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ON CYCLIC GROUP ACTIONS OF EVEN ORDER ON THE THREE DIMENSIONAL TORUS

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In this paper, we prove that if h is a generator of a Z_{2n} action on $S^1 \times S^1 \times S^1$, and $\mathrm{Fix}(h^n)$ consists of two disjoint tori, one torus, four simple closed curves, or two simple closed curves, then h is equivalent to the obvious actions.

0. Introduction

A homeomorphism $h: M \to M$ of a space M onto itself is called a periodic map on M with period n if $h^n =$ identity and $h^i \neq$ identity for $1 \leqslant i < n$. A periodic map h on M is weakly equivalent to a periodic map h' on M' if there exists a homeomorphism $t: M \to M'$ such that $t^{-1}ht = (h')^i$ for some $1 \leqslant i < n$. if $i = 1$, then h and h' are equivalent.

In this paper we consider the classification problem of Z_{2n} actions on $S^1 \times S^1 \times S^1$. Let h be a periodic map which generates the Z_{2n} action. We solve the problem when $\mathrm{Fix}(h^n)$, the fixed point set of h^n, is a torus, two disjoint tori, four simple closed curves, or two simple closed curves. We investigate the actions when $\mathrm{Fix}(h^n)$ consists of eight points. We extend the results of Hempel [**3**] concerning free cyclic actions on $S^1 \times S^1 \times S^1$, and Showers [**7**] and Kwun and Tollefson [**5**] of the involutions of $S^1 \times S^1 \times S^1$. We obtain the following classification theorems for periodic maps $h: S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$ of period $2n$, $n > 1$.

Theorem 3. *If $\mathrm{Fix}(h^n) = T_1 \cup T_2$, the union of two tori, then n is odd and there is a periodic map $g: T \to T$ of period n such that h is equivalent to h_1, where $h_1(x,y,z) = (g(x,y), \bar{z})$. For $n = 3$, there are two such actions, up to weak equivalence. For each $n \geqslant 5$, there exists a unique action up to weak equivalence.*

Theorem 4. *If $\mathrm{Fix}(h^n) = T_1$, a torus, then n is odd and for each n, h is unique up to weak equivalence.*

Theorem 5. *If $\mathrm{Fix}(h^n) = S_1 \cup S_2 \cup S_3 \cup S_4$, the disjoint union of four simple closed curves, then (up to weak equivalence) for $n = 2$ there are three actions, for*

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$n = 3$ there are three actions, and for $n \geqslant 4$ there is a unique action for every odd n, and there is no action for any even n.

THEOREM 6. *If $Fix(h^n) = S_1 \cup S_2$, the disjoint union of two simple closed curves, then (up to weak equivalence) for $n = 2$ there are three actions, and for $n \geqslant 3$, there is a unique action for every odd n and there is no action for any even n.*

Throughout this paper we work in the PL category. We divide the paper into six sections. In Section 1 we list all standard Z_n actions on T, and all nonfree involutions on $S^1 \times S^1 \times S^1$. In Section 2, 3, 4 and 5 we prove Theorems 3, 4, 5 and 6 respectively. In Section 6 we investigate Z_{2n} actions on $S^1 \times S^1 \times S^1$ when $\mathrm{Fix}(h^n) =$ eight points.

Let h be a periodic map of period $n = ml$ on a space M. Then h^m has period l. Let $q\colon M \to M/h^m$ be the orbit map induced by h^m. Then there exists a homeomorphism $\bar{h}$ on M/h^m of period m, uniquely determined by h such that $\bar{h}q = qh$. $\bar{h}$ is called the periodic map on M/h^m induced by h. Throughout this paper we denote $S^1 \times S^1 \times S^1$ by T^3, the torus $S^1 \times S^1$ by T, the Klein bottle by K and the Mobius band by Mb. We view S^1 as the set of complex numbers z with $|z| = 1$.

1.

In this section we give a list of standard cyclic actions on T. We also write a list of standard nonfree actions on T^3. The proof of Theorem 1 may be found [**6**] and [**9**]. The proof of Theorem 2 is in [**4**] and [**7**].

THEOREM 1. *Let h be a periodic map of period n, acting on T. Then h is weakly equivalent to one of the following maps.*

I. h preserves orientation.

a) $h(x,y) = (x, \omega y)$, $\quad \omega = e^{2\pi i/n}$
$Fix(h^i) = \emptyset \quad 1 \leqslant i < n$
$T/h \approx T$.

b) $h(x,y) = (\bar{y}, xy)$, $\quad n = 6$
$Fix(h) = \{(1,1)\}$
$Fix(h^2) = \{(1,1), (\omega,\omega), (\omega^2,\omega^2)\}$, $\quad \omega = e^{2\pi i/3}$
$Fix(h^3) = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$
$T/h \approx S^2$.

c) $h(x,y) = (y, \bar{x})$, $\quad n = 4$
$Fix(h) = \{(1,1), (-1,-1)\}$
$Fix(h^2) = \{(1,1), (-1,-1), (1,-1), (-1,1)\}$
$T/h \approx S^2$.

d) $h(x,y) = (\bar{x}\,\bar{y}, x)$, $\quad n = 3$
$Fix(h) = Fix(h^2) = \{(1,1), (w,w), (w^2,w^2)\}$, $\quad w = e^{2\pi i/3}$
$T/h \approx S^2$.

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e) $h(x,y) = (\bar{x},\bar{y})$, $n = 2$
$Fix(h) = \{(1,1),(-1,-1),(1,-1),(-1,1)\}$
$T/h \approx S^2$.

II. h reverses orientation, and hence n is even, $n = 2k$,

a) $h(x,y) = (x\omega,\bar{y})$, $\omega = e^{2\pi i/n}$
$Fix(h^i) = \emptyset$, $1 \leqslant i < n$
$T/h \approx K$.

b) $h(x,y) = (x\omega, x\bar{y})$, $w = e^{2\pi i/k}$, k even,
$Fix(h^i) = \emptyset$, $1 \leqslant i < n$
$T/h \approx K$.

c) $h(x,y) = (\bar{x},\omega y)$, $\omega = e^{2\pi i/k}$, k odd,
$Fix(h^i) = \emptyset$, $1 \leqslant i < k$
$Fix(h^k) = S^1_1 \cup S^1_2$
$T/h \approx S^1 \times I$.

d) $h(x,y) = (xy\omega,\bar{y})$, $\omega = e^{2\pi i/k}$, k odd,
$Fix(h^i) = \emptyset$, $1 \leqslant i < k$
$Fix(h^k) = S^1$
$T/h \approx Mb$.

THEOREM 2. *The following is a standard list of nonfree involutions on T^3.*

(1) $h_1(x,y,z) = (x,y,\bar{z})$, $Fix(h_1) = T_1 \cup T_2$,

(2) $h_2(x,y,z) = (y,x,z)$, $Fix(h_2) = T$,

(3) $h_3(x,y,z) = (\bar{x},\bar{y},z)$, $Fix(h_3) = S^1_1 \cup S^1_2 \cup S^1_3 \cup S^1_4$,

(4) $h_4(x,y,z) = (xy,\bar{y},\bar{z})$, $Fix(h_4) = S^1_1 \cup S^1_2$,

(5) $h_5(x,y,z) = (\bar{x},\bar{y},\bar{z})$, $Fix(h_5) =$ *eight points.*

2. PROOF OF THEOREM 3

(2.1) $\mathrm{Fix}(h^n) = T_1 \cup T_2$. In fact we may view h^n as given by $h^n(x,y,z) = (x,y,\bar{z}), T_1 = T\times\{1\}, T_2 = T\times\{-1\}$. $T_1\cup T_2$ is invariant under h. $T_1 \cup T_2$ separates T^3 into two components A and B, each of which is homeomorphic to $T\times I$. Since $T_1\cup T_2$ is invariant under h, we have $h(A) = B$ or $h(A) = A$, but $h^n(A) = B$, hence $h(A) = B$ and n is odd. Moreover $h(T_i) = T_i$, $i = 1,2$. Let $q\colon T^3 \to T^3/h^n \approx T \times I$ be the quotient map. h induces $\bar{h}\colon T^3/h^n \to T^3/h^n$, $\bar{h}$ is a periodic map of period n, which keeps each of the two boundary components invariant, and is orientation preserving. Hence it is equivalent to $h'\colon T\times I \to T\times I$, $h'(x,y,t) = (g(x,y),t)$, where g is a periodic map on T with period n [**5**]. Now let $h\colon T^3 \to T^3$ be given by $h_1(x,y,z) = (g(x,y),\bar{z})$, then $\bar{h}_1\colon T^3/h_1^n \to T^3/h_1^n$ may be given by $\bar{h}_1(x,y,t) = (g(x,y),\bar{t})$. Hence $\bar{h}$ is

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equivalent to $\bar{h}_1$. Therefore there exists a homeomorphism $t\colon T^3/h_1^n \to T^3/h^n$ such that $\bar{h}t = t\bar{h}_1$. Define $\bar{t}\colon T^3 \to T^3$ as follows: for each $x_1 \in A_1 \subseteq T^3$, let $x_2 = h_1^n(x_1)$, then $q_1(x_1) = q_1(x_2) = x \in T^3/h_1^n$. If $t(x) = y \in T^3/h^n$, then there exists $y_1 \in A$, $y_2 \in B$ such that $q(y_1) = q(y_2) = y$. Let $\bar{t}(x_1) = y_1$. Similarly define $\bar{t}(x)$ for $x \in B$. It is easy to check that $h\bar{t} = \bar{t}h_1$ and h equivalent to h_1.

(2.2). For $n = 3$ there are two cases - (a) $\mathrm{Fix}(g)$ consists of three points, and (b) $\mathrm{Fix}(g) = \emptyset$.

Case (a). h is given by the following formula (see Section 1).

$$\begin{aligned}
&h(x, y, z) = (\bar{x}\bar{y}, x, \bar{z})\\
&\mathrm{Fix}(h) = \text{six points}\\
&\mathrm{Fix}(h^2) = \{(1,1), (\omega,\omega), (\omega^2,\omega^2)\} \times S^1, \quad \omega = e^{2\pi i/3}\\
&\mathrm{Fix}(h^3) = T_1 \cup T_2.
\end{aligned}$$

h is unique up to weak equivalence.

Case (b). See (2.3).

(2.3) For $n \geqslant 3$, n odd. From Section 1, h is given by

$$\begin{aligned}
&h(x, y, z) = (x, \omega y, \bar{z}), \quad \omega = e^{2\pi i/n}\\
&\mathrm{Fix}(h^i) = \emptyset, \quad 1 \leqslant i < n\\
&\mathrm{Fix}(h^n) = T_1 \cup T_2.
\end{aligned}$$

For each n, h is unique up to weak equivalence. ∎

3. Proof of Theorem 4

$\mathrm{Fix}(h^n) = T_1$, hence $h(T_1) = T_1$ and $h^n(x) = x$ for all $x \in T_1$. h^n interchanges the sides of T_1, therefore h interchanges the sides of T_1 and n is odd. Cut T^3 along T_1 to get a manifold $M \approx T \times I$ and an induced homeomorphism $\bar{h}\colon T \times I \to T \times I$ of period $2n$, where $\bar{h}(T \times \{0\}) = T \times \{1\}$ and $\mathrm{Fix}(\bar{h}) = \emptyset$.

Now $\bar{h}^2$ is orientation preserving of period n which keeps each of the boundary components invariant. Hence there exists a periodic map $g\colon T \to T$ of period n, which is orientation preserving such that $\bar{h}$ is equivalent to h' where $h'(x, y, t) = (g(x, y), t)$ [**5**]. Without loss of generality we may assume $\bar{h}^2(x, y, t) = (g(x, y), t)$ (after parametrising $M \approx T \times I$). Now we have two cases - (a) $\mathrm{Fix}(\bar{h}^2) = \emptyset$, and (b) $\mathrm{Fix}(\bar{h}^2) \neq \emptyset$.

Case (a). $\mathrm{Fix}(\bar{h}^2) = \emptyset$, hence $\mathrm{Fix}(g) = \emptyset$ and g is weakly equivalent to $g(x, y) = (x, uy), u = e^{2\pi i/n}$. $\bar{h}^2\colon T \times I \to T \times I$ induces an involution $h'\colon T \times I/\bar{h}^2 \to T \times I/\bar{h}^2 \approx T \times I$, where h' interchanges the two sides of $T \times I$ and $\mathrm{Fix}(h') = \emptyset$. Hence

$h'(x,y,t) = (x,-y,1-t)$ (after parametrising $T \times I/\bar{h}^2 \approx T \times I$). From this it is easy to show that $\bar{h}(x,y,z) = (x,\omega y,1-t)$, $\omega = e^{\pi i/n}$. Identifying $(x,y,0)$ with $(x,-y,1)$ in $T \times I$ we get $h\colon T^3 \to T^3$ with $\mathrm{Fix}(h^i) = \emptyset$, $1 \leqslant i < n$, $\mathrm{Fix}(h^n) = T_1$. h is unique up to weak equivalence.

Case (b). $\mathrm{Fix}(\bar{h}^2) \neq \emptyset$. Hence $\mathrm{Fix}(g) \neq \emptyset$ and from Section 1, $n = 3$ and $\bar{h}^2(x,y) = (\bar{x}\bar{y},x)$, up to weak equivalence. $\bar{h}$ induces an involution $h''\colon T \times I/\bar{h}^2 \to T \times I/\bar{h}^2 \approx S^2 \times I$, such that $\mathrm{Fix}(h'') \subseteq I_1 \cup I_2 \cup I_3$ the union of three simple arcs, and $h''(I_1 \cup I_2 \cup I_3) = I_1 \cup I_2 \cup I_3$. But there is no such involution on $S^2 \times I$ with these properties. Indeed there is no involution on S^2 with an invariant three point set and fix point set consisting of two points or empty. ∎

4. Proof of Theorem 5

(4.1). $\mathrm{Fix}(h^n) = S_1 \cup S_2 \cup S_3 \cup S_4$, the union of four simple closed curves. Without loss of generality we may view h^n as given by $h^n(x,y,z) = (\bar{x},\bar{y},z)$. Let $q\colon T^3 \to T^3/h^n$ be the quotient map. Now $T^3/h^n \approx S^2 \times S^1$ and h induces a periodic map $\bar{h}\colon \left(S^2 \times S^1, \bigcup_1^4 q(S_i)\right) \to \left(S^2 \times S^1, \bigcup_1^4 q(S_i)\right)$ of period n. ∎

LEMMA 4.2. *h is equivalent to a periodic homeomorphism h_1 given by $h_1(x,y,z) = (g(x,y),\beta(z))$, where $g^n(x,y) = (\bar{x},\bar{y})$ and $\beta^n(z) = z$.*

PROOF: Let $q_1\colon T^3 \to T^3/h_1^n \approx S^2 \times S^1$. h_1 induces $\bar{h}_1\colon T^3/h_1^n \to T^3/h_1^n$ of period n, where $\bar{h}_1([x,y],z) = (\bar{g}([x,y]),\beta(z))$, where $\bar{g}\colon T/g^n \to T/g^n \approx S^2$ is the induced map by g, and $[x,y]$ is the image under the quotient map $q_2\colon T \to T/g^n$. Now $\bar{h}$ and $\bar{h}_1$ are equivalent [**1**]. Hence we can define a homeomorphism $t\colon T^3 \to T^3$ such that $th_1 = ht$ in exactly the same way as we did in Theorem 3. From this it follows that h is equivalent to h_1. ∎

(4.3). For $n = 2$, then by Section 1, $g(x,y) = (y,\bar{x})$ and $\beta(z)$ equals (a) $\bar{z}$, (b) z, (c) $-z$. In each case h is unique up to weak equivalence and is given by:

(a) $h(x,y,z) = (y,\bar{x},\bar{z})$
$\mathrm{Fix}(h) = \{(1,1,1),(1,1,-1),(-1,-1,1),(-1,-1,-1)\}$
$\mathrm{Fix}(h^2) = \{(1,1),(1,-1),(-1,1),(-1,-1)\} \times S^1$.

(b) $h(x,y,z) = (y,\bar{x},z)$
$\mathrm{Fix}(h) = \{(1,1),(-1,-1)\} \times S^1$
$\mathrm{Fix}(h^2) = \{(1,1),(-1,-1),(1,-1),(-1,1)\} \times S^1$.

(c) $h(x,y,z) = (y,\bar{x},-z)$
$\mathrm{Fix}(h) = \emptyset$
$\mathrm{Fix}(h^2) = \{(1,1),(-1,-1),(1,-1),(-1,1)\} \times S^1$.

(4.4) For $n = 3$, by Section 1, g is given by $g(x,y) = (\bar{y},xy)$ or $g(x,y) = (\bar{x},\bar{y})$.

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$\beta(x) = \omega x$, $\omega = e^{2\pi i/3}$. Hence we get the following cases. In each case h is unique up to weak equivalence.

(a) $h(x,y,z) = (\bar{y}, xy, z)$
$\mathrm{Fix}(h) = \{(1,1)\} \times S^1$
$\mathrm{Fix}(h^2) = \{(1,1),(\omega,\omega),(\omega^2,\omega^2)\} \times S^1, \quad \omega = e^{2\pi i/3}$
$\mathrm{Fix}(h^3) = \{(1,1),(-1,-1),(1,-1),(-1,1)\} \times S^1$.

(b) $h(x,y,z) = (\bar{y}, xy, \omega z), \quad \omega = e^{2\pi i/3}$
$\mathrm{Fix}(h) = \mathrm{Fix}(h^2) = \emptyset$
$\mathrm{Fix}(h^3) = \{(1,1),(-1,-1),(1,-1),(-1,1)\} \times S^1$.

(c) $h(x,y,z) = (\bar{z}, \bar{y}, \omega z), \quad \omega = e^{2\pi i/3}$
$\mathrm{Fix}(h) = \mathrm{Fix}(h^2) = \emptyset$
$\mathrm{Fix}(h^3) = \{(1,1),(-1,-1),(1,-1),(-1,1)\} \times S^1$.

(4.5). For $n > 3$ by Section 1, $g(x,y) = (\bar{x}, \bar{y})$ and n has to be odd. Hence for every odd $n > 3$ there is a unique action up to weak equivalence, and there is no action for any even $n > 3$. h is given by the following standard formula

$$h(x,y,z) = (\bar{x}, \bar{y}, \omega z), \quad \omega = e^{2\pi i/n}$$
$$\mathrm{Fix}(h^i) = \emptyset, \quad 1 \leqslant i < n$$
$$\mathrm{Fix}(h^n) = \{(1,1),(-1,-1),(1,-1),(-1,1)\} \times S^1.$$

5. Proof of Theorem 6

(5.1). $\mathrm{Fix}(h^n) = S_1 \cup S_2$, the union of two simple closed curves. Without loss of generality we may take $T^3 = T \times I/\sim \quad (x,y,0) \sim (A(x,y),1) = (-x,-y,1)$ and $h^n(x,y,t) = (\bar{x}, \bar{y}, t)$. Let $q\colon T^3 \to T^3/h^n \approx S^2 \times S^1$ be the quotient map. h induces a period n homeomorphism $\bar{h}\colon (T^3/h^n, q(S_1, \cup S_2)) \to (T^3/h^n, q(S_1 \cup S_2))$. In the same way in the proof of Lemma (4.2), h is equivalent to h_1, where $h_1(x,y,t) = (g(x,y), \beta(t))$, where $g^n(x,y) = (\bar{x}, \bar{y})$ and $\beta^n(t) = t$.

(5.2). For $n = 2$, $g(x,y) = (y, \bar{x})$ and β has three different forms. Hence we have three different cases. In each case h is unique up to weak equivalence. A standard h is given by

(a) $h([x,y,t]) = [y, \bar{x}, 1-t]$
$\mathrm{Fix}(h) = \{[1,1,\frac{1}{2}],[-1,-1,\frac{1}{2}],[1,-1,0],[-1,1,0]\}$
$\mathrm{Fix}(h^2) = \{(1,1),(1,-1),(-1,1),(-1,-1)\} \times I/\sim \approx S_1 \cup S_2$,

(b) $h([x,y,t]) = [y, \bar{x}, t]$
$\mathrm{Fix}(h) = \{(1,1),(-1,-1)\} \times I \sim \approx S_1$
$\mathrm{Fix}(h^2) = S_1 \cup S_2$,

(c) $$h([x,y,t]) = \begin{cases} [y, \bar{x}, t+\frac{1}{2}], & 0 \leqslant t \leqslant \frac{1}{2} \\ [y, \bar{x}, t-\frac{1}{2}], & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

$$\mathrm{Fix}(h) = \emptyset$$
$$\mathrm{Fix}(h^2) = S_1 \cup S_2 .$$

(5.3). For $n = 3$, then $g(x,y) = (\bar{y}, xy)$ or $g(x,y) = (\bar{x}, \bar{y})$, and $\beta(t)$ has two forms. Also we need $gA = Ag$ hence $g(x,y) = (\bar{x}, \bar{y})$ and there is a unique action up to weak equivalence which may be given by:

$$h([x,y,t]) = \begin{cases} [\bar{x}, \bar{y}, t + \frac{1}{3}], & 0 \leqslant t \leqslant \frac{2}{3} \\ [-\bar{x}, -\bar{y}, t - \frac{2}{3}], & \frac{2}{3} \leqslant t \leqslant 1 \end{cases}$$
$$\mathrm{Fix}(h) = \mathrm{Fix}(h^2) = \emptyset$$
$$\mathrm{Fix}(h^3) = S_1 \cup S_2 .$$

(5.4). For $n > 3$, $g(x,y) = (\bar{x}, \bar{y})$ and n is odd. Hence there is a unique action for every odd $n > 3$, up to weak equivalence and there is no action for any even $n > 3$. A standard h may be given by:

$$h([x,y,t]) = \begin{cases} [\bar{x}, \bar{y}, t + \frac{1}{n}], & 0 \leqslant t \leqslant \frac{n-1}{n} \\ [-\bar{x}, -\bar{y}, t - \frac{n-1}{n}, & \frac{n-1}{n} \leqslant t \leqslant 1 \end{cases}$$
$$\mathrm{Fix}(h^i) = \emptyset, \qquad 1 \leqslant i < n$$
$$\mathrm{Fix}(h^n) = S_1 \cup S_2 .$$

6. FIX(h^n) = EIGHT POINTS

(6.1). Without loss of generality h^n may be given by

$$h^n(x,y,z) = (\bar{x}, \bar{y}, \bar{z}).$$

Hence h is orientation reversing and n is odd. If there exists an invariant torus T, then h may be viewed as a product $h(x,y,z) = (g(x,y), \bar{z})$, $g^n(x,y) = (\bar{x}, \bar{y})$.

(6.2). For $n = 3$, $g(x,y) = (\bar{y}, xy)$ and h is unique up to weak equivalence. h may be given by

$$h(x,y,z) = (\bar{y}, xy, \bar{z})$$
$$\mathrm{Fix}(h) = \{(1,1,1), (1,1,-1)\}$$
$$\mathrm{Fix}(h^2) = \{(1,1), (\omega, \omega), (\omega^2, \omega^2)\} \times S^1$$
$$\mathrm{Fix}(h^3) = \text{ eight points } .$$

(6.3). For $n > 3$, the only action g on T such that $g^n(x,y) = (\bar{x}, \bar{y})$ is $g(x,y) = (\bar{x}, \bar{y})$, but then the period of h would be 2. Hence there is no such action.

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(6.4). There is a nonstandard action h which may be given by

$$\begin{aligned} h(x,y,z) &= (\bar{y},\bar{z},\bar{x}) \\ \mathrm{Fix}(h) &= \{(1,1,1),(-1,-1,-1)\} \\ \mathrm{Fix}(h^2) &= S_1 \\ \mathrm{Fix}(h^3) &= \text{ eight points }. \end{aligned}$$

Hence the proof of the case $\mathrm{Fix}(h^n)$ = eight points is not complete.

References

[1] M.J. Dunwoody, 'An equivariant sphere theorem', *Bull. London Math. Soc.* **17** (1985), 437–448.

[2] J. Hempel, *3-manifolds* (Ann. of Math. Studies, no. 86, Princeton University Press, Princeton, N.J., 1976).

[3] J. Hempel, 'Free cyclic actions on $S^1 \times S^1 \times S^1$', *Proc. Amer. Math. Soc.* **48** (1975), 221–227.

[4] K.W. Kwun and J.L. Tollefson, 'PL involutions of $S^1 \times S^1 \times S^1$', *Amer. Math. Soc.* **203** (1975), 97–106.

[5] W. Meeks and P. Scott, 'Finite group actions on 3-manifolds', (Preprint).

[6] J.H. Przytycki, 'Action of Z_n on some surface-bundles over S^1', *Colloq. Math.* **47** (1982), 221–239.

[7] D.K. Showers, *Thesis*, (Michigan State Univeristy, 1973).

[8] J.L. Tollefson, 'Involutions on $S^1 \times S^2$ and other 3-manifolds', *Trans. Amer. Math. Soc.* **183** (1973), 139–152.

[9] K. Yokoyama, 'Classification of periodic maps on compact surfaces I', *Tokyo J. Math.* **6** (1983), 75–94.

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