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AN INEQUALITY FOR COMPLETE SYMMETRIC FUNCTIONS

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Consider the identity

$$\prod_{j=1}^{m}(1-a_jt)^{-1}=\sum_{r=0}^{\infty} T_r(a_1,\ldots,a_m)t^r,$$

where $a_j,\ldots,a_m$ are positive real numbers. Then for $r=1,2,3,\ldots$ $T_r=T_r(a_1,\ldots,a_m)$ is called the rth complete symmetric function in $a_1,\ldots,a_m$ $(T_0=1)$.

$$T_r=\sum a_1^{k_1}\cdots a_m^{k_m},$$

where the summation is over all permutations $(k_1,\ldots,k_m)$ satisfying the conditions $0\le k_i\le r$ $(1\le i\le m)$ and $\sum k_i=r$. We then define the rth complete symmetric mean q_r as

$$q_r=\binom{m+r-1}{r}^{-1}T_r,$$

where $(m+r-1\quad r)$ is the number of terms in T_r. Then by Theorems 220 and 221 in [1] we have the inequalities:

$$(q_r)^2<q_{r-1}q_{r+1} \tag{1}$$

for $r=1,2,3,\ldots$ unless all the a are equal, and

$$(q_r)^{1/r}<(q_{r+1})^{1/r+1} \tag{2}$$

for $r=1,2,3,\ldots$ unless all the a are equal. The inequalities (1) and (2) also follow from part (b) of Theorem 2 and its Corollary in [3] where $k=-1$.

The purpose of this note is to generalize the inequality (1) in the same way as Menon did for elementary symmetric functions in [2]. Define for $r=1,2,3,\ldots$ and $0\le t\le 1$, the functions

$$q_r(t)=\frac{T_r}{\begin{bmatrix} m+r-1 \\ r\end{bmatrix}},$$

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where $[n \quad k]$ is the t-binomial coefficient defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-t^n)(1-t^{n-1})\cdots(1-t^{n-k+1})}{(1-t)(1-t^2)\cdots(1-t^k)},$$

$[n \quad 0]=1$ and $[n \quad k]=0$, for $k<0$ (cf. [2]). Note that $q_r(0)=T_r$ and $q_r(1)=q_r$. We then have the following

THEOREM. *For $r=1,2,3,\ldots$ and positive real numbers $a_1,\ldots,a_m$ we have*

$$\{q_r(t)\}^2 < \left(\frac{r+1}{r}\right) q_{r-1}(t) q_{r+1}(t), \tag{3}$$

$(0 \le t \le 1)$ *unless all the a are equal.*

Proof

$$\frac{\begin{bmatrix} m+r-2 \\ r-1 \end{bmatrix}\begin{bmatrix} m+r \\ r+1 \end{bmatrix}\binom{m+r-1}{r}^2 q_r^2}{\begin{bmatrix} m+r-1 \\ r \end{bmatrix}^2\binom{m+r-2}{r-1}\binom{m+r}{r+1} q_{r-1}q_{r+1}}$$

$$= \frac{(m+r-1)(r+1)(1-t^{m+r})(1-t^r)q_r^2}{(m+r)r(1-t^{m+r-1})(1-t^{r+1})q_{r-1}q_{r+1}} \le \left(\frac{r+1}{r}\right)\frac{q_r^2}{q_{r-1}q_{r+1}},$$

(since

$$\frac{1-t^r}{1-t^{r+1}} \le 1 \quad \text{and} \quad \frac{(m+r-1)(1-t^{m+r})}{(m+r)(1-t^{m+r-1})} \le 1 \Bigg) < \frac{r+1}{r}, \quad \text{(using (1))}.$$

Therefore,

$$\frac{\{q_r(t)\}^2}{q_{r-1}(t)q_{r+1}(t)} < \frac{r+1}{r},$$

and this proves (3).

REMARK. When $t=1$, the inequality (3) is less sharp than the inequality (1).

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