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POINTWISE ESTIMATES OF THE SIZE OF CHARACTERS OF COMPACT LIE GROUPS

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Abstract

Pointwise bounds for characters of representations of the classical, compact, connected, simple Lie groups are obtained which allow us to study the singularity of central measures. For example, we find the minimal integer k such that any continuous orbital measure convolved with itself k times belongs to L^2. We also prove that if $k = \operatorname{rank} G$ then $\mu^{2k} \in L^1$ for all central, continuous measures μ. This improves upon the known classical result which required the exponent to be the dimension of the group G.

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1. Introduction

In this paper sharp, pointwise bounds for characters of representations of the classical, compact, connected, simple Lie groups are obtained. Our prime motivation is to use these estimates to study the singularity of central, continuous measures.

In [8] Ragozin proved the striking fact that if G was such a group and μ was a central, continuous measure on G, then $\mu^{\dim G} \in L^1(G)$ (the product here is convolution). This implies, in particular, that if g is not in the centre of G, then $\operatorname{Tr} \lambda(g)/\deg \lambda \to 0$ as the degree of the representation λ tends to infinity [11]. Ragozin's result was improved by one of the authors in [2] where it was shown that if g does not belong to the centre

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of G, then

$$\left|\frac{\operatorname{Tr}\lambda(g)}{\deg\lambda}\right| \le c(g)\,(\deg\lambda)^{-2/(\dim G-\operatorname{rank} G)}.$$

A consequence of this bound on the trace function is that if $k > \dim G/2$ and μ is a continuous orbital measure, then $\mu^k \in L^2(G)$, while if μ is any central, continuous measure then $\mu^k \in L^1(G)$.

In this paper we improve these results, obtaining the following theorem for classical Lie groups of rank n.

THEOREM 1.1. *Let G be a compact, connected, simple Lie group of type A_n, B_n, C_n or D_n. For every g not in the centre of G there is a constant $c(g)$ such that*

$$\left|\frac{\operatorname{Tr}\lambda(g)}{\deg\lambda}\right| \le c(g)(\deg\lambda)^{-s}$$

for all representations λ if and only if

$$s \le \begin{cases} 1/(n-1) & \text{if } G \text{ is type } A_{n-1} \text{ or } D_n;\\ 1/(2n-1) & \text{if } G \text{ is type } B_n;\\ 2/(2n-1) & \text{if } G \text{ is type } C_n,\ n \ne 3;\\ 1/3 & \text{if } G \text{ is type } C_3.\end{cases}$$

(In contrast, $\dim G - \operatorname{rank} G = O(n^2)$.)

From Theorem 1.1 we are able to show that if G is type A_{n-1}, C_n for $n \ne 3$ or D_n, and μ is any continuous, orbital measure, then μ^k belongs to $L^2(G)$ if and only if $k \ge \operatorname{rank} G = n$. Furthermore, if μ is any continuous, central measure, then μ^n belongs to $L^1(G)$. For type B_n the condition is $k \ge 2n$.

Key to proving Theorem 1.1 is to understand the structural properties of maximal subroot systems. These are discussed in Section 2. In Section 3 we use these properties and computational arguments based on the Weyl character formula to establish the specified pointwise upper bounds on the trace function. Examples are found in Section 4 which prove these upper bounds are best possible. Applications to the study of the singularity of central measures can be found in Section 5.

2. Notation and structural properties of subroot systems

2.1. Notation Let G be a compact, connected, simple, non-exceptional Lie group of rank n. Let $Z(G)$ denote its centre and W be its Weyl group. Denote by $e_1, \ldots, e_m$

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the usual unit vectors in $\mathbb{R}^m$, where $m = n+1$ in type A_n and $m = n$ otherwise. We take a maximal torus T with Φ the set of roots for (G, T) described below.

Type	**Root system Φ**	**Base $\Delta = \{\alpha_j : j = 1, \dots, n\}$**
A_n	$\{e_i - e_j : 1 \le i \ne j \le n+1\}$	$\alpha_j = e_j - e_{j+1}$
B_n	$\{\pm e_i, \pm(e_i \pm e_j) : 1 \le i \ne j \le n\}$	$\alpha_j = e_j - e_{j+1}$ for $j \ne n$ $\alpha_n = e_n$
C_n	$\{\pm 2e_i, \pm(e_i \pm e_j) : 1 \le i \ne j \le n\}$	$\alpha_j = e_j - e_{j+1}$ for $j \ne n$ $\alpha_n = 2e_n$
D_n	$\{\pm(e_i \pm e_j) : 1 \le i \ne j \le n\}$	$\alpha_j = e_j - e_{j+1}$ for $j \ne n$ $\alpha_n = e_{n-1} + e_n$

The set of positive roots associated with the base of simple roots Δ is denoted by Φ^+, the fundamental dominant weights relative to Δ are denoted by $\lambda_1, \dots, \lambda_n$, and Λ^+ is the set of all dominant weights. The set Λ^+ is in a 1-1 correspondence with $\widehat{G}$; $\sigma_\lambda \in \widehat{G}$ is indexed by its highest weight $\lambda \in \Lambda^+$. The degree of σ_λ is denoted by d_λ. The weights of $\lambda \in \Lambda^+$ are given by

$$\Pi(\lambda) = \{\mu \in \Lambda : w(\mu) < \lambda \text{ for all } w \in W\},$$

where $\mu < \lambda$ means $\lambda - \mu$ is a non-negative integral sum of positive roots. We set $\rho = \sum_{j=1}^n \lambda_j$. According to the Weyl dimension formula [13] the degree of λ is given by

$$\prod_{\alpha\in\Phi^+} (\rho + \lambda, \alpha)/(\rho, \alpha). \tag{2.1}$$

For general facts about root systems we refer the reader to [5].

Given $g \in T$ we let $\Phi(g) = \{\alpha \in \Phi : \alpha(g) \in 2\pi\mathbb{Z}\}$ and let $\Phi^+(g) = \Phi(g) \bigcap \Phi^+$. It is easily seen that $\Phi(g)$ is a subroot system of Φ and that $\Phi^+(g)$ is a complete set of positive roots of this subroot system. It is known that $\Phi(g) = \Phi$ if and only if $g \in Z(G)$ [1, page 189]. When $\Phi(g)$ is empty g is called a regular element of G.

For g in the torus, the Weyl character formula ([13]) states

$$\operatorname{Tr}\lambda(g) = \frac{e^{i\rho(g)} \sum_{w\in W} \det w \exp i(\rho + \lambda, w(g))}{\prod_{\alpha\in\Phi^+}(e^{i\alpha(g)} - 1)}.$$

This determines $\operatorname{Tr}\lambda$ on G as characters are class functions.

When $g \in Z(G)$ an application of Schur's lemma shows that $|\operatorname{Tr}\lambda(g)| = d_\lambda$, hence the interest is when $g \notin Z(G)$. It was shown in [2] how one can evaluate the Weyl character formula (by considering suitable directional derivatives if $\Phi^+(g)$ is not empty) to obtain

$$\frac{|\operatorname{Tr}\lambda(g)|}{d_\lambda} = c(g) \frac{\left|\sum_{w\in W} \det w \prod_{\alpha\in\Phi^+(g)} (\rho + \lambda, w(\alpha)) \exp i(\rho + \lambda, w(g))\right|}{\prod_{\alpha\in\Phi^+}(\rho + \lambda, \alpha)}. \tag{2.2}$$

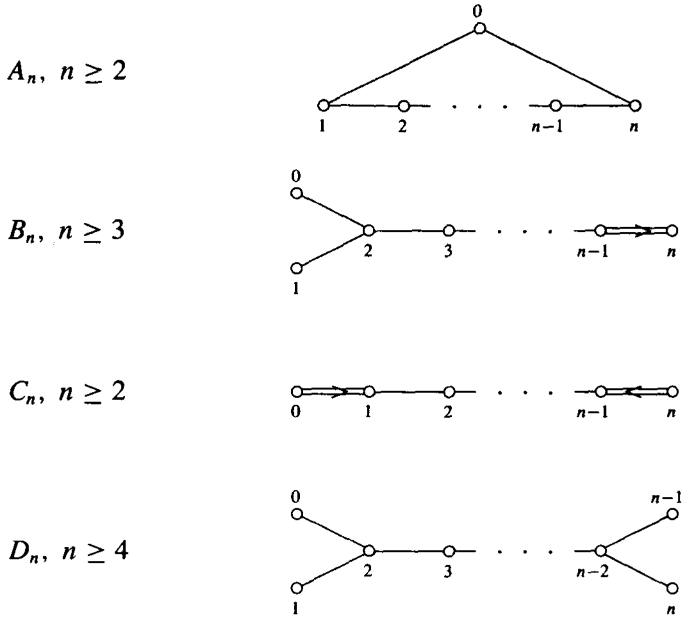

FIGURE 1. Extended Dynkin diagrams.

Consequently,

$$\frac{|\operatorname{Tr}\lambda(g)|}{d_\lambda} \leq c(g)\frac{\sum_{w\in W}\left|\prod_{\alpha\in\Phi^+(g)}(\rho+\lambda, w(\alpha))\right|}{\prod_{\alpha\in\Phi^+}(\rho+\lambda,\alpha)}. \tag{2.3}$$

Thus in order to find pointwise bounds on the trace functions off the centre of G it is useful to understand the structures of the subroot systems properly contained in Φ. It clearly suffices to analyze those subroot systems which are maximal in the sense that there is no other proper subroot system containing it. These subroot systems are always associated with regular subalgebras, (although not always of maximal rank) and hence their diagrams are subdiagrams of the extended diagram of the original root system (see Figure 1). Note that the additional vertex, labelled 0, is identified with the highest root α_0.) Once all these subdiagrams have been identified we can determine all possible sets of positive roots associated with maximal subroot systems by considering Weyl conjugates of the bases corresponding to the subdiagrams.

We illustrate how to do this to find the positive roots of all maximal subroot systems for type B_n. The other types are summarized below.

2.2. Maximal subroot systems of type B_n Consider the extended diagram of Figure 2. Notice that if vertex 0 or 1 is removed the remaining subgraph is still type B_n and thus is not proper. If vertex 2 is removed we are left with type $A_1 \times A_1 \times B_{n-2}$.

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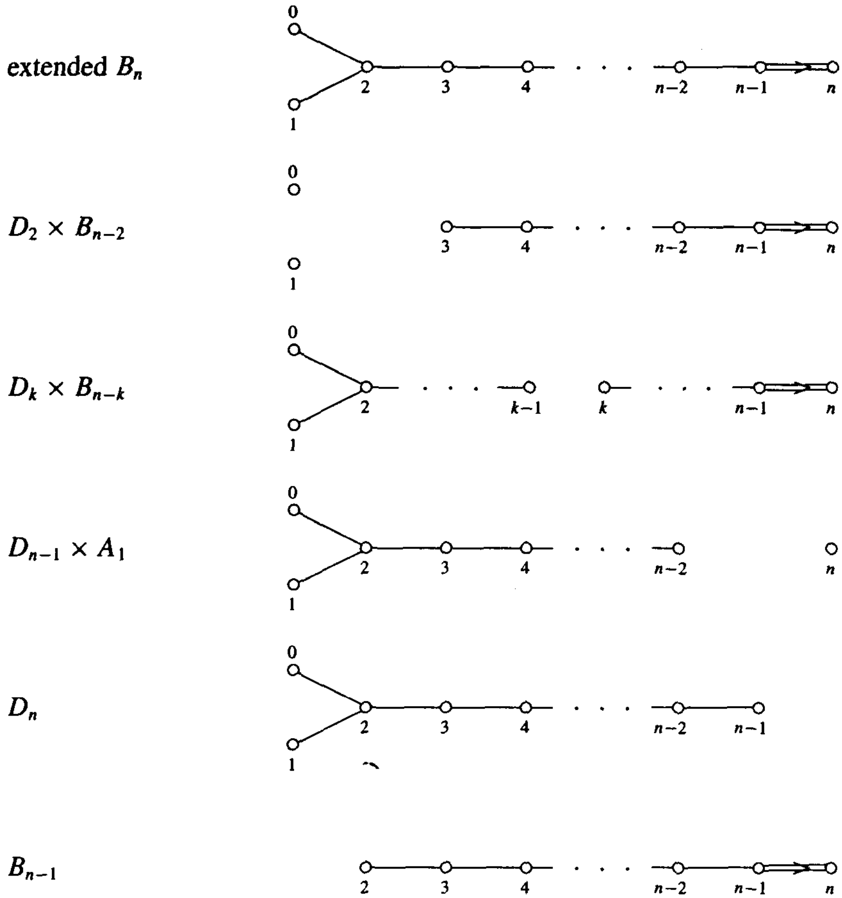

FIGURE 2. Maximal subdiagrams for B_n.

Because the highest root is $e_1 + e_2$, the two roots making up $A_1 \times A_1$ (in the base we have chosen) are $\{e_1 \pm e_2\}$, which for simplicity are referred to as D_2. If any of vertices 3 through $n-2$ are removed, say vertex k, we have type $D_k \times B_{n-k}$, where $k \geq 3$, $n-k \geq 2$ and D_3 is understood to be the obvious root system. It has base

$$\{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\} \cup \{\alpha_{k+1}, \ldots, \alpha_n\}$$

which in terms of Δ may be expressed as

$$\{e_1 \pm e_2, e_i - e_{i+1} : 2 \leq i \leq k-1\} \cup \{e_i - e_{i+1}, e_n : k+1 \leq i \leq n-1\}.$$

The Weyl group acts as the group of permutations and sign changes of the set $\{e_1, \ldots, e_n\}$. Thus any set of positive roots associated with the subroot systems

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of type $D_k \times B_{n-k}$ is of the form

$$\{e_i \pm e_j : i < j ; i, j \in J_1\} \cup \{e_l, e_i \pm e_j : i < j ; i, j, l \in J_2\},$$

where J_1 and J_2 are disjoint subsets of $\{1, \ldots, n\}$ of sizes k and $n-k$ respectively.

If vertex $n-1$ is removed the subroot system is type $A_1 \times D_{n-1}$ and the root in A_1 is short. The sets of positive roots associated with this type of maximal subroot system are of the form

$$\{e_i\} \cup \{e_l \pm e_j : l < j ; l, j \neq i\}.$$

When vertex n is removed we are left with type D_n and positive roots

$$\{e_i \pm e_j : 1 \leq i < j \leq n\}.$$

If vertices 0 and 1 are both removed we are left with the maximal system B_{n-1} and the sets of positive roots are Weyl conjugates of

$$\{e_l, e_i \pm e_j : 1 < i < j \leq n, l \neq 1\}$$

and thus are of the form

$$\{e_l, e_i \pm e_j : i < j ; i, j, l \neq n_0\}.$$

If any other two (or more) vertices are removed from the extended graph we clearly do not have a maximal subroot system.

Notice that of all these maximal subroot systems only types $D_k \times B_{n-k}$ and $A_1 \times D_{n-1}$ are also of maximal rank.

2.3. Summary of maximal subroot systems In the charts which follow J_1 and J_2 denote disjoint subsets of $\{1, \ldots, n\}$ in types B_n, C_n and D_n; and disjoint subsets of $\{1, \ldots, n+1\}$ in type A_n.

Type	Maximal subroot systems	Positive roots of the maximal subroot systems
$\mathbf{A}_n$	A_{n-1}	$\{e_i - e_j : 1 \leq i < j \leq n+1 ; i, j \neq n_0\}$
	$A_k \times A_{n-k-1}$	$\{e_i - e_j : i < j ; i, j \in J_1\} \cup \{e_i - e_j : i < j ; i, j \in J_2\}$, where $\lvert J_1\rvert = k+1 \geq 2$, $\lvert J_2\rvert = n-k \geq 2$
$\mathbf{B}_n$	B_{n-1}	$\{e_l, e_i \pm e_j : i < j ; i, j, l \neq n_0\}$
	D_n	$\{e_i \pm e_j : 1 \leq i < j \leq n\}$
	$D_k \times B_{n-k}$	$\{e_i \pm e_j : i < j ; i, j \in J_1\} \cup \{e_l, e_i \pm e_j : i < j ; i, j, l \in J_2\}$, where $\lvert J_1\rvert = k \geq 2$, $\lvert J_2\rvert = n-k \geq 2$
	$A_1 \times D_{n-1}$	$\{e_i\} \cup \{e_l \pm e_j : l < j ; l, j \neq i\}$

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$\mathbf{C}_n$	A_{n-1}	$\{s_i e_i - s_j e_j : 1 \le i < j \le n\}$, where $s_j = \pm 1$
	$C_k \times C_{n-k}$	$\{2e_l, e_i \pm e_j : i < j; i, j, l \in J_1\}$ $\cup \{2e_l, e_i \pm e_j : i < j; i, j, l \in J_2\}$, where $\lvert J_1 \rvert = k \ge 1$, $\lvert J_2 \rvert = n - k \ge 1$
$\mathbf{D}_n$	D_{n-1}	$\{e_i \pm e_j : i < j; i, j \ne n_0\}$
	A_{n-1}	$\{s_i e_i - s_j e_j : 1 \le i < j \le n\}$, where $s_j = \pm 1$ and an even number of $s_j = -1$
	$D_k \times D_{n-k}$	$\{e_i \pm e_j : i < j; i, j \in J_1\} \cup \{e_i \pm e_j : i < j; i, j \in J_2\}$, where $\lvert J_1 \rvert = k \ge 2$, $\lvert J_2 \rvert = n - k \ge 2$

Here D_2 is understood to mean $\{e_1 \pm e_2\}$ and $C_1 = \{2e_i\}$. C_2 and D_3 are the obvious root systems.

3. Upper bounds for the trace function

In this section we establish the sufficiency of the choice of s in our main result. Each Lie group type must be handled separately, taking into account the possible choices for $\Phi^+(g)$.

THEOREM 3.1. *Let G be a compact, connected, simple Lie group of type A_n, B_n, C_n or D_n. For every $g \notin Z(G)$ there is a constant $c(g)$ such that*

$$\left| \frac{\operatorname{Tr} \lambda(g)}{\deg \lambda} \right| \le c(g)(\deg \lambda)^{-s}$$

for all $\lambda \in \widehat{G}$ provided

$$s \le \begin{cases} 1/(n-1) & \text{if } G \text{ is type } A_{n-1} \text{ or } D_n; \\ 1/(2n-1) & \text{if } G \text{ is type } B_n; \\ 2/(2n-1) & \text{if } G \text{ is type } C_n, n \ne 3; \\ 1/3 & \text{if } G \text{ is type } C_3. \end{cases}$$

PROOF. Inequality (2.3) together with the Weyl dimension formula (2.1) show that it is sufficient to prove that there is some constant c such that for all $w \in W$ and representations λ,

$$\frac{\left| \prod_{\alpha \in \Phi^+(g)} (\rho + \lambda, w(\alpha)) \right|}{\prod_{\alpha \in \Phi^+} (\rho + \lambda, \alpha)^{1-s}} \le c.$$

Indeed, as $\alpha \in \Phi(g)$ if and only if $w(\alpha) \in \Phi(w^{-1}(g))$, it suffices to prove there is a constant c such that

$$(3.1)\qquad \frac{\prod_{\alpha\in\Phi^{+\prime}}(\rho+\lambda,\alpha)}{\prod_{\alpha\in\Phi^{+}}(\rho+\lambda,\alpha)^{1-s}} = \prod_{\alpha\in\Phi^{+\prime}}(\rho+\lambda,\alpha)^{s} \prod_{\alpha\in\Phi^{+}\setminus\Phi^{+\prime}}(\rho+\lambda,\alpha)^{s-1} \le c$$

whenever $\Phi^{+\prime}$ is the set of positive roots of some maximal subroot system, and this is what we show in each case.

Throughout this proof we assume $\rho+\lambda$ can be expressed in terms of the fundamental dominant weights as $\sum_{i=1}^{n} m_i\lambda_i$. We also assume $m_k = \max_{i=1,\dots,n} m_i$. The letter c denotes a constant which may vary from one line to another.

One common technique we use is an induction argument. We often partition Φ^+ (and $\Phi^{+\prime}$) into two sets, one of which is a positive root system (subroot system) of smaller type. The product we need to study corresponding to these roots of smaller type are handled by the induction assumption. Another common technique is to count the number of positive roots α, from some appropriate set, such that $(\rho+\lambda,\alpha)$ is (essentially) maximal and see that there are enough of these terms occuring in the product with a negative exponent to make the product suitably small. Both these ideas are used in Case 1.1 below (when the maximal subroot system is type A_{n-1} in type A_n). In other cases, the arguments are slightly more delicate, but always they are of an elementary, combinatorial nature.

Type A_n

Case 1.1 Maximal subroot system is type A_{n-1}.

We proceed by induction on n. If $n=1$, then $\Phi^{+\prime}$ is empty and consequently $s=1$ suffices. So assume inductively that (3.1) is satisfied with $s=1/(n-1)$ whenever Φ^+ is the set of positive roots of type A_{n-1} and $\Phi^{+\prime}$ is the set of positive roots of a subroot system of type A_{n-2}.

Let Φ^+ be the set of positive roots of type A_n and let $\Phi^{+\prime}$ be the set of positive roots of a subroot system of type A_{n-1}; $\Phi^{+\prime}$ is a set of the form $\{e_i - e_j : 1 \le i < j \le n+1; i, j \ne n_0\}$.

First assume $k \le n_0 - 1$ (which implies, in particular, that $n_0 \ne 1$). Partition Φ^+ as $\Phi_1^+ \cup \Phi_2^+$ with

$$\Phi_1^+ = \{e_i - e_j : 2 \le i < j \le n+1\} \quad\text{and}\quad \Phi_2^+ = \{e_1 - e_j : 2 \le j \le n+1\}.$$

Similarly partition $\Phi^{+\prime}$ as $\Phi_1^{+\prime} \cup \Phi_2^{+\prime}$, where

$$\Phi_1^{+\prime} = \{e_i - e_j : 2 \le i < j \le n+1;\ i, j \ne n_0\}$$

and

$$\Phi_2^{+\prime} = \{e_1 - e_j : 2 \le j \le n+1;\ j \ne n_0\}.$$

The set Φ_1^+ may be viewed as the positive roots of type A_{n-1} and $\Phi_1^{+\prime}$ as the positive roots of a subroot system of type A_{n-2} (considering the vectors to be in $\mathbb{R}^n$ by omitting the first (zero) coordinate). When $\alpha \in \Phi_1^+$, then $(\rho+\lambda, \alpha)$ is equal to $\left(\sum_{i=2}^n m_i\lambda_i, \alpha\right)$, thus the inductive hypothesis may be applied to conclude that if $s \leq 1/(n-1)$, then

$$\prod_{\alpha\in\Phi_1^{+\prime}} (\rho+\lambda, \alpha)^s \prod_{\alpha\in\Phi_1^+\backslash\Phi_1^{+\prime}} (\rho+\lambda, \alpha)^{s-1} \leq c.$$

Since the cardinality of $\Phi_2^{+\prime}$ is $n-1$ we clearly have

$$\prod_{\alpha\in\Phi_2^{+\prime}} (\rho+\lambda, \alpha)^s \leq c m_k^{s(n-1)}.$$

Recall that $e_1 - e_{n_0} = \lambda_1 + \cdots + \lambda_{n_0-1}$. As $k \leq n_0 - 1$ this means that $(\rho+\lambda, e_1 - e_{n_0}) \geq m_k$, and since $e_1 - e_{n_0} \in \Phi_2^+\backslash\Phi_2^{+\prime}$ we obtain the inequality

$$\prod_{\alpha\in\Phi_2^+\backslash\Phi_2^{+\prime}} (\rho+\lambda, \alpha)^{s-1} \leq m_k^{s-1}.$$

Therefore,

$$\prod_{\alpha\in\Phi_2^{+\prime}} (\rho+\lambda, \alpha)^s \prod_{\alpha\in\Phi_2^+\backslash\Phi_2^{+\prime}} (\rho+\lambda, \alpha)^{s-1} \leq c m_k^{s(n-1)+s-1}.$$

This is bounded if $s(n-1)+s-1 \leq 0$, that is, when $s \leq 1/n$, giving the desired result.

Otherwise $k \geq n_0$ (and $n_0 \neq n+1$). In this case we partition $\Phi^{+\prime}$ into $\Phi_1^{+\prime} \cup \Phi_2^{+\prime}$, where $\Phi_1^{+\prime}$ is the subset of $\Phi^{+\prime}$ consisting of all the words $e_i - e_j$ with $i, j \neq n+1$, and

$$\Phi_2^{+\prime} = \{e_j - e_{n+1} : 1 \leq j \leq n; j \neq n_0\}.$$

Similarly partition Φ^+ so that $\Phi_2^+\backslash\Phi_2^{+\prime} = \{e_{n_0} - e_{n+1}\}$. Again the inductive hypothesis can be applied to the factors of the product corresponding to $\alpha \in \Phi_1^+$ and $\Phi_1^{+\prime}$, and this observation reduces the problem to proving

$$\prod_{\alpha\in\Phi_2^{+\prime}} (\rho+\lambda, \alpha)^s \prod_{\alpha\in\Phi_2^+\backslash\Phi_2^{+\prime}} (\rho+\lambda, \alpha)^{s-1} \leq c$$

for $s \leq 1/n$. As

$$(\rho+\lambda, e_{n_0} - e_{n+1}) = (\rho+\lambda, \lambda_{n_0} + \cdots + \lambda_n) \geq m_k$$

the required inequality can be established in the same manner as the first part.

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Case 1.2 Maximal subroot system is type $A_k \times A_{n-k-1}$, where $k, n-k-1 \geq 1$. We again proceed by induction on n. Notice that a maximal subroot system of this type is not found in type A_1 or A_2, and consequently the initial step of the inductive hypothesis is with $n = 3$ and $\Phi^{+\prime}$ the set of positive roots of type $A_1 \times A_1$. We leave it to the reader to verify the hypothesis for this initial condition.

We assume inductively that (3.1) holds with $s = 1/(n-1)$ whenever Φ^+ is the set of positive roots of type A_{n-1} and $\Phi^{+\prime}$ is type $A_k \times A_{n-k-2}$ for some k and $n-k-2 \geq 1$, and proceed to verify the induction step for type A_n.

From Section 2.3 we know that any set of positive roots of type $A_k \times A_{n-k-1}$ in A_n is of the form $\Phi^{+\prime} = \Phi_1^{+\prime} \cup \Phi_2^{+\prime}$, where

$$\Phi_1^{+\prime} = \{e_i - e_j : i < j;\ i, j \in J_1\},$$
$$\Phi_2^{+\prime} = \{e_i - e_j : i < j;\ i, j \in J_2\}$$

and J_1, J_2 are disjoint sets whose union is $\{1, \dots, n+1\}$, of sizes $k+1$ and $n-k$ respectively. Without loss of generality we may assume $1 \in J_1$.

Let

$$\Psi_1' = \{e_i - e_j : 1 < i < j;\ i, j \in J_1\}$$

(Ψ_1' is taken to be empty if the cardinality of J_1 is two) and $\Psi_2' = \Phi_1^{+\prime} \backslash \Psi_1'$. Let Ψ_1 be the set of words $e_i - e_j$, $i < j$, on the letters $\{2, \dots, n+1\}$, and $\Psi_2 = \{e_1 - e_j : j \neq 1\}$. Then Ψ_1 may be viewed as the set of positive roots of type A_{n-1}, with $\Psi_1' \cup \Phi_2^{+\prime}$ a subroot system of type $A_{k-1} \times A_{n-k-1}$. Thus the inductive hypothesis may be applied to yield

$$\prod_{\alpha \in \Psi_1' \cup \Phi_2^{+\prime}} (\rho + \lambda, \alpha)^s \prod_{\alpha \in \Psi_1 \backslash \Psi_1' \cup \Phi_2^{+\prime}} (\rho + \lambda, \alpha)^{s-1} \leq c$$

when $s \leq 1/(n-1)$. (If Ψ_1' is empty, then this is actually Case 1.1 which has already been done.)

It remains to prove that for $\Psi_2' = \{e_1 - e_j : j \in J_1 \backslash \{1\}\}$ and $s \leq 1/n$,

$$\prod_{\alpha \in \Psi_2'} (\rho + \lambda, \alpha)^s \prod_{\alpha \in \Psi_2 \backslash \Psi_2'} (\rho + \lambda, \alpha)^{s-1} \leq c.$$

If there exists some $j \in J_2$ such that $j \geq k+1$, then for some $\alpha \in \Psi_2 \backslash \Psi_2'$

$$(\rho + \lambda, \alpha) = (\rho + \lambda, e_1 - e_j) = (\rho + \lambda, \lambda_1 + \cdots + \lambda_{j-1}) \geq m_k.$$

Combining this with the fact that the cardinality of J_1 is at most $n-1$ we obtain the inequalities

$$\prod_{\alpha \in \Psi_2'} (\rho + \lambda, \alpha)^s \prod_{\alpha \in \Psi_2 \backslash \Psi_2'} (\rho + \lambda, \alpha)^{s-1} \leq c m_k^{s(|J_1|-1)} m_k^{s-1} \leq c$$

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when $s \leq 1/n$.

Otherwise, $n+1 \in J_1$ and there is some $j \leq k$ which belongs to J_2. (Indeed, all $j \in J_2$ must satisfy $j \leq k$.) Redefine

$$\Psi_1' = \{e_i - e_j : 1 \leq i < j \leq n;\ i, j \in J_1\},$$
$$\Psi_1 = \{e_i - e_j : 1 \leq i < j \leq n\}$$

and Ψ_2, Ψ_2' correspondingly. The argument now follows from the fact that $(\rho+\lambda, \alpha) \geq m_k$ for $\alpha = e_j - e_{n+1} \in \Psi_2 \backslash \Psi_2'$.

Type B_n

Case 2.1 Maximal subroot system is type B_{n-1}.

The maximal subroot system $\Phi^{+\prime} = \{e_l, e_i \pm e_j : i < j;\ i, j, l \neq n_0\}$. We consider the cases $n_0 = 1$ and $n_0 \neq 1$ separately and assume $s \leq 1/(2n-1)$.

$n_0 = 1$: Notice $(\rho+\lambda, \alpha) = O(m_k)$ for all $\alpha = e_1,\ e_1 + e_j$ and these roots all belong to $\Phi^+ \setminus \Phi^{+\prime}$. Also, $|\Phi^{+\prime}| = (n-1)^2$, and thus

$$\prod_{\alpha \in \Phi^{+\prime}} (\rho+\lambda, \alpha)^s \prod_{\alpha \in \Phi^+ \setminus \Phi^{+\prime}} (\rho+\lambda, \alpha)^{s-1} \leq m_k^{s(n-1)^2} m_k^{(s-1)n}. \tag{3.2}$$

Since

$$\frac{1}{2n-1} \leq \frac{n}{n^2-n+1}$$

it follows that (3.2) is bounded whenever $s \leq 1/(2n-1)$.

$n_0 \neq 1$: Here we proceed by induction, leaving the initial step with $n = 3$ to the reader. The words from $\Phi^{+\prime}$ in Φ^+ with letters from $\{2, \ldots, n\}$ are the positive roots of a subroot system of type B_{n-2} in type B_{n-1}. Thus the inductive hypothesis reduces the problem to consideration of

$$\prod_{\alpha \in \Psi'} (\rho+\lambda, \alpha)^s \prod_{\alpha \in \Psi \setminus \Psi'} (\rho+\lambda, \alpha)^{s-1},$$

where Ψ' and Ψ are the remaining roots in $\Phi^{+\prime}$ and Φ^+ respectively.

Set $a_j = \max\{m_l : l < j\}$. Notice that $\{a_j\}$ is an increasing sequence and that $(\rho+\lambda, e_1 - e_j) = O(a_j)$. Also, both $(\rho+\lambda, e_1)$ and $(\rho+\lambda, e_1+e_j)$ are $O(m_k)$. As $\Psi' = \{e_1, e_1 \pm e_j : j \neq 1, n_0\}$, this implies

$$\begin{aligned}
\prod_{\alpha \in \Psi'} (\rho+\lambda, \alpha)^s &= (\rho+\lambda, e_1)^s \prod_{\alpha = e_1+e_j, j \neq 1, n_0} (\rho+\lambda, \alpha)^s \prod_{\alpha = e_1-e_j, j \neq 1, n_0} (\rho+\lambda, \alpha)^s \\
&\leq c m_k^{s(n-1)} \prod_{j=2}^{n_0-1} a_j^s \prod_{j=n_0+1}^{n} a_j^s \leq c m_k^{s(n-1)} a_{n_0}^{s(n_0-2)} m_k^{s(n-n_0)} \\
&\leq c m_k^{s(2n-n_0-1)} a_{n_0}^{s(n_0-2)}.
\end{aligned}$$

Moreover, $\Psi \setminus \Psi' = \{e_1 \pm e_{n_0}\}$, therefore

$$\prod_{\alpha\in\Psi\setminus\Psi'} (\rho+\lambda,\alpha)^{s-1} = cm_k^{s-1} a_{n_0}^{s-1}.$$

Hence

$$\prod_{\alpha\in\Psi'} (\rho+\lambda,\alpha)^{s} \prod_{\alpha\in\Psi\setminus\Psi'} (\rho+\lambda,\alpha)^{s-1} \le cm_k^{s(2n-n_0)-1} a_{n_0}^{s(n_0-1)-1},$$

and noting that both exponents are negative completes the argument.

Case 2.2 Maximal subroot system is type D_n.

In this case $\Phi^{+\prime} = \{e_i \pm e_j : 1 \le i < j \le n\}$ and therefore $\Phi^+ \setminus \Phi^{+\prime}$ is the set of all words of length one in Φ^+. Let $b_i = \max\{m_l : l \ge i\}$. Then $(\rho+\lambda,\alpha) = O(b_i)$ if $\alpha = e_i$ or $e_i + e_j$ for any $j > i$. Also, $(\rho+\lambda, e_i - e_j) \le O(b_i)$ whenever $j > i$. Thus

$$\begin{aligned}\prod_{\alpha\in\Phi^{+\prime}} (\rho+\lambda,\alpha)^{s} \prod_{\alpha\in\Phi^{+}\setminus\Phi^{+\prime}} (\rho+\lambda,\alpha)^{s-1} &= \prod_{\alpha=e_i\pm e_j, i<j} (\rho+\lambda,\alpha)^{s} \prod_{i=1}^{n} (\rho+\lambda,e_i)^{s-1} \\ &\le c\prod_{i=1}^{n-1} b_i^{2s(n-i)} \prod_{i=1}^{n} b_i^{s-1} = c\prod_{i=1}^{n} b_i^{s(2n-2i+1)-1},\end{aligned}$$

and this is clearly bounded for $s \le 1/(2n-1)$.

Case 2.3 Maximal subroot system is type $A_1 \times D_{n-1}$.

The argument is essentially the same as Case 2.2.

Case 2.4 Maximal subroot system is type $D_m \times B_{n-m}$; $m, n-m \ge 2$.

The positive roots of type D_m in type B_m were already treated in Case 2.2, so it suffices to show

$$P \equiv \left| \prod_{\alpha\in\Psi'} (\rho+\lambda,\alpha)^{s} \prod_{\alpha\in\Psi} (\rho+\lambda,\alpha)^{s-1} \right|$$

is bounded when

$$\Psi' = \{e_l, e_i \pm e_j : i<j;\ i,j,l \in J_2\} \quad \text{and} \quad \Psi = \{e_i \pm e_j : i \in J_1, j \in J_2\}.$$

We consider the cases $1 \in J_1$ and $1 \in J_2$ separately. The argument is much easier when $1 \in J_1$ and hinges on the fact that in this case $\Psi \supseteq \{e_1 \pm e_j : j \in J_2\}$. Thus

$$P \le \prod_{i\in J_2} (\rho+\lambda,e_i)^{s} \prod_{i<j\in J_2} (\rho+\lambda,e_i\pm e_j)^{s} \prod_{j\in J_2} (\rho+\lambda,e_1\pm e_j)^{s-1}.$$

Let $b_i = \max\{m_l : l \ge i\}$ and $a_i = \max\{m_l : l < i\}$. With this notation, for $i < j$ we have

$$(\rho+\lambda, e_i+e_j) = O(b_i), \quad (\rho+\lambda, e_i) = O(b_i)$$

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$$(\rho+\lambda, e_i - e_j) \leq O(a_j), \quad (\rho+\lambda, e_1 - e_j) = O(a_j).$$

Hence we can further bound P by

$$P \leq c \prod_{i\in J_2} b_i^s b_i^{s(|J_2|-1)} \prod_{j\in J_2} a_j^{s(|J_2|-1)} b_1^{(s-1)|J_2|} \prod_{j\in J_2} a_j^{s-1} \leq c b_1^{s(|J_2|^2+|J_2|)-|J_2|} \prod_{j\in J_2} a_j^{s|J_2|-1}.$$

The final product is bounded over all λ since $|J_2| \leq n-2$ and $s \leq 1/(2n-1)$.

Now assume $1 \in J_2$. Here a further induction argument is useful. Partition Ψ' as $X_1 \cup X_2$ and Ψ as $Y_1 \cup Y_2$, where

$$X_1 = \{e_l, e_i \pm e_j : i<j, 1 \neq i, j, l \in J_2\}, \quad X_2 = \{e_1, e_1 \pm e_j : 1 \neq j \in J_2\}$$

and

$$Y_1 = \{e_i \pm e_j : i \in J_1, 1 \neq j \in J_2\}, \quad Y_2 = \{e_1 \pm e_i : i \in J_1\},$$

and assume inductively that

$$\left| \prod_{\alpha\in X_1} (\rho+\lambda, \alpha)^s \prod_{\alpha\in Y_1} (\rho+\lambda, \alpha)^{s-1} \right| \leq c$$

for $s \leq 1/(2n-1)$. (The initial case is left for the reader). We need to check that

$$\left| \prod_{\alpha\in X_2} (\rho+\lambda, \alpha)^s \prod_{\alpha\in Y_2} (\rho+\lambda, \alpha)^{s-1} \right| \leq c$$

to complete the induction step. Since $(\rho+\lambda, e_1+e_i) = O(m_k)$ for all $i \in J_1$, and $(\rho+\lambda, e_1-e_i)^{s-1} \leq 1$, the product above is bounded by $m_k^{s|X_2|} m_k^{(s-1)|J_1|}$. As $|X_2| = 2|J_2| - 1$ and J_1 has at least two elements the desired result is obtained.

This completes type B_n.

Type C_n

Case 3.1 Maximal subroot system is type A_{n-1}.

When $k=n$ then $(\rho+\lambda, 2e_i) = O(m_k)$ for all $i = 1, \ldots, n$ and as these roots belong to $\Phi^+ \backslash \Phi^{+\prime}$ it follows that for $s \leq 2/(2n-1)$,

$$P = \prod_{\alpha\in\Phi^{+\prime}} (\rho+\lambda, \alpha)^s \prod_{\alpha\in\Phi^+\backslash\Phi^{+\prime}} (\rho+\lambda, \alpha)^{s-1} \leq m_k^{s\binom{n}{2}} m_k^{(s-1)n} \leq c.$$

When $k \neq n$ we proceed inductively. The words from Φ^+ and $\Phi^{+\prime}$ built on the letters $\{2, \ldots, n\}$ form a subroot system of type A_{n-2} in C_{n-1} and thus our standard induction argument reduces the problem to showing that

$$P = \prod_{\alpha\in\Psi'} (\rho+\lambda, \alpha)^s \prod_{\alpha\in\Psi\backslash\Psi'} (\rho+\lambda, \alpha)^{s-1} \leq c,$$

where Ψ' and Ψ are the remaining words of $\Phi^{+'}$ and Φ^{+} respectively.

As Ψ' contains only one of $e_1 \pm e_n$, it follows that $(\rho+\lambda,\alpha) = O(m_k)$ for at least two $\alpha \in \Psi\backslash\Psi'$, namely, $\alpha = 2e_1$ and the one of $e_1 \pm e_n$ which is not in Ψ'. Furthermore, $|\Psi'| = n-1$, hence

$$P \le m_k^{s(n-1)} m_k^{2(s-1)},$$

and this is certainly bounded for $s \le 2/(2n-1)$.

Case 3.2 Maximal subroot system is type $C_1 \times C_{n-1}$.

This case is much more delicate than any of the others. When $n = 3$ it can be done by explicit calculation and we leave this for the reader. So we begin with $n \ge 4$ and take $s \ge 2/(2n-1)$.

As the maximal subroot system is

$$\Phi^{+'} = \{2e_{i_0}\} \cup \{2e_l, e_i \pm e_j : i < j;\ i,j,l \ne i_0\},$$

(3.1) can be written as

$$P = \prod_{l=1}^{n} (\rho+\lambda, 2e_l)^s \prod_{i<j\ne i_0} (\rho+\lambda, e_i \pm e_j)^s \prod_{j \ne i_0} \left|(\rho+\lambda, e_{i_0} \pm e_j)\right|^{s-1}.$$

Let $b_i = \max\{m_l : l \ge i\}$. When $i < j$ then $(\rho+\lambda, e_i + e_j) = O(b_i) = (\rho+\lambda, 2e_i)$. Thus

$$P = c\prod_{i=1}^{n} b_i^s \prod_{i<i_0} b_i^{s(n-i-1)} \prod_{i>i_0} b_i^{s(n-i)} \prod_{j<i_0} b_j^{s-1} b_{i_0}^{(n-i_0)(s-1)} Q, \tag{3.3}$$

where

$$Q = \prod_{i<j\ne i_0} (\rho+\lambda, e_i - e_j)^s \prod_{j\ne i_0} \left|(\rho+\lambda, e_{i_0} - e_j)\right|^{s-1}.$$

Notice that Q is the product we considered for the problem of the maximal subroot system of type A_{n-2} in type A_{n-1} (Case 1.1), and thus is bounded provided $s \le 1/(n-1)$. This is true in our situation since we have the stronger inequality $s \le 2/(2n-1)$.

Simplifying, and using the fact that when $i > i_0$ then $b_i \le b_{i_0}$, we obtain

$$P \le c b_{i_0}^{s+(n-i_0)(s-1)} \prod_{i<i_0} b_i^{s(n-i+1)-1} \prod_{i>i_0} b_{i_0}^{s(n-i+1)} Q$$

and hence

$$P \le c b_{i_0}^{s(n-i_0+1+(n-i_0+1)(n-i_0)/2)-(n-i_0)} \prod_{i<i_0} b_i^{s(n-i+1)-1} Q. \tag{3.4}$$

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To continue, we split the problem into two cases. First, suppose $k \geq i_0$. Then $b_i = m_k$ whenever $i \leq i_0$. Recall also that Q is bounded, thus

$$P \leq cm_k^{s(n-i_0+1+(n-i_0+1)(n-i_0)/2)-(n-i_0)} \prod_{i<i_0} m_k^{s(n-i+1)-1}.$$

Routine calculations reduce this to the inequality

$$P \leq m_k^{s(n^2+n)/2-n+1}$$

which one can check is bounded for our choices of n and s.

Now, suppose $k < i_0$. A standard argument with inequalities shows that the exponent of b_{i_0} in (3.4) is negative (for $s \leq 2/(2n-1)$) if $i_0 \neq n$. Also, $s(n-i+1)-1 \leq 0$ if $i > 1$. Consequently,

$$P \leq \begin{cases} cb_n^s b_1^{sn-1} b_{n-1}^{2s-1} Q & \text{if } i_0 = n; \\ cb_1^{sn-1} Q & \text{if } i_0 \neq n. \end{cases}$$

We factor Q as

$$Q = \prod_{j \neq 1, i_0} (\rho + \lambda, e_1 - e_j)^s (\rho + \lambda, e_1 - e_{i_0})^{s-1} Q_1,$$

where

$$Q_1 = \prod_{i<j \neq 1, i_0} (\rho + \lambda, e_i - e_j)^s \prod_{j \neq 1, i_0} |(\rho + \lambda, e_j - e_{i_0})|^{s-1}.$$

Q_1 is bounded being the product we consider for the problem of a maximal subroot system of type A_{n-3} in type A_{n-2} (on the letters $\{2, \ldots, n\}$; note that the assumption $k < i_0$ implies $i_0 \neq 1$). Also, as $k < i_0$, $(\rho + \lambda, e_1 - e_{i_0}) = O(m_k)$, thus

$$P \leq \begin{cases} cb_n^s b_1^{sn-1} b_{n-1}^{2s-1} m_k^{s(n-2)} m_k^{s-1} Q_1 & \text{if } i_0 = n; \\ cb_1^{sn-1} m_k^{s(n-2)} m_k^{s-1} Q_1 & \text{if } i_0 \neq n. \end{cases}$$

But $b_n \leq b_{n-1}$ and $b_1 = m_k$, hence

$$P \leq \begin{cases} cb_{n-1}^{3s-1} m_k^{s(2n-1)-2} & \text{if } i_0 = n; \\ cm_k^{s(2n-1)-2} & \text{if } i_0 \neq n. \end{cases}$$

As $n \geq 4$ we have $s \leq 1/3$, and thus P is bounded in either case.

Case 3.3 Maximal subroot system is type $C_k \times C_{n-k}$; $k, n-k \geq 2$.

This is similar to Case 1.2 (but easier because of the fact that $(\rho + \lambda, e_1 + e_j) = O(m_k)$ for all j).

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Type D_n

Case 4.1 Maximal subroot system is type D_{n-1}.

Assume $s \leq 1(n-1)$ and

$$\Phi^{+\prime} = \{e_i \pm e_j : 1 \leq i < j \leq n;\ i, j \neq n_0\}.$$

The case when $k \geq n_0$ can be done directly by counting, but is slightly different from the earlier cases because of the fact that $(e_1 + e_n, \lambda_{n-1}) = 0$. Observe that $\Phi^+ \backslash \Phi^{+\prime} = \{e_i \pm e_{n_0} : i \neq n_0\}$ (where $e_i - e_{n_0}$ should be understood to mean $e_{n_0} - e_i$ when $i > n_0$). Because $k \geq n_0$,

$$(\rho + \lambda, \alpha) \geq m_k \quad \text{for } \alpha = \begin{cases} e_i + e_{n_0} & \forall i \neq n_0, \text{ provided } k \neq n-1; \\ e_i + e_{n_0} & \forall i \neq n_0 \text{ or } n, \text{ if } k = n-1; \\ e_{n_0} - e_n & \text{if } k = n-1 \text{ (so that } n_0 \neq n). \end{cases}$$

Thus for all $k \geq n_0$, $(\rho + \lambda, \alpha) \geq m_k$ for at least $n-1$ elements in $\Phi^+ \backslash \Phi^{+\prime}$, and so

$$\prod_{\alpha \in \Phi^+ \backslash \Phi^{+\prime}} (\rho + \lambda, \alpha)^{s-1} \leq m_k^{(n-1)(s-1)}.$$

Combined with the fact that $|\Phi^{+\prime}| = 2\binom{n-1}{2}$, this yields

$$\prod_{\alpha \in \Phi^{+\prime}} (\rho + \lambda, \alpha)^s \prod_{\alpha \in \Phi^+ \backslash \Phi^{+\prime}} (\rho + \lambda, \alpha)^{s-1} \leq c m_k^{s(n-1)(n-2)+(s-1)(n-1)},$$

which is clearly bounded when $s \leq 1/(n-1)$.

If $k = n-1$ and $n_0 = n$, then $(\rho + \lambda, e_i - e_{n_0}) = O(m_k)$ for all $i = 1, \ldots, n-1$ and so the argument is similar.

Otherwise we proceed inductively. The words from Φ^+ and $\Phi^{+\prime}$ based on the letters $\{2, \ldots, n\}$ are a subroot system of type D_{n-2} in D_{n-1} and so are handled by the inductive hypothesis, leaving us to show that

$$\prod_{j \neq 1, n_0} (\rho + \lambda, e_1 \pm e_j)^s \prod_{\alpha = e_1 \pm e_{n_0}} (\rho + \lambda, \alpha)^{s-1}$$

is bounded. But this is quite routine because the assumptions $k \leq n_0 - 1$ and $k \neq n-1$ ensure that $(\rho + \lambda, e_1 \pm e_{n_0}) \geq m_k$.

Case 4.2 Maximal subroot system is type A_{n-1}.

It is convenient for the induction argument used in this case to assume $\Phi^{+\prime} = \{s_i e_i - s_j e_j : i < j\}$, taking no consideration for the parity of the signs, s_i. We leave the initial case of $n = 4$ for the reader, so assume $n > 4$ and proceed inductively.

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Suppose first that $k \le n-2$. Applying the induction argument one can see that it suffices to establish the boundedness of

$$\left|\prod_{\alpha\in\Psi'}(\rho+\lambda,\alpha)^s \prod_{\alpha\in\Psi\setminus\Psi'}(\rho+\lambda,\alpha)^{s-1}\right|, \tag{3.5}$$

where $\Psi = \{e_1 \pm e_j : 1 < j \le n\}$ and $\Psi' = \Psi \cap \Phi^{+\prime}$. Because $(\rho+\lambda,\alpha) \ge m_k$ for $\alpha = e_1 \pm e_{n-1}$ and $\alpha = e_1 \pm e_n$, at least two of which belong to $\Psi\setminus\Psi'$, and $|\Psi'| = n-1$, the product above is at most $cm_k^{s(n-1)+2(s-1)}$ and hence is bounded when $s \le 1/(n-1)$.

If $k = n$, then let J denote the number of $s_j = +1$. Notice that if s_i and s_j are the same sign, then $|(\rho+\lambda, s_ie_i + s_je_j)| \ge m_n$. A counting argument shows that

$$\left|\prod_{\alpha\in\Phi^{+\prime}}(\rho+\lambda,\alpha)^s \prod_{\alpha\in\Phi^+\setminus\Phi^{+\prime}}(\rho+\lambda,\alpha)^{s-1}\right| \le cm_n^{\binom{n}{2}s} m_n^{(s-1)\left(\binom{J}{2}+\binom{n-J}{2}\right)}$$

and one can readily verify that this exponent is negative for our choices of s and n.

The case $k = n-1$ is similar letting J denote the number of elements of $\{s_1, \dots, s_{n-1}, -s_n\}$ equal to $+1$.

Case 4.3 Maximal subroot system is type $D_k \times D_{n-k}$; $k, n-k \ge 2$.

Here it is convenient for the induction argument to allow k or $n-k$ to equal 1, understanding that D_1 is the empty set. When $n = 3$ we can only have $D_1 \times D_2$, which is actually just D_2, and this was done in Case 4.1 of this section. (Indeed, Case 4.1 does $D_1 \times D_{n-1}$ for general n.) This begins the induction argument.

From the previous remarks one can see there is no loss of generality in assuming k and $n-k \ge 2$. Moreover, we may assume

$$\Phi^{+\prime} = \{e_i \pm e_j : i < j;\ i,j \in J_1\} \cup \{e_i \pm e_j : i < j;\ i,j \in J_2\},$$

where J_1 and J_2 are disjoint subsets of $\{1, \dots, n\}$ of sizes k and $n-k$, and $1 \in J_1$.

The induction argument applies to the factors with $\alpha = e_i \pm e_j$, $i, j \ne 1$, thus we need only consider the product over the remaining words:

$$\prod_{\alpha\in\{e_1\pm e_j : 1\ne j\in J_1\}}(\rho+\lambda,\alpha)^s \prod_{\alpha\in\{e_1\pm e_j : j\in J_2\}}(\rho+\lambda,\alpha)^{s-1}. \tag{3.6}$$

If $k \ne n-1$, then $(\rho+\lambda, e_1+e_j) \ge m_k$ for all $j \in J_2$. If $k = n-1$, it is still true that $(\rho+\lambda, e_1+e_j) \ge m_k$ for all $j \in J_2$ except $j = n$, but then also $(\rho+\lambda, e_1-e_n) \ge m_k$. In either case there are at least $|J_2|$ positive roots $\alpha \in \{e_1 \pm e_j : j \in J_2\}$ such that $(\rho+\lambda,\alpha) \ge m_k$. As $|J_2| \ge 2$ and $|J_1| \le n-2$, this implies (3.6) is bounded when $s \le 1/(n-1)$ and completes the proof for type D_n. □

REMARK 3.1. The expressions obtained for the maximal subroot systems of the exceptional Lie groups, E_6, E_7, and E_8, are too cumbersome for the application of this method.

4. Optimality of the upper bounds

In this section we demonstrate the optimality of the choice of s in the main theorem, in the sense that there exist $g \in G$ and infinitely many representations λ such that $\operatorname{Tr}\lambda(g) = O(d_\lambda^{1-s})$. The elements g in the torus T which we work with, and the corresponding sets $\Phi^+(g)$, are listed below. Notice that the sets $\Phi^+(g)$ are the positive roots of maximal subroot subsystems of type A_{n-1}, D_n, $C_1 \times C_{n-1}$ and D_{n-1} in A_n, B_n, C_n and D_n respectively.

Type	**Element g of T**	**Positive subroot system $\Phi^+(g)$**
A_n	$(-nx, x, \dots, x) \in \mathbb{R}^{n+1}$, where $x = \pi/(n+1)$	$\{e_i - e_j : 2 \le i < j \le n+1\}$
B_n	$(\pi, \dots, \pi)$	$\{e_i \pm e_j : 1 \le i < j \le n\}$
C_n	$(\pi, 0, \dots, 0)$	$\{2e_1\} \cup \{e_i \pm e_j, 2e_k : i < j, k \neq 1\}$
D_n	$(\pi, 0, \dots, 0)$	$\{e_i \pm e_j : 1 < i < j \le n\}$

THEOREM 4.1. *Suppose G is a compact, connected, simple Lie group of type A_n, B_n, C_n or D_n. Let g be the element in T listed in the chart above and let $\lambda = m\lambda_1$ with m an even integer ($\lambda = m\lambda_3$ in type C_3). Then*

$$\left|\frac{\operatorname{Tr}\lambda(g)}{\deg\lambda}\right| \ge \tilde{c}(g)(\deg\lambda)^{-s}$$

for some constant $\tilde{c}(g)$ independent of λ if

$$s = \begin{cases} 1/(n-1) & \text{if } G \text{ is type } A_{n-1} \text{ or } D_n; \\ 1/(2n-1) & \text{if } G \text{ is type } B_n; \\ 2/(2n-1) & \text{if } G \text{ is type } C_n, n \neq 3; \\ 1/3 & \text{if } G \text{ is type } C_3. \end{cases}$$

The strategy of the proof is to first establish that

$$\det w \operatorname{sgn}\left(\prod_{\alpha\in\Phi^+(g)} (\rho+\lambda, w(\alpha))\right) \exp i(\rho+\lambda, w(g))$$

is constant over $w \in W$. This fact, together with (2.2), show that

$$\left|\frac{\operatorname{Tr}\lambda(g)}{d_\lambda}\right| \ge \max_{w\in W} \tilde{c}(g)\frac{\left|\prod_{\alpha\in\Phi^+(g)}(\rho+\lambda, w(\alpha))\right|}{\prod_{\alpha\in\Phi^+}(\rho+\lambda, \alpha)},$$

and we shall see that it is a straightforward matter to prove that the latter ratio is $O(d_\lambda^{-s})$.

First, some preliminary results.

LEMMA 4.2. *Let* λ *be any representation, let* $\Phi^+(g)$ *be as above and let* $w = w_1 w_2 \in W$ *where* w_1 *is a product of sign changes and* w_2 *a permutation.* ($w_1 = 1$ *in type* A_n.) *Then*

$$\operatorname{sgn}\left(\prod_{\alpha\in\Phi^+(g)} (\rho+\lambda, w(\alpha))\right) = \begin{cases} \det w_2 & \text{in type } B_n; \\ (-1)^{w_2(1)-1}\det w & \text{in type } A_n, C_n \text{ or } D_n. \end{cases}$$

PROOF. Obviously $\operatorname{sgn}((\rho+\lambda, w_2(\alpha))) = 1$ when $\alpha = e_i + e_j$, e_i or $2e_i$. If $i < j$ and $w_2(i) < w_2(j)$, then $\operatorname{sgn}((\rho+\lambda, w_2(e_i - e_j))) = 1$, while if w_2 reverses their order the sign is negative. Thus if we let

$$X = \{(i,j) : e_i - e_j \in \Phi^+(g), i < j \text{ and } w_2(i) > w_2(j)\},$$

then

$$\operatorname{sgn}\left(\prod_{\alpha\in\Phi^+(g)} (\rho+\lambda, w(\alpha))\right) = (-1)^{|X|}.$$

In type B_n,

$$X = \{(i,j) : 1 \le i < j \le n \text{ and } w_2(i) > w_2(j)\},$$

hence $(-1)^{|X|} = \det w_2$. For the other types the pairs $(1, j)$ are never included in X and therefore

$$\det w_2 = (-1)^{|X|+|\{j : j>1 \text{ and } w_2(1)>w_2(j)\}|} = (-1)^{|X|+w_2(1)-1}.$$

Hence,

$$(-1)^{|X|} = (-1)^{w_2(1)-1}\det w_2$$

in types A_n, C_n or D_n. This completes the proof for type A_n as $w_2 = w$.

Next, assume w_1 is a simple sign change, say $w_1(e_i) = -e_i$ if $i = i_0$ and $w_1(e_i) = e_i$ otherwise. Then

$$(\rho+\lambda, w_1(e_{i_0}+e_k))(\rho+\lambda, w_1(e_{i_0}-e_k)) = (\rho+\lambda, e_{i_0}+e_k)(\rho+\lambda, e_{i_0}-e_k),$$

while of course $(\rho+\lambda, w_1(e_{i_0})) = -(\rho+\lambda, e_{i_0})$. Since $\Phi^+(g)$ only contains words of the form $e_i \pm e_j$ in types B_n and D_n,

$$\prod_{\alpha\in\Phi^+(g)} (\rho+\lambda, w_1(\alpha)) = \begin{cases} -\prod_{\alpha\in\Phi^+(g)}(\rho+\lambda, \alpha) & \text{in type } C_n; \\ +\prod_{\alpha\in\Phi^+(g)}(\rho+\lambda, \alpha) & \text{in type } B_n \text{ or } D_n. \end{cases}$$

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We can determine the effect of an arbitrary sign change by repeating this argument the appropriate number of times:

$$\operatorname{sgn}\left(\prod_{\alpha\in\Phi^+(g)}(\rho+\lambda, w_1(\alpha))\right) = \begin{cases}(-1)^{\#\text{sign changes}} = \det w_1 & \text{in type } C_n;\\ +1 & \text{in type } B_n \text{ or } D_n.\end{cases}$$

This is also the determinant of w_1 in type D_n since only an even number of sign changes are allowed in the Weyl group.

Combining these observations completes the proof. □

LEMMA 4.3. *Let $g \in G$ be as above and let $\lambda = \sum m_i\lambda_i$ with m_i even (and $m_n \equiv m_{n-1} \equiv 0 \mod 4$ in type D_n). Let $w \in W$, $w = w_1w_2$, where w_1 is a product of sign changes ($w_1 = 1$ in type A_n) and w_2 a permutation. Then*

$$\exp i(\rho+\lambda, w(g)) = \begin{cases}(-1)^{w_2(1)-1}\theta & \text{in type } A_n, C_n \text{ or } D_n;\\ \det w_1\theta & \text{in type } B_n\end{cases}$$

for some complex numbers θ of modulus one which do not depend on w.

PROOF. Type B_n. Here w_2 is clearly irrelevant. Expressed in terms of the standard basis vectors the j'th entry of $\rho + \sum m_i\lambda_i$ is

$$\begin{cases}\sum_{i=j}^{n-1} m_i + m_n/2 + n - j + 1/2 & \text{if } j \neq n;\\ (m_n+1)/2 & \text{if } j = n.\end{cases}$$

The reader can easily check from this that if w_1 changes k signs, then

$$\exp i(\rho+\lambda, w(g)) = (-1)^k\theta = \det w_1\theta$$

for an appropriate choice of θ.

Type C_n. In terms of the standard basis vectors

$$\rho + \sum m_i\lambda_i = \left(\sum_{i=1}^n m_i + n, \sum_{i=2}^n m_i + n - 1, \dots, m_n + 1\right).$$

Suppose $w_2(1) = j$. Then,

$$(\rho+\lambda, w_2(g)) = \left(\sum_{i=j}^n m_i + n - j + 1\right)\pi.$$

As all m_i are assumed even,

$$\exp i(\rho+\lambda, w_2(g)) = (-1)^{j-1}\exp in\pi.$$

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Because $g = -g$, the sign changes have no effect on g and thus the argument is complete.

Type D_n. One can verify that if $w_2(1) = j$, then

$$(\rho+\lambda, w(g)) = \begin{cases} \pm\left(\sum_{i=j}^{n-2} m_i + (m_{n-1}+m_n)/2 + n - j\right)\pi & \text{if } j \le n-2; \\ \pm(1 + (m_{n-1}+m_n)/2)\pi & \text{if } j = n-1; \\ \pm(m_n - m_{n-1})\pi/2 & \text{if } j = n \end{cases}$$

with the choice of $\pm$ depending on w_1. As these are all integer multiples of π the choice of $\pm$ does not affect the parity of $(\rho+\lambda, w(g))$, and since $m_1, \dots, m_{n-2}$ and $1/2(m_{n-1} \pm m_n)$ are even integers, it follows that for all choices of j we have

$$\exp i(\rho+\lambda, w(g)) = (-1)^j \exp in\pi.$$

Type A_n. The j'th entry of $\rho + \sum m_i\lambda_i$ is

$$\frac{1}{n+1}\left(-(m_1+1) - 2(m_2+1) + \cdots + (m_j+1)(n-j+1) + \cdots + (m_n+1)\right).$$

The same kinds of calculations as used for the other types show that if $w_2(1) = j$ then

$$\exp i(\rho+\lambda, w(g)) = \exp i(\rho+\lambda, g)(-1)^{j-1}. \qquad \square$$

PROOF OF THEOREM 4.1. Combining these lemmas we clearly obtain

$$\det w \operatorname{sgn}\left(\prod_{\alpha\in\Phi^+(g)} (\rho+\lambda, w(\alpha))\right) \exp i(\rho+\lambda, w(g)) = \theta$$

and this is independent of w. Thus

$$\frac{|\operatorname{Tr}\lambda(g)|}{d_\lambda} \ge \tilde{c}(g)\frac{\max_{w\in W}\left|\prod_{\alpha\in\Phi^+(g)}(\rho+\lambda, w(\alpha))\right|}{\prod_{\alpha\in\Phi^+}(\rho+\lambda,\alpha)}.$$

For type B_n notice that $\Phi^+(g) \supseteq \{e_1 \pm e_j : j \neq 1\}$. As $\lambda = m\lambda_1$ we have $(\rho+\lambda, e_1 \pm e_j) \ge m$, thus

$$\max_{w\in W}\left|\prod_{\alpha\in\Phi^+(g)} (\rho+\lambda, w(\alpha))\right| \ge cm^{2(n-1)}.$$

Since also $(\rho+\lambda, e_1) = O(m)$ and $(\rho+\lambda,\alpha)$ is bounded independently of m for all other $\alpha \in \Phi^+$, it follows that $d_\lambda = cm^{2n-1}$ and hence

$$\frac{|\operatorname{Tr}\lambda(g)|}{d_\lambda} \ge cd_\lambda^{-1/(2n-1)}$$

as claimed.

The other cases are similar. $\square$

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5. Singularity of central, continuous measures

A measure μ on G is called central if μ commutes with all other measures on G under the action of convolution. Central measures are characterized by the fact that their Fourier transforms are scalar multiples of identity matrices:

$$\widehat{\mu}(\lambda) = a_\lambda I_{d_\lambda}, \quad \text{where } a_\lambda = \int_G \frac{\operatorname{Tr}\lambda(x)}{d_\lambda}\, d\mu.$$

We simply write $\widehat{\mu}(\lambda)$ in place of a_λ.

An interesting class of singular, central measures are the orbital measures. The orbital measure μ_g, supported on the conjugacy class $C(g)$ containing $g \in G$, is defined by

$$\int_G f\, d\mu_g = \int_G f(tgt^{-1})\, dm_G(t) \quad \text{for } f \in C(G).$$

Orbital measures are continuous if and only if $g \notin Z(G)$, the centre of G.

In [8] Ragozin proved that if $g \notin Z(G)$, then $\mu_g^{\dim G} \in L^1(G)$. One can easily see that $\widehat{\mu_g}(\lambda) = \operatorname{Tr}\lambda(g)/d_\lambda$, and using this fact it was shown in [2] that if $k > \dim G/2$, then $\mu_g^k \in L^2$. By appealing to the sharper results of this paper we can now prove:

PROPOSITION 5.1. *The measures μ_g^k belong to $L^2(G)$ for all $g \notin Z(G)$ if and only if $k \geq k_0$, where*

$$k_0 = \begin{cases} n & \text{if } G \text{ is type } A_{n-1};\ C_n, n \neq 3;\ \text{or } D_n; \\ 2n & \text{if } G \text{ is type } B_n; \\ 4 & \text{if } G \text{ is type } C_3. \end{cases}$$

PROOF. From the Peter-Weyl theorem we know $\mu_g^k \in L^2$ if and only if

$$\sum_{\lambda\in\widehat{G}} d_\lambda \left|\widehat{\mu}_g(\lambda)\right|^{2k} \operatorname{Tr}\left|I_{d_\lambda}\right|^2 = \sum_{\lambda\in\widehat{G}} d_\lambda^2 \left|\frac{\operatorname{Tr}\lambda(g)}{d_\lambda}\right|^{2k} < \infty.$$

It was shown in Corollary 9 of [2] that $\sum_{\lambda\in\widehat{G}} d_\lambda^t < \infty$ when $t < -\operatorname{rank} G/|\Phi^+|$. This fact, combined with Theorem 3.1, proves the sufficiency of the choice of k.

Necessity is a consequence of Theorem 4.1. For example, when G is type A_n and $g = (-nx, x, \dots, x)$ for $x = \pi/(n+1)$ we know

$$\begin{aligned}\sum_{\lambda\in\widehat{G}} d_\lambda^2 \left|\frac{\operatorname{Tr}\lambda(g)}{d_\lambda}\right|^{2k} &\geq \sum_{m \text{ even}} d_{m\lambda_1}^2 \left|\frac{\operatorname{Tr} m\lambda_1(g)}{d_{m\lambda_1}}\right|^{2k} \\ &\geq \tilde{c}(g) \sum_{m \text{ even}} d_{m\lambda_1}^2 d_{m\lambda_1}^{-2k/n} = c \sum_{m \text{ even}} m^{n(2-2k/n)},\end{aligned}$$

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which is finite only if $2n - 2k < -1$. Thus we require $k > n + 1/2$, but as $k \in \mathbb{N}$ this means $k \geq n + 1$ is a necessary condition. The other types are similar. □

REMARK 5.1. Of course, if $\mu_g^k \in L^2$ then μ_g^{2k} is a continuous function.

The same result can be proved for central, continuous measures compactly supported on the conjugates of a set of the form $\{x \in T : \Phi^+(x) = \Phi^{+\prime}\}$ for some fixed set $\Phi^{+\prime}$ as such measures μ also have the property that $|\widehat{\mu}(\lambda)| \leq O(d_\lambda^{-s})$ for s as in the main theorem (see [2]). We should point out, in contrast, that for any $a < 1$ there are central, continuous measures μ such that $\widehat{\mu}(\lambda) \geq d_\lambda^{a-1}$ for infinitely many λ. This is shown in [3] and is a consequence of the fact that although compact Lie groups do not admit infinite central Sidon sets (an application of Ragozin's original work) they do admit central $(a, 1)$-Sidon sets for all $a < 1$.

Finally, we are ready to improve upon Ragozin's result on convolutions of arbitrary central, continuous measures.

PROPOSITION 5.2. *Suppose $\mu_1, \ldots, \mu_k$ are central continuous measures and $k \geq k_0$. Then $\mu_1 * \cdots * \mu_k \in L^1(G)$.*

PROOF. The proof is essentially the same as Theorem 11 of [2] but uses the stronger results obtained in Proposition 5.1. □

REMARK 5.2. Ragozin observed that μ_g^k is singular to Haar measure on G for all $k < \dim G / \dim C(g)$. As $\dim C(g) = 2(|\Phi^+| - |\Phi^+(g)|)$ ([7]) this means, for instance, that if G is type A_n, then μ_g^k is singular to Haar measure when $k < n/2 + 1$. It remains open as to whether or not $\mu_g^k \in L^1$ for all $g \in G \backslash Z(G)$, when k is between $n/2 + 1$ and $n + 1$ (other than for the trivial case A_1, where clearly $k = 2$ is the best possible result).

A measure μ is called L^p-improving if there is some $p < 2$ such that $\mu * L^p \subseteq L^2$. Young's inequality implies that all functions in L^q, for some $q > 1$, are examples of L^p-improving measures. A question of current interest is to understand which singular measures on compact groups are L^p-improving. For example, surface measures on analytic manifolds which generate G were shown to be L^p-improving in [9]. In [10] it was shown that if g was a regular element, then $\mu_g * L^p \subseteq L^2$ if and only if $p \geq 1 + r/(2 \dim G - r)$. For arbitrary continuous, orbital measures we can prove:

PROPOSITION 5.3. *If $g \notin Z(G)$, then μ_g is L^p-improving. Indeed, for any $g \notin Z(G)$, $\mu_g * L^p \subseteq L^2$ for $p > 2 - 2/(n+1)$ when G is type A_{n-1}, D_n or C_n, $n \neq 3$; $p > 2 - 2/(2n+1)$ in type B_n; and $p > 8/5$ for C_3.*

PROOF. Proposition 5.1 tells us that the operator $T_{k_0}(f) = \mu_g^{k_0} * f$ maps $L^1(G)$ into $L^2(G)$ whenever $g \notin Z(G)$. Since the identity map obviously maps $L^2(G)$ into $L^2(G)$ an application of Stein's interpolation theorem [12] (see also [4]) gives that $\mu_g * L^p \subseteq L^2$ for the choices of p listed. □

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