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ON THE NONSTANDARD DUALITY THEORY OF LOCALLY CONVEX SPACES

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This paper continues the nonstandard duality theory of locally convex, topological vector spaces begun in Section 5 of [**3**]. In Section 1, we isolate an external property, called the *pseudo monad*, that appears to be one of the central concepts of the theory (Definition 1.2). In Section 2, we relate the pseudo monad to the Fin operation. For example, it is shown that the pseudo monad of a μ-saturated subset A of $^*E$, the nonstandard model of the vector space E, is the smallest subset of A that generates $\text{Fin}(A)$ (Proposition 2.7).

The nonstandard model of a dual system of vector spaces is considered in Section 3. In this section, we use pseudo monads to establish relationships among infinitesimal polars, finite polars (see (3.1) and (3.2)) and the Fin operation (Theorem 3.7). These relationships, along with pseudo monads, are used to obtain a necessary and sufficient condition for the nonstandard hulls of a locally convex, topological vector space to be invariant (Proposition 4.4). Also in Section 4, we examine the pseudo monad of $F(\mathscr{E})$, the union monad of equicontinuous sets. We show, in Proposition 4.5, for Schwartz spaces this pseudo monad, denoted by $\hat{\mu}(F(\mathscr{E}))$, has a nice characterization. Also, we give examples that tend to support the conjecture that the characterization of Proposition 4.5 is only true for Schwartz spaces.

Preliminaries. Throughout this paper, $\mathbf{K}$ will denote either the real or complex numbers and E will symbolize an infinite dimensional vector space over $\mathbf{K}$. It is assumed that E and $\mathbf{K}$ are entities of a full set-theoretical structure

$$B_\Gamma = \{B_\sigma | \quad \sigma \in \Gamma\},$$

where Γ is the set of types. We will assume that the nonstandard structure $^*B_\Gamma$ is a higher-order, κ-saturated ultrapower of B_Γ, where κ is the cardinality of $\bigcup_{\sigma \in \Gamma} B_\Gamma$. Note that the cardinality of any entity of B_Γ is strictly less than κ. Also due to a theorem of Kenneth Kunen, κ-saturated ultrapowers exist, without the assumption of the generalized continuum hypothesis ([**1**], Theorem 10.4, page 239 and [**6**], Theorem 1.6.4, page 32).

We make the usual definitions and extensions for $^*E$ and $^*\mathbf{K}$ as found in the preliminaries of [**2**], [**3**] and [**4**]; e.g., $\mu(\mathfrak{F})$ denotes the *monad* of a filter $\mathfrak{F}$, $\mu_\theta(x)$ denotes the monad of the filter of θ-neighborhoods of $x \in E$ for topology

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θ on E, $\mu(0)$ symbolizes the set of infinitesimals of $^*\mathbf{K}$, etc. For $A \subset {}^*E$, define $\text{Fin}(A)$ as follows:

$$\text{Fin}(A) = \{z \in {}^*E| \quad \lambda z \in A \text{ for each } \lambda \in \mu(0)\}.$$

The set $\text{Fin}(A)$ is nonempty if and only if $0 \in A$. It can be shown that

(0.1) $\quad \text{Fin}(\text{Fin}(A)) = \text{Fin}(A)$.

Also it is easy to see that $\text{Fin}(\mu(0))$ is the set of finite numbers of $^*\mathbf{K}$. For further properties of $\text{Fin}(A)$, the reader is referred to [**2**].

Actually, the above definitions and extensions can be made for any entity F of B_Γ that is a vector space over $\mathbf{K}$; therefore, we make these assumptions for such an entity F without further elucidation.

1. Pseudo monads. We begin this section by illustrating one of the most useful properties of monads of sub-additive filters on vector spaces. The proof of the following proposition is essentially due to Henson and Moore (cf. [**3**], Theorem 1.6).

PROPOSITION 1.1. *Let F be an entity of B_Γ such that F is a vector space over* $\mathbf{K}$. *If $\mathfrak{F}$ is a filter on F for which $\mu(\mathfrak{F}) + \mu(\mathfrak{F}) \subset \mu(\mathfrak{F})$ then for $z \in \mu(\mathfrak{F})$ there exists an infinite $\omega \in {}^*\mathbf{K}$ such that $\omega z \in \mu(\mathfrak{F})$.*

Proof. Let $z \in \mu(\mathfrak{F})$. For each $n \in \mathbf{N}$ and $x \in \mathfrak{F}$, define the *internal set* $A(n, x)$ as follows:

$$A(n, x) = \{m| \ m \in {}^*\mathbf{N}, n < m \text{ and } mz \in {}^*X\}.$$

Since $\mu(\mathfrak{F}) + \mu(\mathfrak{F}) \subset \mu(\mathfrak{F})$ implies $m\mu(\mathfrak{F}) \subset \mu(\mathfrak{F})$ for each $m \in \mathbf{N}$, we have that each set $A(n, x)$ is nonempty. Also the collection

$$\mathscr{E} = \{A(n, x)| \ n \in \mathbf{N}, x \in \mathfrak{F}\}$$

has the finite intersection property since $\mathfrak{F}$ is a filter. By Theorem 2.7.12 of [**6**] and the saturation of $^*B_\Gamma$, there exists $\omega \in \bigcap \mathscr{E}$. We thus infer $\omega z \in \mu(\mathfrak{F})$ and ω is infinite.

The main idea of this note is to exploit subsets of $^*F$ that satisfy the conclusion of Proposition 1.1. Since most of the subsets of $^*F$ are not filter monads, we need a way of extending the conclusion of Proposition 1.1 to arbitrary subsets of $^*F$. The following definition provides such a procedure.

Definition 1.2. Let F be an entity of B_Γ such that F is a vector space over $\mathbf{K}$. For $A \subset {}^*F$ define $\hat{\mu}(A) \subset A$ as follows: $a \in \hat{\mu}(A)$ if and only if $a \in A$ and there exists an infinite $\beta \in {}^*\mathbf{K}$ such that $\beta a \in A$. The set $\hat{\mu}(A)$ is called the *pseudo monad of A*.

Thus if F and a filter $\mathfrak{F}$ on F satisfy the hypotheses of Proposition 1.1 then $\hat{\mu}(\mu(\mathfrak{F})) = \mu(\mathfrak{F})$.

Observe that $\hat{\mu}(A) \neq \emptyset$ if $0 \in A$. We shall see that $\hat{\mu}(A)$ is non trivial if A satisfies the following condition.

Definition 1.3. Let F be an entity of B_Γ that is a vector space over $\mathbf{K}$. A set $A \subset {}^*F$ is *μ-saturated* if and only if $\lambda A \subset A$ for each $\lambda \in \mu(0)$.

For the remainder of this section, all vector spaces over $\mathbf{K}$ considered are entities of B_Γ.

PROPOSITION 1.4. *Let F be a vector space over $\mathbf{K}$. If $A \subset {}^*F$ is μ-saturated and has a non zero element then $\hat{\mu}(A)$ has a non zero element.*

Proof. Let $z \in A$ such that $z \neq 0$, let $\lambda_0 \in \mu(0)$ for which $\lambda_0 \neq 0$ and consider $z_0 = \lambda_0 z$. Note that $z_0 \neq 0$ and $z_0 \in A$. Also $\lambda_0 \in \mu(0)$ implies $\lambda_0^{2/3} \in \mu(0)$ and $\lambda_0^{-1/3}$ is infinite. Hence

$$\lambda_0^{-1/3} z_0 = \lambda_0^{-1/3}(\lambda_0 z) = \lambda_0^{2/3} z \in A$$

since A is μ-saturated. Therefore, z_0 is a non zero element of $\hat{\mu}(A)$ since $z_0 \in A$, $\lambda_0^{-1/3} z_0 \in A$ and $\lambda_0^{-1/3}$ is infinite.

PROPOSITION 1.5. *Let F be a vector space over $\mathbf{K}$ and let $A \subset {}^*F$ be μ-saturated. If $z \in \hat{\mu}(A)$ then there exists a positive, infinite $\beta \in {}^*\mathbf{R}$ for which $\alpha z \in \hat{\mu}(A)$ for $\alpha \in {}^*\mathbf{K}$ such that $|\alpha| \leqq \beta$.*

Proof. Let $z \in \hat{\mu}(A)$ which implies $z \in A$ and there exists an infinite $\beta_1 \in {}^*\mathbf{K}$ such that $\beta_1 z \in A$. Now, β_1 being infinite implies β_1^{-1} and $\beta_1^{-2/3}$ are infinitesimals. Thus, there exist $\omega_1, \omega_2 \in {}^*\mathbf{N}\backslash\mathbf{N}$ for which $\omega_1\beta_1^{-1}$ and $\omega_2\beta_1^{-2/3}$ are infinitesimals. Let $\beta = \min\{\omega_1, \omega_2\}$. Hence β is a positive, infinite element of ${}^*\mathbf{R}$.

Now, consider $\alpha \in {}^*\mathbf{K}$ for which $|\alpha| \leqq \beta$. Since

$$|\alpha\beta_1^{-1}| \leqq \beta|\beta_1^{-1}| \leqq |\omega_1\beta_1^{-1}|$$

we have $\alpha\beta_1^{-1} \in \mu(0)$. Similarly,

$$|\alpha\beta_1^{-2/3}| \leqq \beta|\beta_1^{-2/3}| \leqq |\omega_2\beta_1^{-2/3}|$$

implies $\alpha\beta_1^{-2/3} \in \mu(0)$.

Therefore, if $\alpha \in {}^*\mathbf{K}$ such that $|\alpha| \leqq \beta$ then

$$\alpha z = \alpha\beta_1^{-1}(\beta_1 z) \in A$$

since $\beta_1 z \in A$, A is μ-saturated and $\alpha\beta_1^{-1} \in \mu(0)$. Also,

$$\beta_1^{1/3}(\alpha z) = \beta_1^{1/3}\alpha\beta_1^{-1}(\beta_1 z) = \alpha\beta_1^{-2/3}(\beta_1 z) \in A$$

since $\alpha\beta_1^{-2/3} \in \mu(0)$. We thus infer that $\alpha z \in \hat{\mu}(A)$ since β_1 being infinite implies $\beta_1^{1/3}$ is infinite.

PROPOSITION 1.6. *Let F be a vector space over $\mathbf{K}$. If A and B are μ-saturated subsets of ${}^*F$ then $\hat{\mu}(A \cap B) = \hat{\mu}(A) \cap \hat{\mu}(B)$.*

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Proof. Clearly $\hat{\mu}(A \cap B) \subset \hat{\mu}(A) \cap \hat{\mu}(B)$. Let $z \in \hat{\mu}(A) \cap \hat{\mu}(B)$ which implies $z \in A \cap B$ and there exist positive, infinite $\beta_1, \beta_2 \in {}^*\mathbf{R}$ such that $\alpha z \in A$ for $|\alpha| \leqq \beta_1$ and $\xi z \in B$ for $|\xi| \leqq \beta_2$ by Proposition 1.5. Let $\beta = \min\{\beta_1, \beta_2\}$. Consequently, $\beta z \in A \cap B$ which implies $z \in \hat{\mu}(A \cap B)$ since $\beta \in {}^*\mathbf{K}$ is infinite. Therefore $\hat{\mu}(A) \cap \hat{\mu}(B) \subset \hat{\mu}(A \cap B)$.

Using Propositions 1.1 and 1.5, we can give conditions for a filter, in a vector space, to have a basis of balanced sets.

PROPOSITION 1.7. *Let F be a vector space over $\mathbf{K}$ and let $\mathfrak{F}$ be a filter on F for which $\mu(\mathfrak{F}) + \mu(\mathfrak{F}) \subset \mu(\mathfrak{F})$. If $\mu(\mathfrak{F})$ is μ-saturated then $\mathfrak{F}$ has a filter basis of balanced sets.*

Proof. Let $z \in \mu(\mathfrak{F})$ and let $\lambda \in {}^*\mathbf{K}$ such that $|\lambda| \leqq 1$. Proposition 1.1 implies $\hat{\mu}(\mu(\mathfrak{F})) = \mu(\mathfrak{F})$; therefore, there exists a positive, infinite $\beta \in {}^*\mathbf{R}$ such that $\alpha z \in \mu(\mathfrak{F})$ for $|\alpha| \leqq \beta$ by Proposition 1.5. In particular, $\lambda z \in \mu(\mathfrak{F})$ since $|\lambda| \leqq 1 < \beta$. Consequently $\mu(\mathfrak{F})$ is $*$-balanced which implies $\mathfrak{F}$ has a filter basis of balanced sets ([**2**], Proposition 2.6).

From the above proposition, we derive the following standard results.

PROPOSITION 1.8. *Let F be a vector space over $\mathbf{K}$ and let $\varphi : \mathbf{K} \times F \rightarrow F$ denote the scalar multiplication map. Let θ be a topology on F for which vector addition is continuous. If φ is continuous at $(0, 0)$ then the map $x \rightarrow \lambda x$ is θ-continuous on F for each $\lambda \in \mathbf{K}$ such that $|\lambda| \leqq 1$.*

Proof. Let $\mathcal{N}_\theta(0)$ denote the filter of θ-neighborhoods of $0 \in F$ and let $\mu_\theta(0)$ denote the monad of $\mathcal{N}_\theta(0)$. By the continuity of vector addition, we have that for $V \in \mathcal{N}_\theta(0)$ there exists $W \in \mathcal{N}_\theta(0)$ such that $W + W \subset V$; therefore, $\mu_\theta(0) = \mu_\theta(0) + \mu_\theta(0)$ ([**2**], Proposition 2.8). Also, Theorem 4.2.7 of [**7**] and the continuity of φ at $(0, 0)$ imply ${}^*\varphi[\mu(0) \times \mu_\theta(0)] \subset \mu_\theta(0)$; i.e., $\mu_\theta(0)$ is μ-saturated. Consequently, there exists a filter basis $\mathcal{E} \subset \mathcal{N}_\theta(0)$ such that each $V \in \mathcal{E}$ is balanced by Proposition 1.7. Since $\lambda(x + V) = \lambda x + \lambda V \subset \lambda x + V$ for $x \in F$, $V \in \mathcal{E}$ and $\lambda \in \mathbf{K}$ such that $|\lambda| \leqq 1$, we infer that the map $x \rightarrow \lambda x$ is θ-continuous for each $\lambda \in \mathbf{K}$ such that $|\lambda| \leqq 1$.

COROLLARY 1.9. *If θ and F satisfy the hypotheses of Proposition 1.8, then (F, θ) is a topological group with respect to vector addition.*

Proof. Apply Proposition 1.8 with $\lambda = -1$.

PROPOSITION 1.10. *Let θ and F satisfy the hypotheses of Proposition 1.8 and let $\mathcal{N}_\theta(0)$ denote the filter of θ-neighborhoods of $0 \in F$. If each $V \in \mathcal{N}_\theta(0)$ is absorbing then (F, θ) is a topological vector space.*

Proof. Arguing in the manner of Proposition 1.8, we infer from Proposition 1.7 the existence of a filter basis $\mathcal{E} \subset \mathcal{N}_\theta(0)$ of balanced sets. Therefore (F, θ)

is a topological vector space since the members of $\mathscr{E}$ are absorbing and, for $V \in \mathscr{E}$, there exists $W \in \mathscr{E}$ such that $W + W \subset V$ ([**5**], Theorem 1, page 81).

2. Pseudo monads and the Fin operation. In this section, we establish the relationship between the pseudo monad and the Fin operation. The objective is to show that $\hat{\mu}(A)$ is the smallest subset of a μ-saturated set A that generates $\text{Fin}(A)$.

PROPOSITION 2.1. *Let F be a vector space over* **K**. *If $A \subset {}^*F$ such that $0 \in A$ then*

$$\text{Fin}(A) = \text{Fin}(\hat{\mu}(A)).$$

Proof. Since $\hat{\mu}(A) \subset A$ implies $\text{Fin}(\hat{\mu}(A)) \subset \text{Fin}(A)$, it suffices to show that $\text{Fin}(A) \subset \text{Fin}(\hat{\mu}(A))$.

Let $z \in \text{Fin}(A)$ and let $\lambda \in \mu(0)$. There exists an infinite $\beta \in {}^*\mathbf{K}$ such that $\beta\lambda \in \mu(0)$. Hence $\lambda z \in A$ and $\beta(\lambda z) = (\beta\lambda)z \in A$. Therefore $\lambda z \in \hat{\mu}(A)$. We thus infer that $z \in \text{Fin}(\hat{\mu}(A))$ since $\lambda \in \mu(0)$ was arbitrary.

PROPOSITION 2.2. *Let F be a vector space over* **K**. *If $A \subset {}^*F$ such that $0 \in A$ then*

$$\hat{\mu}(\text{Fin}(\text{A})) \subset \hat{\mu}(A).$$

Proof. Let $z \in \hat{\mu}(\text{Fin}(A))$. Hence $z \in \text{Fin}(A)$ and there exists an infinite $\beta \in {}^*\mathbf{K}$ for which $\beta z \in \text{Fin}(A)$. Now, $\beta z \in \text{Fin}(A)$ implies $\lambda(\beta z) \in A$ for each $\lambda \in \mu(0)$. In particular $z = \beta^{-1}(\beta z) \in A$ since $\beta^{-1} \in \mu(0)$. Also

$$\beta^{-1/3} \in \mu(0)$$

which implies $\beta^{2/3}z = \beta^{-1/3}(\beta z) \in A$. Consequently, $z \in \hat{\mu}(A)$.

COROLLARY 2.3. *Let F be a vector space over* **K**. *If $A \subset {}^*F$ is μ-saturated then*

$$\hat{\mu}(\text{Fin}(A)) = \hat{\mu}(A).$$

Proof. A being μ-saturated implies $A \subset \text{Fin}(A)$ ([**2**], Proposition 1.3); therefore, $\hat{\mu}(A) \subset \hat{\mu}(\text{Fin}(A))$. Proposition 2.2 implies $\hat{\mu}(\text{Fin}(A)) \subset \hat{\mu}(A)$.

COROLLARY 2.4. *Let F be a vector space over* **K**. *If $A \subset {}^*F$ is μ-saturated then $\hat{\mu}(\hat{\mu}(A)) = \hat{\mu}(A)$.*

Proof. A being μ-saturated implies $0 \in \hat{\mu}(A)$; therefore,

$$\hat{\mu}(A) = \hat{\mu}(\text{Fin}(A)) = \hat{\mu}(\text{Fin}(\hat{\mu}(A))) \subset \hat{\mu}(\hat{\mu}(A))$$

by Corollary 2.3 and Propositions 2.1 and 2.2. By definition, $\hat{\mu}(\hat{\mu}(A)) \subset \hat{\mu}(A)$; therefore, $\hat{\mu}(A) = \hat{\mu}(\hat{\mu}(A))$.

Now we are in a position to show that $\hat{\mu}(A)$ is the smallest subset of a μ-saturated set A that generates $\text{Fin}(A)$.

LEMMA 2.5. *Let F be a vector space over* $\mathbf{K}$ *and let* $A \subset {}^*F$ *be* μ*-saturated. If* $B \subset {}^*F$ *such that* $\mathrm{Fin}(A) \subset \mathrm{Fin}(B)$, *then* $\hat{\mu}(A) \subset \hat{\mu}(B)$.

Proof. A being μ-saturated implies $0 \in A \subset \mathrm{Fin}(A)$; therefore, $\mathrm{Fin}(A) \subset \mathrm{Fin}(B)$ implies $0 \in \mathrm{Fin}(B)$ which implies $0 \in B$. Consequently,

$$\hat{\mu}(A) = \hat{\mu}(\mathrm{Fin}(A)) \subset \hat{\mu}(\mathrm{Fin}(B)) \subset \hat{\mu}(B)$$

by Proposition 2.2 and Corollary 2.3.

PROPOSITION 2.6. *Let F be a vector space over* $\mathbf{K}$. *If A and B are* μ*-saturated subsets of* ${}^*F$ *for which* $\mathrm{Fin}(A) = \mathrm{Fin}(B)$ *then* $\hat{\mu}(A) = \hat{\mu}(B)$.

Proof. $\mathrm{Fin}(A) \subset \mathrm{Fin}(B)$ implies $\hat{\mu}(A) \subset \hat{\mu}(B)$ by Lemma 2.5. Conversely, $\mathrm{Fin}(B) \subset \mathrm{Fin}(A)$ implies $\hat{\mu}(B) \subset \hat{\mu}(A)$ again by Lemma 2.5. Therefore $\hat{\mu}(A) = \hat{\mu}(B)$.

PROPOSITION 2.7. *Let F be a vector space over* $\mathbf{K}$ *and let* $A \subset {}^*F$ *be* μ*-saturated. If* $B \subset {}^*F$ *for which* $\mathrm{Fin}(B) = \mathrm{Fin}(A)$ *then* $\hat{\mu}(A) \subset B$.

Proof. $\mathrm{Fin}(A) \subset \mathrm{Fin}(B)$ implies $\hat{\mu}(A) \subset \hat{\mu}(B) \subset B$ by Lemma 2.5.

Example 2.8. We now exhibit a Fin invariant set of infinitesimals that agrees with its own pseudo monad.

Let $I = [0, 1]$, let $F = \mathbf{K}^I$ and define

$$\mathfrak{A} = \{x \in F \mid 0 < x(i) \text{ for each } i \in I\}.$$

Consider an infinitesimal $\iota \in {}^*\mathbf{R}$ such that $0 < \iota$ which implies $\iota \in {}^*I$. Following the notation of [**2**] we define

$$\nu_\iota(\mathfrak{A}) = \{\lambda \in {}^*\mathbf{K} \mid |\lambda| \leqq {}^*x(\iota) \text{ for each } x \in \mathfrak{A}\}.$$

Since $\{{}^*x \mid x \in \mathfrak{A}\}$ is an external subset of some *-finite subset of ${}^*\mathfrak{A}$, it can be shown that $\nu_\iota(\mathfrak{A})$ is non trivial, i.e., $\nu_\iota(\mathfrak{A}) \neq \{0\}$ (see [**6**], Example 1.5.3). By Proposition 4.9 of [**2**], $\nu_\iota(\mathfrak{A})$ is Fin invariant, i.e., $\nu_\iota(\mathfrak{A}) = \mathrm{Fin}(\nu_\iota(\mathfrak{A}))$, since $\mathfrak{A}$ satisfies Definition 4.4 of [**2**] (see [**2**], Theorem 6.6).

Consider $\lambda_0 \in \nu_\iota(\mathfrak{A})$. For $x \in \mathfrak{A}$ and $n \in \mathbf{N}$, define $D(x, n) \subset {}^*\mathbf{N}$ as follows:

$$D(x, n) = \{m \in {}^*\mathbf{N} \mid |m\lambda_0| \leqq {}^*x(\iota) \text{ and } n < m\}.$$

Now, $x \in \mathfrak{A}$ implies $m^{-1}x \in \mathfrak{A}$ for each $m \in \mathbf{N}$ which implies $|\lambda_0| \leqq m^{-1}{}^*x(\iota)$ for each $m \in \mathbf{N}$; therefore, $\mathscr{D} = \{D(x, n) \mid x \in \mathfrak{A}, n \in \mathbf{N}\}$ is a collection of non empty internal subsets of ${}^*\mathbf{N}$. Furthermore, we infer that $\mathscr{D}$ has the finite intersection property since $\{x_1, \ldots, x_j\} \subset \mathfrak{A}$ implies $x = x_1 \wedge \ldots \wedge x_j \in \mathfrak{A}$. By Theorem 2.7.12 of [**6**] and the saturation of ${}^*B_\Gamma$, there exists $\omega \in \bigcap \mathscr{D}$. Consequently, ω is infinite and $\omega\lambda_0 \in \nu_\iota(\mathfrak{A})$. Therefore, $\hat{\mu}(\nu_\iota(\mathfrak{A})) = \nu_\iota(\mathfrak{A})$ since $\lambda_0 \in \nu_\iota(\mathfrak{A})$ was arbitrary.

3. Pseudo monads and dual systems. Let F be a vector space over $\mathbf{K}$ and let $\langle \ldots, \ldots \rangle$ be a bilinear functional on $E \times F$. The collection $(E, F, \langle \ldots, \ldots \rangle)$ is said to form a *dual system* if and only if the following conditions are satisfied:

(i) If $x \in E$ and $x \neq 0$ then $\langle x, y \rangle \neq 0$ for some $y \in F$, and
(ii) If $y \in F$ and $y \neq 0$ then $\langle x, y \rangle \neq 0$ for some $x \in E$.

Throughout this section, it will be assumed that the bilinear functional $\langle \ldots, \ldots \rangle$ and the vector space E and F form a dual system.

Let A be a subset of $*E$ (internal or external). Following the notation of [**3**] we define

$$(3.1) \qquad A^i = \{q \in *F | \langle p, q, \rangle \in \mu(0) \text{ for all } p \in A\},$$

$$(3.2) \qquad A^f = \{q \in *F | \langle p, q \rangle \in \mathrm{Fin}(\mu(0)) \text{ for all } p \in A\}.$$

We define A^i and A^f similarly for $A \subset *F$. Also we denote $(A^i)^i$ by A^{ii} and $(A^f)^f$ by A^{ff}. Immediately from the definition we derive

$$(3.3) \qquad (A \cup B)^i = A^i \cap B^i \quad \text{and} \quad (A \cup B)^f = A^f \cap B^f$$

for subsets A, B of either $*E$ or $*F$. Also we will make use of the properties of A^i and A^f listed in Lemma 5.5 of [**3**].

LEMMA 3.4. *Let A and B be subsets of G, where G is either $*E$ or $*F$.*
(a) *If $A^{ii} = A$ and $B^{ii} = B$ then $(A \cap B)^{ii} = A \cap B$.*
(b) *If $A^{ff} = A$ and $B^{ff} = B$ then $(A \cap B)^{ff} = A \cap B$.*

Proof. $A \cap B = A^{ii} \cap B^{ii} = (A^i \cup B^i)^i$ by (3.3); therefore,

$$(A \cap B)^{ii} = (A^i \cup B^i)^{iii} = (A^i \cup B^i)^i = A \cap B$$

by Lemma 5.5 (vii) of [**3**]. A similar argument proves (b).

We will say that a topology θ on E is *compatible with the dual system* $(E, F, \langle \ldots, \ldots \rangle)$ if and only if θ is a Hausdorff, locally convex linear topology and $\sigma(E, F) \subset \theta \subset \tau(E, F)$, where $\sigma(E, F)$ and $\tau(E, F)$ are respectively the weak and Mackey topologies on E generated by F and $\langle \ldots, \ldots \rangle$ (see [**5**], Proposition 4, page 206).

In [**3**], Henson and Moore showed that if θ is a linear topology on E compatible with the dual system then

$$(3.5) \qquad [\mu_\theta(0)]^i = F(\mathscr{E}), \quad [F(\mathscr{E})]^i = \mu_\theta(0)$$

$$(3.6) \qquad (m_\theta)^i = \mathrm{Fin}(\mu_\theta(0)) \quad \text{and} \quad F(\mathscr{E}) = \mathrm{Fin}(m_\theta)$$

where $m_\theta = [\mathrm{Fin}(\mu_\theta(0))]^i$, $\mathscr{E}$ is the collection of all θ-equicontinuous subsets of F and $F(\mathscr{E}) = \bigcup \{*A | A \in \mathscr{E}\}$ (see [**3**], Theorems 5.8, 5.9 and 5.11). The central concept in the above results is the fact that $\mu_\theta(0)$ is a *filter monad*. In the following theorem, we extend Henson and Moore's results to pseudo monads.

THEOREM 3.7. *If* $A \subset {}^*E$ *such that* $\hat{\mu}(A) = A$ *then*:
(a) $A^{i} = \mathrm{Fin}([\mathrm{Fin}(A^{ii})]^{i}) = A^{f}$,
(b) $\mathrm{Fin}(A^{ii}) = [\mathrm{Fin}(A^{ii})]^{ii}$,
(c) $A^{i} = [\mathrm{Fin}(A^{ii})]^{f}$,
(d) $A^{if} = \mathrm{Fin}(A^{ii}) = [\mathrm{Fin}(A^{ii})]^{ff}$ *and*
(e) $\mathrm{Fin}(A^{ii}) = [\mathrm{Fin}(A^{ii})]^{if}$.

Proof. If $z \in \mathrm{Fin}([\mathrm{Fin}(A^{ii})]^{i})$ then $\lambda z \in [\mathrm{Fin}(A^{ii})]^{i}$ for each $\lambda \in \mu(0)$ which implies $\langle q, \lambda z \rangle \in \mu(0)$ for $\lambda \in \mu(0)$ and $q \in \mathrm{Fin}(A^{ii})$.

Let $z \in \mathrm{Fin}([\mathrm{Fin}(A^{ii})]^{i})$ and consider an arbitrary $p \in A$. $A = \hat{\mu}(A)$ implies the existence of an infinite $\beta p \in A$. Consequently,

$$\langle p, z \rangle = \langle \beta^{-1}\beta p, z \rangle = \langle \beta p, \beta^{-1} z \rangle \in \mu(0)$$

since $\beta^{-1} \in \mu(0)$ and $A^{ii} \subset \mathrm{Fin}(A^{ii})$, i.e., $\beta p \in \mathrm{Fin}(A^{ii})$ and $\beta^{-1} z \in [\mathrm{Fin}(A^{ii})]^{i}$. Therefore $z \in A^{i}$.

Let $z \in A^{i}$ and consider an arbitrary $q \in \mathrm{Fin}(A^{ii})$. If $\lambda \in \mu(0)$ then $\langle q, \lambda z \rangle = \langle \lambda q, z \rangle \in \mu(0)$ since $\lambda q \in A^{ii}$. So, $\lambda z \in [\mathrm{Fin}(A^{ii})]^{i}$ for each $\lambda \in \mu(0)$; thus, $z \in \mathrm{Fin}([\mathrm{Fin}(A^{ii})]^{i})$. Therefore, $A^{i} = \mathrm{Fin}([\mathrm{Fin}(A^{ii})]^{i})$.

By Lemma 5.7 of [**3**], we have $\mathrm{Fin}(A^{i}) = A^{f}$. Consequently,

$$A^{f} = \mathrm{Fin}(A^{i}) = \mathrm{Fin}(\mathrm{Fin}([\mathrm{Fin}(A^{ii})]^{i})) = \mathrm{Fin}([\mathrm{Fin}(A^{ii})]^{i}) = A^{i}$$

by (0.1). Hence, (a) is established.

Let $q \in [\mathrm{Fin}(A^{ii})]^{ii}$ and let $\lambda \in \mu(0)$. If $z \in A^{i}$ then $\lambda z \in [\mathrm{Fin}(A^{ii})]^{i}$, by (a), which implies $\langle \lambda q, z \rangle = \langle q, \lambda z \rangle \in \mu(0)$ since $[\mathrm{Fin}(A^{ii})]^{iii} = [\mathrm{Fin}(A^{ii})]^{i}$ ([**3**], Lemma 5.5(vii)). Consequently $\lambda q \in A^{ii}$ which implies $q \in \mathrm{Fin}(A^{ii})$ since $\lambda \in \mu(0)$ was arbitrary. We infer $\mathrm{Fin}(A^{ii}) = [\mathrm{Fin}(A^{ii})]^{ii}$ which establishes (b) since $\mathrm{Fin}(A^{ii}) \subset [\mathrm{Fin}(A^{ii})]^{ii}$ ([**3**], Lemma 5.5(vi)).

Let $z \in A^{i}$ and let $q \in \mathrm{Fin}(A^{ii})$ be arbitrary. For $\lambda \in \mu(0)$ we have $\langle q, \lambda z \rangle = \langle \lambda q, z \rangle \in \mu(0)$ since $\lambda q \in A^{ii}$; consequently, $\langle q, z \rangle \in \mathrm{Fin}(\mu(0))$ which implies $z \in [\mathrm{Fin}(A^{ii})]^{f}$.

Let $z \in [\mathrm{Fin}(A^{ii})]^{f}$ and let $p \in A$ be arbitrary. Since $\hat{\mu}(A) = A \subset A^{ii} \subset \mathrm{Fin}(A^{ii})$, we have that there exists an infinite $\beta \in {}^*\mathbf{K}$ such that $\beta p \in A$ which implies $\beta p \in \mathrm{Fin}(A^{ii})$. So,

$$\langle p, z \rangle = \langle \beta^{-1}\beta p, z \rangle = \beta^{-1} \langle \beta p, z \rangle \in \mu(0)$$

since $\beta^{-1} \in \mu(0)$ and $\langle \beta p, z \rangle \in \mathrm{Fin}(\mu(0))$. Consequently $z \in A^{i}$; therefore, (c) is established.

Let $q \in A^{if}$ and let $\lambda \in \mu(0)$. For $z \in A^{i}$ we have $\langle \lambda q, z \rangle = \lambda \langle q, z \rangle \in \mu(0)$ since $\langle q, z \rangle \in \mathrm{Fin}(\mu(0))$. Hence $\lambda q \in A^{ii}$ which implies $q \in \mathrm{Fin}(A^{ii})$ since $\lambda \in \mu(0)$ was arbitrary.

Let $q \in \mathrm{Fin}(A^{ii})$ which implies $\lambda q \in A^{ii}$ for each $\lambda \in \mu(0)$. If $z \in A^{i}$ then

$\lambda\langle q, z\rangle = \langle \lambda q, z\rangle \in \mu(0)$ for each $\lambda \in \mu(0)$ which implies $\langle q, z\rangle \in \mathrm{Fin}(\mu(0))$. Hence $q \in A^{if}$; therefore, $A^{if} = \mathrm{Fin}(A^{ii})$.

Also $A^{if} = [\mathrm{Fin}(A^{ii})]^{ff}$ since $[\mathrm{Fin}(A^{ii})]^{f} = A^{i}$ from (c). Consequently (d) is established. Finally, we obtain (e) by observing

$$[\mathrm{Fin}(A^{ii})]^{if} = \mathrm{Fin}([\mathrm{Fin}(A^{ii})]^{ii}) = \mathrm{Fin}(\mathrm{Fin}(A^{ii})) = \mathrm{Fin}(A^{ii})$$

from Lemma 5.7 of [**3**], part (b) and (0.1).

Remark. In figure 1, we summarize the results of Theorem 3.7 with a flow diagram. Observe that once (3.5) is established, (3.6) and the other results of Theorems 5.8, 5.9 and 5.11 of [**3**] are obtained from Proposition 1.1 and Theorem 3.7.

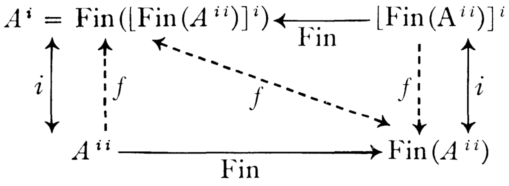

FIGURE 1. A flow diagram of Theorem 3.7.

In the next proposition and corollary, we relate the pseudo monad of A^{i} to $\mathrm{Fin}(A^{ii})$.

PROPOSITION 3.8. *Let $A \subset {}^*E$ such that $\hat{\mu}(A) = A$. If $B = \mathrm{Fin}(A^{ii})$ then $[\hat{\mu}(B^{i})]^{i} = B$.*

Proof. Assume that $x_0 \in [\hat{\mu}(B^{i})]^{i}$ such that $x_0 \notin B = \mathrm{Fin}(A^{ii})$. Hence, there exists $\lambda_0 \in \mu(0)$ for which $\lambda_0 x_0 \notin A^{ii}$. Thus, there exists $z_0 \in A^{i}$ for which

$$\langle \lambda_0 x_0, z_0\rangle \notin \mu(0). \tag{3.9}$$

Now, $A^{i} = \mathrm{Fin}(B^{i})$ by Theorem 3.7(a); therefore, $A^{i} = \mathrm{Fin}(\hat{\mu}(B^{i}))$ by Proposition 2.1. Thus, $z_0 \in A^{i}$ and $\lambda_0 \in \mu(0)$ imply $\lambda_0 z_0 \in \hat{\mu}(B^{i})$ which implies $\langle x_0, \lambda_0 z_0\rangle \in \mu(0)$, since $x_0 \in [\hat{\mu}(B^{i})]^{i}$, contradicting $\langle x_0, \lambda_0 z_0\rangle = \langle \lambda_0 x_0, z_0\rangle \notin \mu(0)$. We thus infer $[\hat{\mu}(B^{i})]^{i} \subset B$.

By Theorem 3.7(b) we have $B^{ii} = B$; therefore, $\hat{\mu}(B^{i}) \subset B^{i}$ implies $B = B^{ii} \subset [\hat{\mu}(B^{i})]^{i}$ (see [**3**], Lemma 5.5(iv)).

Consequently, $[\hat{\mu}(B^{i})]^{i} = B$.

COROLLARY 3.10. *If $A \subset {}^*E$ such that $\hat{\mu}(A) = A$ then $[\hat{\mu}(A^{i})]^{i} = \mathrm{Fin}(A^{ii})$.*

Proof. Observe that $\hat{\mu}(A^{i}) = \hat{\mu}([\mathrm{Fin}(A^{ii})]^{i})$ by Theorem 3.7(a) and Corollary 2.3 since $[\mathrm{Fin}(A^{ii})]^{i}$ is μ-saturated by Lemma 5.5(ii) of [**3**]. Therefore, $[\hat{\mu}(A^{i})]^{i} = \mathrm{Fin}(A^{ii})$ by Proposition 3.8.

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4. Applications. Let θ be a compatible topology on E, let $\mathscr{E}$ denote the collection of θ-equicontinuous subsets of F and let

$$F(\mathscr{E}) = \cup \{{}^*A \mid A \in \mathscr{E}\}.$$

From (3.5), we infer $[F(\mathscr{E})]^{ii} = F(\mathscr{E})$ and $[\mu_{\sigma(F,E)}(0)]^{ii} = \mu_{\sigma(F,E)}(0)$; consequently,

$$F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0) = [F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0)]^{ii}$$

by Lemma 3.4(a).

From the fact that $\text{pns}_\theta({}^*E) \subset \text{Fin}\ (\mu_\theta(0))$, we have

$$[\hat{\mu}(F(\mathscr{E}))]^{ii} = [\text{Fin}\ (\mu_\theta(0))]^i \subset [\text{pns}_\theta({}^*E)]^i$$
$$= [F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0)]^{ii} = F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0)$$

by Corollary 3.10, Lemma 5.5 (iv) of [**3**] and Theorem 6.1 of [**3**]. Therefore, we have

$$(4.1) \quad \hat{\mu}(F(\mathscr{E})) \subset [\hat{\mu}(F(\mathscr{E}))]^{ii} \subset F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0)$$

(see [**3**], Lemma 5.5 (vi)).

In this section, we give a sufficient condition for the sets of (4.1) to coincide (Proposition 4.5) and we exhibit examples for each of the three cases when the sets of (4.1) do not coincide (see Examples 5.1, 5.2 and 5.3).

For a balanced convex, $\sigma(F, E)$-compact subset A of F, let

$$F_A = \cup \{nA \mid n \in N_+\}.$$

Consequently, F_A is a vector subspace of F. Let g_A be the gauge (Minkowski functional) of A on F_A. Hence g_A is a norm on F_A, since $\sigma(F, E)$ is Hausdorff. Let τ_A denote the normed topology on F_A generated by g_A. The notation $\mu_A(0)$ will denote the monad of the filter, in F_A, of τ_A-neighborhoods of $0 \in F_A$. Thus,

$$(4.2) \quad \mu_A(0) \subset {}^*A \subset {}^*F_A \quad \text{and}$$
$$z \in \mu_A(0) \text{ if and only if } {}^*g_A(z) \in \mu(0).$$

Our first proposition of this section gives the structure of $\hat{\mu}(F(\mathscr{E}))$ when $\mathscr{E}$ is the collection of θ-equicontinuous subsets of F for a compatible topology θ on E.

PROPOSITION 4.3. *Let θ be a compatible topology on E. If $\mathscr{E}_0$ is the collection of all $\sigma(F, E)$-closed, balanced convex θ-equicontinuous subsets of F, then*

$$\hat{\mu}(F(\mathscr{E})) = \cup \{\mu_A(0) \mid A \in \mathscr{E}_0\}$$
$$= \cup \{({}^*V)^i \mid V \in \mathscr{N}_\theta(0)\}$$

where $\mathscr{N}_\theta(0)$ is the filter of θ-neighborhoods of $0 \in E$.

Proof. If $z \in \mu_A(0)$ then there exists an infinite $\lambda \in {}^*\mathbf{K}$ such that $\lambda z \in \mu_A(0)$ by Proposition 1.1. Thus, from (4.2), we infer $\cup \{\mu_A(0) \mid A \in \mathscr{E}_0\} \subset \hat{\mu}(F(\mathscr{E}))$.

Now, consider $z \in \hat{\mu}(F(\mathscr{E}))$ which implies $z \in F(\mathscr{E})$ and $\lambda z \in F(\mathscr{E})$ for some infinite $\lambda \in {}^*\mathbf{K}$. Consequently, there exist A_1, $A_2 \in \mathscr{E}$ such that $z \in {}^*A$, and $\lambda z \in {}^*A_2$. Let A be the bipolar of $A_1 \cup A_2$, i.e., $A = (A_1 \cup A_2)^{00}$; therefore, $A \in \mathscr{E}_0$ ([**5**], Theorem 1, p. 192). Also, $z \in {}^*A$ and $\lambda z \in {}^*A$.

If $p \in \mathbf{K}$ such that $p > 0$ then ${}^*g_A(z) < p$. Indeed, if $g_A(z) \geqq p$ then

$$ {}^*g_A(\lambda z) = |\lambda| {}^*g_A(z) \geqq |\lambda| p $$

which would imply ${}^*g_A(\lambda z)$ is infinite, contradicting $\lambda z \in {}^*A$. Hence $z \in \mu_A(0)$. We conclude $\hat{\mu}(F(\mathscr{E})) = \cup \{\mu_A(0) | A \in \mathscr{E}_0\}$.

Now, consider $V \in \mathscr{N}_\theta(0)$ and let $A = V^0$ which implies $A \in \mathscr{E}_0$ ([**5**], Proposition 1(e), p. 190). For $z \in \mu_A(0)$ and $n \in \mathbf{N}_+$ we have $z \in {}^*(n^{-1}A) = {}^*[(nV)^0]$ which implies $|\langle v, z \rangle| \leqq n^{-1}$ for each $v \in {}^*V$. Thus, we infer $\mu_A(0) \subset ({}^*V)^i$. If $z \in ({}^*V)^i$ then $\langle v, z \rangle \in \mu(0)$ for each $v \in {}^*V$ which implies $|\langle v, z \rangle| \leqq n^{-1}$ for each $v \in {}^*V$ and each $n \in \mathbf{N}_+$. Hence $z \in {}^*(n^{-1}V^0) = {}^*(n^{-1}A)$ for each $n \in \mathbf{N}_+$.

We conclude $\mu_A(0) = ({}^*V)^i$. Consequently,

$$ \cup \{\mu_A(0) | A \in \mathscr{E}_0\} = \cup \{({}^*V)^i | V \in \mathscr{N}_\theta(0)\}. $$

Using the latter two sets of (4.1), we now give a necessary and sufficient condition for a compatible topology θ on E to have invariant nonstandard hulls. Essentially, the condition can be considered as the dual statement of Lemma 1(v) of [**4**].

PROPOSITION 4.4. *Let θ be a compatible topology on E and let $\mathscr{E}$ be the collection of θ-equicontinuous subsets of F. (E, θ) has invariant nonstandard hulls if and only if*

$$ [\hat{\mu}(F(\mathscr{E}))]^{ii} = F(\mathscr{E}) \cap \mu_{\theta(F,E)}(0). $$

Proof. Assume (E, θ) has invariant nonstandard hulls. By Lemma 1(i and v) of [**4**], we have $\text{Fin}\ (\mu_\theta(0)) = \text{pns}_\theta({}^*E)$; therefore, $\text{pns}_\theta({}^*E) = [\hat{\mu}(F(\mathscr{E}))]^i$ by Corollary 3.10 since $\mu_\theta(0) = [\mu_\theta(0)]^{ii}$. Now, $[F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0)]^i = \text{pns}_\theta({}^*E)$ by Theorem 6.1 of [**3**]; therefore,

$$ \begin{aligned} [\hat{\mu}(F(\mathscr{E}))]^{ii} &= [\text{pns}_\theta({}^*E)]^i \\ &= [F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0)]^{ii} \\ &= F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0). \end{aligned} $$

Conversely, assume $F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0) = [\hat{\mu}(F(\mathscr{E}))]^{ii}$. Thus,

$$ \begin{aligned} \text{Fin}\ (\mu_\theta(0)) &= [\hat{\mu}(F(\mathscr{E}))]^i = [\hat{\mu}(F(\mathscr{E}))]^{iii} \\ &= [F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0)]^i = \text{pns}_\theta({}^*E) \end{aligned} $$

by Corollary 3.10, Lemma 5.5 (vii) of [**3**] and Theorem 6.1 of [**3**]. Therefore, (E, θ) has invariant nonstandard hulls by Lemma 1(i and v) of [**4**].

The next proposition gives a sufficient condition for the three sets of (4.1) to coincide.

PROPOSITION 4.5. *Let θ be a compatible topology on E and let $\mathscr{E}$ be the collection of θ-equicontinuous subsets of F. If (E, θ) is a Schwartz space, then*

$$\hat{\mu}(F(\mathscr{E})) = F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0).$$

Proof. Let $z \in F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0)$. There exists a balanced, convex $A \in \mathscr{E}$ such that $z \in {}^*A \cap \mu_{\sigma(F,E)}(0)$. Also, there exists a balanced, convex $B \in \mathscr{E}$ such that $A \subset B$ and A is a τ_B-compact subset of (F_B, τ_B) (see [**5**], Theorem 1 and Proposition 5, p. 277). Consequently, each element of ${}^*A$ is τ_B-near-standard ([**7**], Theorem 4.1.13, p. 93) which implies

$$(4.6) \qquad {}^*A \cap \mu_{\sigma(F,E)}(0) \subset \mu_B(0).$$

Indeed, if $y \in {}^*A \cap \mu_{\sigma(F,E)}(0)$, then $y - {}^*a \in \mu_B(0)$, for some $a \in A$, which implies ${}^*a - y \in \mu_{\sigma(F,E)}(0)$, since $\mu_B(0) \subset \mu_{\sigma(F,E)}(0)$; therefore $a = 0$, since $\sigma(F, E)$ is Hausdorff, $y + ({}^*a - y) = {}^*a$ and $y \in \mu_{(F,E)}(0)$.

Thus, by (4.6), we have $z \in \mu_B(0)$, which implies

$$F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0) \subset \hat{\mu}(F(\mathscr{E}))$$

by Proposition 4.3. From (4.1) we infer $F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0) = \hat{\mu}(F(\mathscr{E}))$.

Remark. It is a conjecture of the author that Proposition 4.5 is true only for Schwartz spaces.

5. Examples. We now give examples for the cases when the sets of (4.1) are not the same. In Example 1 we exhibit a space for which the last two sets of (4.1) are equal and properly contain the first set. Example 2 yields a space for which the three sets of (4.1) are distinct. Finally, in Example 3, we observe a class of spaces for which the first two sets of (4.1) are equal and are proper subsets of the third set. Since the first two examples are the function spaces of [**2**] (see [**2**], Section 4), we begin by establishing the necessary notation and definitions.

Unless stated otherwise, J will denote the closed interval $[0, 1]$ equipped with the usual topology. Let $G = \mathbf{K}^J$, the vector space of all $\mathbf{K}$-valued functions on J. For $x \in G$, define

$$(5.1) \qquad s(x) = \{j \in J \mid x(j) \neq 0\}.$$

Let $E = \mathbf{K}^{(J)}$, the set of all $x \in G$ for which $s(x)$ is finite. Hence, E is a proper vector subspace of G (it is assumed that $\emptyset$ is a finite set).

For $j \in J$, define $e_j \in E$ as follows: $e_j(j) = 1$ and $e_j(k) = 0$ for $k \in J$ such that $k \neq j$. Clearly, $\{e_j \mid j \in J\}$ is a Hamel basis for E. Let $\mathfrak{A}(G)$ be the col-

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lection of all $x \in G$ for which $x(j) > 0$ for each $j \in J$. For $x \in E$ and $y \in G$ define

$$(5.2) \qquad \langle x, y \rangle = \sum_{j \in s(x)} x(j)y(j).$$

Clearly, $\langle . , . \rangle$ is a bilinear form on $E \times G$. For $B \subset E$, define $B^0 \subset G$ as follows: $y \in B^0$ if and only if $|\langle x, y \rangle| \leqq 1$ for each $x \in B$. We define $B^0 \subset E$, for $B \subset G$, in a similar manner.

For $x \in G$, define $[x] \subset G$, as follows: $z \in [x]$ if and only if $|z(j)| \leqq |x(j)|$ for each $j \in J$. If $A \subset G$ is non empty, then let

$$(5.3) \qquad [A] = \{[x] | \, x \in A\}.$$

Definition 5.4. A set $A \subset G$ is said to have the *linear topology property* (LTP) if and only if A is non empty, $A \subset \mathfrak{A}(G)$ and the following conditions hold:

1. For $x, y \in A$, there exists $z \in A$ such that $z(j) \leqq \min(x(j), y(j))$ for each $j \in J$.
2. For $x \in A$, there exists $y \in A$ such that $2y(j) \leqq x(j)$ for each $j \in J$.
3. For $\delta > 0$ and $j \in J$, there exists $x \in A$ for which $x(j) < \delta$.

Remark. For $A \subset G$, we will say that A has LTP whenever A satisfies Definition 5.4.

If $A \subset G$ has LTP, then the filter $\mathfrak{F}$ on G generated by $[A]$ induces a unique Hausdorff, locally convex linear topology θ on E such that $\mathfrak{F}_E$, the trace of $\mathfrak{F}$ on E, is the filter of θ-neighborhoods 0 in E. Therefore, such a topology will be called the *linear topology induced on* E *by* A, whenever A has LTP.

For $A \subset G$ having LTP and θ, the linear topology induced on E by A, we need to characterize the linear subspace F of G that is the dual of (E, θ) via the bilinear form (5.2). This characterization is derived from the following two propositions.

PROPOSITION 5.5. *Let* $x \in \mathfrak{A}(G)$. *If* $y \in ([x] \cap E)^0$, *then* $s(y)$ *is countable.*

Proof. Assume $s(y)$ is uncountable. Hence, there exists $n \in \mathbf{N}_+$ such that

$$J_1 = \{j \in J | \, n^{-1} < |y(j)|\}$$

is uncountable. Also, $0 < x(j)$ for each $j \in J_1$; therefore, there exists $m \in \mathbf{N}_+$ such that

$$J_2 = \{j \in J_1 | \, m^{-1} < x(j)\}$$

is uncountable. Let $p = 2mn$, let $\{j_k\}_{k=1}^p \subset J_2$ and define

$$\beta_k = x(j_k)|y(j_k)|(y(j_k))^{-1}$$

for $k = 1, \ldots, p$. Thus for $z_0 = \sum_{k=1}^p \beta_k e_{j_k}$, we have $|z_0(j_k)| = |\beta_k| = x(j_k)$

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for $k = 1, \ldots, p$; therefore, $z_0 \in [x] \cap E$. Observe,

$$|\langle z_0, y\rangle| = \left|\sum_{k=1}^{p} \beta_k y(j_k)\right| = \left|\sum_{k=1}^{p} x(j_k)|y(j_k)|\right|$$

$$= \sum_{k=1}^{p} x(j_k)|y(j_k)| \geqq m^{-1}n^{-1}p = 2$$

which contradicts $y \in ([x] \cap E)^0$. Consequently, $s(y)$ is countable.

PROPOSITION 5.6. *If $x \in \mathfrak{A}(G)$ then $y \in ([x] \cap E)^0$ if and only if $s(y)$ is countable and $\sum_{j \in s(y)} x(j)|y(j)| \leqq 1$.*

Proof. Let $y \in G$. Assume $s(y)$ is countable and $\sum_{j \in s(y)} x(j)|y(j)| \leqq 1$. Consider $z \in [x] \cap E$, which implies $s(z)$ is finite, $z = \sum_{j \in s(z)} z(j)e_j$ and $|z(j)| \leqq x(j)$ for $j \in s(z)$. It can be assumed that $s = s(y) \cap s(z)$ is not empty. Hence,

$$|\langle z, y\rangle| = |\textstyle\sum_{j \in s} z(j)y(j)|$$
$$\leqq \textstyle\sum_{j \in s} |z(j)|(x(j))^{-1}x(j)|y(j)|$$
$$\leqq \textstyle\sum_{j \in s} x(j)|y(j)| \leqq 1$$

since $|z(j)|(x(j))^{-1} \leqq 1$ for $j \in s$. We infer $y \in ([x] \cap E)^0$.

Conversely, assume $y \in ([x] \cap E)^0$. Thus, $s(y)$ is countable by Proposition 5.5. Let $s(y) = \{j_k\}_{k=1}^{\infty}$ and, for $k \in \mathbf{N}_+$, define

$$\beta_k = x(j_k)|y(j_k)|(y(j_k))^{-1}.$$

For $n \in \mathbf{N}_+$, let $x_n = \sum_{k=1}^{n} \beta_k e_{j_k}$; therefore, $x_n \in [x] \cap E$ for each $n \in \mathbf{N}_+$. Observe,

$$\sum_{k=1}^{n} x(j_k)|y(j_k)| = \left|\sum_{k=1}^{n} x(j_k)|y(j_k)|(y(j_k))^{-1}y(j_k)\right|$$

$$= \left|\sum_{k=1}^{n} \beta_k y(j_k)\right| = |\langle x_n, y\rangle|$$

for each $n \in \mathbf{N}_+$. So, $y \in ([x] \cap E)^0$ implies $\sum_{k=1}^{n} x(j_k)|y(j_k)| = |\langle x_n, y\rangle| \leqq 1$ for each $n \in \mathbf{N}_+$. Therefore,

$$\textstyle\sum_{j \in s(y)} x(j)|y(j)| \leqq 1.$$

Let $A \subset G$ have LTP and let θ be the linear topology induced on E by A. If $F \subset G$ is defined as follows:

$$y \in F \text{ if and only if } s(y) \text{ is countable and } \textstyle\sum_{j \in s(y)} x(j)|y(j)| < \infty \text{ for some } x \in A$$

then it can be inferred, from Proposition 5.6, that F is the dual of (E, θ) via the

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bilinear form (5.2) (see [**5**], Proposition 3, p. 204). In other words, $(E, F\ \langle .\,, . \rangle)$ is a dual system and θ is a compatible topology on E.

Next, we need a description of the infinitesimal polars, in $^*F$, generated by $^*([x] \cap E)$ for $x \in A$. Recall that a subset S of $^*J$ is **-countable* if and only if S is internal and there exists an internal mapping of $^*\mathbf{N}_+$ onto S.

PROPOSITION 5.7. *If $x \in \mathfrak{A}(G)$ then $z \in [^*([x] \cap E)]^i$ if and only if $s(z)$ is *-countable and $\sum_{\iota \in s(z)} {}^*x(\iota)|z(\iota)|$ is an infinitesimal.*

Proof. Let $z \in {}^*G$. Assume $s(z)$ is *-countable and $\sum_{\iota \in s(z)} {}^*x(\iota)|z(\iota)| = \beta \in \mu(0)$. Consider $v \in {}^*([x] \cap E)$. If $s(v) \cap s(z) = \emptyset$ then $|\langle v, z \rangle| = 0$. If $s = s(v) \cap s(z) \neq \emptyset$, then

$$\begin{aligned} |\langle v, z \rangle| &= |\textstyle\sum_{\iota \in s} v(\iota) z(\iota)| \\ &\leqq \textstyle\sum_{\iota \in s} |v(\iota)|({}^*x(\iota))^{-1}\,{}^*x(\iota)|z(\iota)| \\ &\leqq \textstyle\sum_{\iota \in s} {}^*x(\iota)|z(\iota)| \leqq \beta \end{aligned}$$

since $|v(\iota)|({}^*x(\iota))^{-1} \leqq 1$ for each $\iota \in {}^*J$, and $s \subset s(z)$. Thus, $|\langle v, z \rangle| \leqq \beta$ and $\beta \in \mu(0)$ imply $\langle v, z \rangle \in \mu(0)$. We infer $z \in [^*([x] \cap E)]^i$. Conversely, assume $z \in [^*([x] \cap E)]^i$. Since $[^*([x] \cap E)]^i \subset {}^*[([x] \cap E)^0]$, we can deduce, from Proposition 5.6 that $s(z)$ is *-countable. Let $s(z) = \{\iota_k\}_{k \in {}^*\mathbf{N}+}$ and for $k \in {}^*\mathbf{N}_+$, define

$$\beta_k = {}^*x(\iota_k)|z(\iota_k)|(z(\iota_k))^{-1}.$$

For $\eta \in \mathbf{N}_+$, let $x_\eta = \sum_{k=1}^{\eta} \beta_k e_{\iota_k}$. Consequently, $x_\eta \in {}^*([x] \cap E)$ for each $\eta \in {}^*\mathbf{N}_+$. Observe,

$$\begin{aligned} &\sum_{k=1}^{\eta} {}^*x(\iota_k)|z(\iota_k)| \\ &= \left| \sum_{k=1}^{\eta} {}^*x(\iota_k)|z(\iota_k)|(z(\iota_k))^{-1} z(\iota_k) \right| \\ &= \left| \sum_{k=1}^{\eta} \beta_k z(\iota_k) \right| = |\langle x_\eta, z \rangle| \in \mu(0) \end{aligned}$$

for each $\eta \in {}^*\mathbf{N}_+$.

For positive $\zeta \in \mathbf{R}$, we have

$$\sum_{k=1}^{\eta} {}^*x(\iota_k)|z(\iota_k)| = |\langle x_\eta, z \rangle| < \zeta$$

for each $\eta \in {}^*\mathbf{N}_+$, which implies $\sum_{\iota \in s(z)} {}^*x(\iota)|z(\iota)| \leqq \zeta$. Since $\zeta > 0$ was arbitrary, we infer $\sum_{\iota \in s(z)} {}^*x(\iota)|z(\iota)| \in \mu(0)$.

Finally, we must define a class of scalar valued functions called funnels.

Definition 5.8. For X a non empty subset of an interval J of real numbers we say that $x \in \mathbf{K}^J$ is an *X-funnel* if and only if x is a positive, real valued

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function and the function $\Psi(x) \in \mathbf{K}^J$, defined by: $\Psi(x)(j) = x(j)$ for $j \in J \backslash X$ and $\Psi(x)(j) = 0$ for $j \in X$, is continuous.

Example 1. Let $J = [0, 1]$, $G = \mathbf{K}^J$ and let $A = [\Delta(J)]$ be the set of all functions in G that are S-funnels for some non empty finite subset S of J. It can be shown that $[\Delta(J)]$ has LTP (see [**2**], Theorem 6.6); therefore, let θ denote the linear topology induced on E by $[\Delta(J)]$. As stated earlier, the dual of (E, θ) is the vector subspace $F \subset G$ defined as follows: $y \in F$ if and only if $s(y)$ is countable and $\sum_{j \in s(y)} x(j)|y(j)| < \infty$ for some $x \in [\Delta(J)]$. Also, the sets of the form $([x] \cap E)^0$, for $x \in [\Delta(J)]$, constitute a fundamental system for $\mathscr{E}$, the collection of θ-equicontinuous subsets of F.

The objective of this example is to produce an element z_0 of $^*F$ such that $z_0 \in [\hat{\mu}(F(\mathscr{E}))]^{ii}$ and $z_0 \notin \hat{\mu}(F(\mathscr{E}))$. We will use the fact that $[\hat{\mu}(F(\mathscr{E}))]^{ii} = [\text{Fin}(\mu_\theta(0))]^i$ (Corollary 3.10), $\hat{\mu}(F(\mathscr{E})) = \cup \{[^*([x] \cap E)]^i | x \in [\Delta(J)]\}$ (Proposition 4.3) and actually exhibit an element z_0 of $[\text{Fin}(\mu_\theta(0))]^i$ such that $z_0 \notin [^*([x] \cap E)]^i$ for each funnel x in $[\Delta(J)]$.

Let λ be a positive infinitesimal. Let $W_0 = \{\lambda\}$ and let

$$W_\eta = \{\lambda + (2k - 1)2^{-(\eta+2)} | k = 1, \dots, 2^{\eta-1}\} \quad \text{for} \quad \eta \in {}^*\mathbf{N}_+.$$

If we define

$$W = \cup \{W_\eta | \eta \in {}^*\mathbf{N}\},$$

then W is a *-countable subset of $^*[0, 1/2]$.

Define $z_0 \in {}^*G$ as follows: $z_0(\iota) = 0$ for $\iota \in {}^*J \backslash W$, $z_0(\lambda) = 2^{-1}$ and

$$(5.9) \qquad z_0(\iota) = 2^{-2\eta} \text{ for } \iota \in W_\eta \text{ and } \eta \in {}^*\mathbf{N}_+.$$

Consequently, $s(z_0) = W$. Observe that $\sum_{\iota \in W_\eta} z_0(\iota) = 2^{-(\eta+1)}$ for each $\eta \in {}^*\mathbf{N}$ and

$$(5.10) \qquad \sum_{\iota \in W} z_0(\iota) = \sum_{\eta \in {}^*\mathbf{N}} [\sum_{\iota \in W_\eta} z_0(\iota)] = \sum_{\eta \in {}^*\mathbf{N}} 2^{-(\eta+1)} = 1.$$

Also, if $V_\xi = \cup_{\eta=0}^{\xi} W_\eta$, for $\xi \in {}^*\mathbf{N}$, then

$$(5.11) \qquad \sum_{\iota \in W \backslash V_\xi} z_0(\iota) = \sum_{\eta = \xi+1} 2^{-(\eta+1)} = 2^{-(\xi+1)}$$

for each $\xi \in {}^*\mathbf{N}$.

As in Example 2.8, we will use the notation of [**2**] and define $\nu_\iota([\Delta(J)]) \subset {}^*\mathbf{K}$, for $\iota \in {}^*J$, as follows: $\delta \in \nu_\iota([\Delta(J)])$ if and only if $|\delta| \leqq {}^*x(\iota)$ for each $x \in [\Delta(J)]$. It can be shown that

$$(5.12) \qquad \nu_\iota([\Delta(J)]) \subset \mu(0)$$

for each $\iota \in {}^*J$ ([**2**], Definition 4.4, Theorem 6.6 and Propositions 4.8 and 4.9). Let $v \in \text{Fin}\,(\mu_\theta(0))$. Hence, there exists a standard finite subset Q of J for which $v(j) \in \text{Fin}(\mu(0))$ for $j \in Q$ and $v(\iota) \in \nu_\iota([\Delta(J)])$ for $\iota \in {}^*J \backslash Q$ ([**2**],

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Proposition 4.13). Since $s(v)$ is *-finite and $^*J\backslash Q$ is internal, we have that

(5.13) $S_0 = s(v) \cap ({}^*J\backslash Q)$

is *-finite and $s(v) = Q \cup S_0$. Also if we denote

(5.14) $S_1 = Q \cap W$ and $S_2 = S_0 \cap W$,

then S_1 and S_2 are disjoint and

(5.15) $s(v) \cap s(z_0) = S_1 \cup S_2$.

Note that S_2 is *-finite and S_1 is a standard finite set since $S_1 \subset Q$ and W is internal.

It can be assumed that S_1 and S_2 are non empty sets.

If $j \in S_1$, then $j \in W_\eta$ for some infinite η of $^*\mathbf{N}$, since $\lambda \in \mu(0)$ implies $W_n \subset {}^*J\backslash J$ for each $n \in \mathbf{N}$; therefore, from (5.9) we infer $z_0(j) \notin \mu(0)$ for each $j \in S_1$. Consequently,

(5.16) $\beta_0 = \sum_{j\in S_1} |v(j)| z_0(j) \in \mu(0)$

since $v(j)$ is finite for each $j \in Q$.

Next, there exists $\iota_0 \in S_2$ such that $|v(\iota)| \leqq |v(\iota_0)|$ for each $\iota \in S_2$ since S_2 is *-finite. Also, $v(\iota_0) \in \nu_{\iota_0}([\Delta(J)])$ implies $v(\iota_0) \in \mu(0)$ by (5.12) therefore, there exists a positive infinite $\gamma \in {}^*\mathbf{R}$ such that

(5.17) $\gamma |v(\iota_0)| \in \mu(0)$

by Proposition 1.1. Consequently, there exist $\xi \in {}^*\mathbf{N}\backslash\mathbf{N}$ for which

$$1 + \sum_{\eta=1}^{\xi} 2^{\eta-1} \leqq \gamma.$$

Let $V_\xi = \bigcup_{\eta=0}^{\xi} W_\eta$ and define

(5.18) $S_3 = S_2 \cap V_\xi$ and $S_4 = S_2 \cap [W\backslash V_\xi]$.

Thus $S_3 \cap S_4 = \emptyset$ and $S_2 = S_3 \cup S_4$. Again, it can be assumed that S_3 and S_4 are non empty *-finite sets. Hence, there exists $\xi_1 \in \{1; \ldots ; 1 + \sum_{\eta=1}^{\xi} 2^{\eta-1}\}$ for which there is an internal bijection of $\{1, \ldots, \xi_1\}$ onto S_3, which implies $\xi_1 \leqq \gamma$. Observe,

$$\begin{aligned}\beta_1 &= \sum_{\iota\in S_3} |v(\iota)| z_0(\iota) \leqq \sum_{\iota\in S_3} |v(\iota)| \leqq \sum_{\iota\in S_3} |v(\iota_0)| \\ &= \xi_1 |v(\iota_0)| \leqq \gamma |v(\iota_0)| \in \mu(0)\end{aligned}$$

since (5.10) implies $z_0(\iota) \leqq 1$ for each $\iota \in s(z)$; therefore,

(5.19) $\beta_1 \in \mu(0)$.

Also

$$\beta_2 = \sum_{\iota\in S_4} |v(\iota)| z_0(\iota) \leqq \sum_{\iota\in S_4} z_0(\iota) \leqq \sum_{\iota\in W\backslash V_\xi} z_0(\iota) = 2^{-(\xi+1)} \in \mu(0)$$

from (5.11) and (5.18) since $v(\iota) \in \mu(0)$ for $\iota \in S_4$ and ξ is infinite; therefore,

(5.20) $\beta_2 \in \mu(0)$.

From the above arguments, we deduce

$$|\langle v, z_0 \rangle| = |\sum_{\iota \in S_1 \cup S_2} v(\iota) z_0(\iota)| \leqq \sum_{\iota \in S_1} |v(\iota)| z_0(\iota) + \sum_{\iota \in S_2} |v(\iota)| z_0(\iota)$$
$$= \beta_0 + \sum_{\iota \in S_3} |v(\iota)| z_0(\iota) + \sum_{\iota \in S_4} |v(\iota)| z_0(\iota) = \beta_0 + \beta_1 + \beta_2$$

which implies $\langle v, z_0 \rangle \in \mu(0)$. We infer $z_0 \in [\text{Fin}(\mu_\theta(0))]^i$ since $v \in \text{Fin}(\mu_\theta(0))$ was arbitrary.

Now, consider an arbitrary funnel $x \in [\Delta(J)]$. Hence x is an S-funnel for some finite subset S of J. Consequently, there exists $n \in \mathbf{N}$ such that $2^{-(n+2)} \notin S$ which implies $0 < x(2^{-(n+2)})$ and x is continuous at $2^{-(n+2)}$. Thus, there exists $\delta > 0$ such that $\delta < {}^*x(\iota)$ for each $\iota \in \mu(0) + 2^{-(n+2)}$. In particular,

$$\delta < {}^*x(\lambda + 2^{-(n+2)}).$$

Note that $\lambda + 2^{-(n+2)} \in W_n$ which implies $z_0(\lambda + 2^{-(n+2)}) = 2^{-2n}$ by (5.9). Let $\zeta = \lambda + 2^{-(n+2)}$. If we define $w_0 = {}^*x(\zeta)e_\zeta$, then $w_0 \in {}^*([x] \cap E)$ and

$$|\langle w_0, z_0 \rangle| = {}^*x(\zeta) z_0(\zeta) > \delta 2^{-2n}$$

therefore, $z_0 \notin [{}^*([x] \cap E)]^i$ by Proposition 5.7.

We infer, from the above arguments, $z_0 \in [\hat{\mu}(F(\mathscr{E}))]^{ii}$ and $z_0 \notin \hat{\mu}(F(\mathscr{E}))$; i.e., $\hat{\mu}(F(\mathscr{E})) \neq [\hat{\mu}(F(\mathscr{E}))]^{ii}$.

Since (E, θ) has invariant nonstandard hulls ([**2**], Example 2, Theorem 6.6 and Theorem 4.17) we have $[\hat{\mu}(F(\mathscr{E}))]^{ii} = F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0)$ by Proposition 4.4.

Remark. In the construction of W, we used an arbitrary positive infinitesimal λ; therefore, there exists an infinite collection of *-linearly independent elements of $[\hat{\mu}(F(\mathscr{E}))]^{ii}$ that are not elements of $\hat{\mu}(F(\mathscr{E}))$.

Example 2. For $J = [0, 1]$, let $[\Delta(J))$ denote the collection of all functions in $G = \mathbf{K}^J$ that are S-funnels for some non empty finite subset S of $[0, 1)$. Note that $[\Delta(J))$ is a proper subset of $[\Delta(J)]$. In particular, $[\Delta(J))$ contains no $\{1\}$-funnels. However, $[\Delta(J))$ does have LTP; therefore, let θ denote the linear topology induced on $E = \mathbf{K}^{(J)}$ by $[\Delta(J))$. Also if F denotes the set of all $y \in G$ for which $s(y)$ is countable and

$$\sum_{j \in s(y)} x(j) |y(j)| < \infty$$

for some $x \in [\Delta(J))$, then F is the dual of (E, θ).

Now consider $z_0 \in {}^*G$ of Example 1. Since $s(z_0) \subset {}^*[0, 1/2]$ it can be shown, by using the arguments of Example 1 and [**2**], Section 4, that $z_0 \notin \hat{\mu}(F(\mathscr{E}))$ and $z_0 \in [\hat{\mu}(F(\mathscr{E}))]^{ii}$. We will now exhibit an element of $F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0)$ that is not an element of $[\hat{\mu}(F(\mathscr{E}))]^{ii}$.

Let λ be a positive infinitesimal. For $\zeta = 1 - \lambda$, consider e_ζ. Clearly

$$e_\zeta \in F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0);$$

therefore we need only to show that $e_\zeta \notin [\text{Fin}(\mu_\theta(0))]^i$ since $[\hat{\mu}(F(\mathscr{E}))]^{ii} = [\text{Fin}(\mu_\theta(0))]^i$ by Corollary 3.10.

It is easy to show that $z \in \mu_\theta(0)$, for $z \in E$, if and only if $z(\iota) \in \nu_\iota([\Delta(J)))$ for each $\iota \in J$ which implies $z \in \text{Fin}(\mu_\theta(0))$ if and only if

$$z(\iota) \in \text{Fin}(\nu_\iota([\Delta(J))))$$

for each $\iota \in J$. Using the fact that x is continuous at 1 and $x(1) > 0$ for each $x \in [\Delta(J))$, it is easily shown that $\nu_\iota([\Delta(J))) = \mu(0)$ for each $\iota \in {}^*[0,1] \cap \mu(1)$. Thus we infer $e_\zeta \in \text{Fin}(\mu_\theta(0))$. However

$$\langle e_\zeta, e_\zeta \rangle = e_\zeta(\zeta) e_\zeta(\zeta) = 1$$

which implies e_ζ is not an element of $[\text{Fin}(\mu_\theta(0))]^i$.

Consequently, (E, θ) is an example of a space for which the three sets of (4.1) are distinct.

Remark. In Example 1, we could have shown, with considerably less effort, that $z_0 \in F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0)$ and $z_0 \notin \hat{\mu}(F(\mathscr{E}))$. However, in Example 2 it is crucial that z_0 is an element of $[\hat{\mu}(F(\mathscr{E}))]^{ii}$. Thus, the work we could have avoided in Example 1 would have to be done in Example 2.

Example 3. Let (E, θ) be an infinite dimensional normed linear space and let F be its dual. Let S and S' be the unit balls in E and F respectively. Also let β denote the normed topology on F generated by S'. Since $\{nS' | n \in \mathbf{N}_+\}$ is a fundamental system of $\mathscr{E}$, the θ-equicontinuous subsets of F, and S' is a β-bounded, β-neighborhood of $0 \in F$, it can be shown that $F(\mathscr{E}) = \text{Fin}(\mu_\beta(0))$ which implies $\hat{\mu}(F(\mathscr{E})) = \mu_\beta(0)$ by Corollary 2, 3 and Proposition 1.1. From the fact that $[\text{Fin}(\mu_\theta(0))]^i = \mu_\beta(0)$ (see [**3**], Theorem 5.12) we deduce

$$[\hat{\mu}(F(\mathscr{E}))]^{ii} = [\mu_\beta(0)]^{ii} = \mu_\beta(0) = \hat{\mu}(F(\mathscr{E})).$$

However, infinite dimensional normed linear spaces do not have invariant non-standard hulls ([**3**], Theorem 4.4); therefore,

$$[\hat{\mu}(F(\mathscr{E}))]^{ii} \neq F(\mathscr{E}) \cap \mu_{\sigma(F,E)}(0)$$

by Proposition 4.4.

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