1+(4r/\pi^2)u^2}\,dt\\ &\le \frac{2A}{\pi}\frac{1}{(1-r)^{1-\alpha}}\int_0^{\infty} \frac{u^{\alpha}}{1+(4r/\pi^2)u^2}\,dt\\ &= O\left(\frac{1}{(1-r)^{1-\alpha}}\right),\end{aligned}$$

because the last integral converges for $\alpha < 1$. As in the proof of Theorem 1, we have

$$|h'| \le \frac{1}{1-k}|F'|, \quad |g'| \le \frac{k}{1-k}|F'|,$$

and therefore,

$$|h'(re^{i\theta})| = O\left(\frac{1}{(1-r)^{1-\alpha}}\right) = |g'(re^{i\theta})|.$$

Then, an appeal to Theorem E implies

$$h(e^{i\theta}) \in \Lambda_{\alpha} \quad \text{and} \quad g(e^{i\theta}) \in \Lambda_{\alpha}.$$

It follows that $f(e^{i\theta}) \in \Lambda_{\alpha}$, and consequently, $v(e^{i\theta}) \in \Lambda_{\alpha}$, as desired. This completes the proof. ∎

The theorem is not true for $\alpha = 1$, even if f is analytic (i.e., 1-quasiregular). The following example is well-known.

Example 1 Let u be the harmonic function in $\mathbb{D}$ with boundary values

$$u(e^{i\theta}) = |\theta| \quad \text{for } \theta \in [-\pi, \pi].$$

Clearly, $u(e^{i\theta})$ is Lipschitz. One can show, by the method of Hilbert transforms, that the boundary values of the conjugate function v behave like

$$v(e^{i\theta}) \sim \theta \log|\theta| \quad \text{near } \theta = 0.$$

It follows that

$$v'(e^{i\theta}) \sim \log|\theta|,$$

which is unbounded as $\theta \to 0$. Thus, $v(e^{i\theta})$ is not Lipschitz.

Remark 1 The Hölder continuity of quasiregular mappings has been widely studied in the literature. Suppose $G \subset \mathbb{R}^n$, $n \ge 2$, is a domain and $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$. It is known (see [15, Theorem 1.11], cf. [9, Theorem 16.13]) that every bounded K-quasiregular mapping $f : G \to \mathbb{R}^n$ is δ-Hölder continuous for some exponent

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$\delta \in (0,1]$ which depends on the *inner dilatation* of f (and therefore, on the constant K). Further, the exponent δ is best possible, as can be seen from the function $f : \mathbb{B}^n \to \mathbb{B}^n$, $f(x) = |x|^{\delta-1}x$ (here $\delta = K^{1/(1-n)}$).

It is important to clarify that Theorem 3 presented herein diverges from this setting. We have shown that if $u(e^{i\theta})$ is α-Hölder continuous, then so is $v(e^{i\theta})$, for any arbitrary $\alpha \in (0,1)$, i.e., the constant K plays no role here. In other words, the primary interest of our result is in showing that the real and imaginary parts of a (planar) harmonic quasiregular mapping essentially behave like "harmonic conjugates."

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