ON SEMIGROUPS WHOSE NONTRIVIAL LEFT CONGRUENCE CLASSES ARE LEFT IDEALS†

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An equivalence relation λ on a semigroup S is called a left congruence of S if $(u, v) \in \lambda$ implies that $(su, sv) \in \lambda$ for every s in S. With every set $\mathscr{L}$ of pairwise disjoint left ideals (i.e. subsets L of S such that $SL \subseteq L$), one can associate the left congruence $\{(u, v) \mid u = v$ or there exists an L in $\mathscr{L}$ such that $u \in L$ and $v \in L\}$. Thus every nonempty left ideal is a left congruence class (i.e. an equivalence class of some left congruence). A left congruence has the form just described if and only if all its nontrivial classes (i.e. its classes containing at least two elements) are left ideals. Such a left congruence is called a Rees left congruence if there is at most one nontrivial class. The identity relation on S is a Rees left congruence since the empty set is a left ideal by definition.

The aim of this paper is to characterize those semigroups S without a zero which have the property that

(A) all left congruences of S are Rees left congruences, or more generally that
(B) all nontrivial left congruence classes of S are left ideals.

These properties coincide in commutative semigroups and in semigroups which have a zero. In the commutative case, the question of which semigroups (with or without a zero) have property (A) has already been answered in 1950 by Ljapin ([**3**]). The semigroups with zero which have property (A) have recently been described ([**2**]). Therefore, the structure of these semigroups can be looked at as known in the present investigation.

In every semigroup S, the set $R(S)$ of right zeros (i.e. elements r such that $xr = r$ for all x in S) is a two-sided ideal. One can also characterize $R(S)$ as the only ideal of S which is a right zero semigroup (i.e. a semigroup such that $xy = y$ for all x, y). If $R(S)$ is nonempty, then the Rees factor semigroup $S/R(S)$ has a zero.

THEOREM 1. *Let S be a semigroup such that $R(S)$ is nonempty. Then S has property* (B) *if and only if $S/R(S)$ has property* (A).

The proof depends on the fact that property (B) is preserved under epimorphisms and that every left congruence class containing a right zero is a left ideal.

The class of semigroups having property (A) is more restricted. In such a semigroup there are at most three right zeros, because otherwise there would exist at least two disjoint two-element left ideals. Any subsemigroup which contains just one of these right zeros is a semigroup with a zero.

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THEOREM 2. *Let S be a semigroup which has exactly two right zeros q and r. Then S has property* (A) *if and only if $S\backslash\{q\}$ or $S\backslash\{r\}$ is a subsemigroup of S which has property* (A).

Proof. Let S have property (A). The right zeros q and r are in different classes of the congruence $\{(s, t) \mid xs = xt \text{ for all } x \in R(S)\}$. This congruence is either the identity relation, or it has exactly one nontrivial class, and this class, being a left ideal, contains a right zero, say r. Let L be the congruence class containing r. The left congruence

$$\{(s, t) \mid \text{for all } x, y \in R(S)\text{: } xs = ys \text{ if and only if } xt = yt\}$$

has one class which is a left ideal, namely $L \cup \{q\}$, and at most one further class, since $|R(S)| = 2$. This class, if it exists, contains just one element, say e, and one has $qe = q$, $re = r$, and $e = e^2$. Furthermore, $eL \subseteq L$ and $Le \subseteq L$ because L is a left ideal and a congruence class. Thus $S\backslash\{q\}$ is a subsemigroup of S. Obviously, a subsemigroup of this form is isomorphic to $S/R(S)$, and $S/R(S)$ has property (A).

Conversely, assume that $S\backslash\{q\}$ is a subsemigroup which has property (A). Every nontrivial left congruence class of S which does not contain q is a left congruence class of $S\backslash\{q\}$, and consequently, a left ideal of this subsemigroup. It therefore contains r, the zero of $S\backslash\{q\}$. Thus every nontrivial left congruence class of S contains r or q. Since every left congruence class containing a right zero is a left ideal, it follows that S has property (B). Now r is contained in every left ideal of S having more than one element, so that the intersection of two such left ideals is nonempty. Therefore S has property (A).

THEOREM 3. *Let S be a semigroup which has exactly three right zeros. Then S has property* (A) *if and only if it is isomorphic to one of the following semigroups*:

	p	q	r
p	p	q	r
q	p	q	r
r	p	q	r

	p	q	r	1
p	p	q	r	p
q	p	q	r	q
r	p	q	r	r
1	p	q	r	1

	p	q	r	a
p	p	q	r	p
q	p	q	r	p
r	p	q	r	q
a	p	q	r	p

	p	q	r	a	1
p	p	q	r	p	p
q	p	q	r	p	q
r	p	q	r	q	r
a	p	q	r	p	a
1	p	q	r	a	1

	p	q	r	e
p	p	q	r	p
q	p	q	r	p
r	p	q	r	r
e	p	q	r	e

	p	q	r	e	b
p	p	q	r	p	p
q	p	q	r	p	r
r	p	q	r	r	p
e	p	q	r	e	p
b	p	q	r	b	p

	p	q	r	e	c
p	p	q	r	p	r
q	p	q	r	p	p
r	p	q	r	r	r
e	p	q	r	e	r
c	p	q	r	c	r

	p	q	r	e	f
p	p	q	r	p	p
q	p	q	r	p	q
r	p	q	r	r	p
e	p	q	r	e	p
f	p	q	r	p	f

	p	q	r	e	g
p	p	q	r	p	p
q	p	q	r	p	r
r	p	q	r	r	r
e	p	q	r	e	e
g	p	q	r	g	g

	p	q	r	e	g	b
p	p	q	r	p	p	p
q	p	q	r	p	r	r
r	p	q	r	r	r	p
e	p	q	r	e	e	p
g	p	q	r	g	g	p
b	p	q	r	b	b	p

Proof. Let S have property (A). The right zeros, say p, q, and r, are in different classes of the congruence $\{(s, t) \mid xs = xt \text{ for all } x \in R(S)\}$. This is the identity relation, because other-

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wise there would be a nontrivial class, say L, which would be a left ideal and would contain just one right zero, say r. But then there would exist two disjoint left ideals each containing more than one element, namely L and $\{p, q\}$, contrary to the assumption that S has property (A). It follows that the homomorphism

$$\rho: s \mapsto \rho_s = \begin{pmatrix} p & q & r \\ ps & qs & rs \end{pmatrix}$$

which maps S onto a subsemigroup $\rho(S)$ of the full transformation semigroup on $\{p, q, r\}$ is one-to-one.

The left congruence

$$\{(s, t) \mid \text{for all } x, y \in R(S): \; xs = ys \text{ if and only if } xt = yt\}$$

has one class which is a left ideal, namely $\{p, q, r\}$, and its further classes, if there are any, contain just one element each. Therefore two different transformations in $\rho(S)$ induce the same partition on $\{p, q, r\}$ only if both of them are constant; i.e. equal to $\rho_p = \begin{pmatrix} p & q & r \\ p & p & p \end{pmatrix}$, $\rho_q = \begin{pmatrix} p & q & r \\ q & q & q \end{pmatrix}$ or $\rho_r = \begin{pmatrix} p & q & r \\ r & r & r \end{pmatrix}$.

For the present it may be assumed that the identity transformation is not in $\rho(S)$, as all the semigroups left out by this can be obtained from the others by adjoining identity elements. Then, since a mapping like $\begin{pmatrix} p & q & r \\ r & r & p \end{pmatrix}$ induces the same partition as its square $\begin{pmatrix} p & q & r \\ p & p & r \end{pmatrix}$, only the following transformations have to be considered: $\rho_e = \begin{pmatrix} p & q & r \\ p & p & r \end{pmatrix}$ and those obtained from ρ_e by permuting p, q, and r (idempotents), and $\rho_a = \begin{pmatrix} p & q & r \\ p & p & q \end{pmatrix}$ and those obtained from ρ_a by permuting p, q, and r (non-idempotents).

If there are no idempotents in $\rho(S)$ except ρ_p, ρ_q and ρ_r, then either $\rho(S) = \{\rho_p, \rho_q, \rho_r\}$ or there is another transformation, say $\rho_a = \begin{pmatrix} p & q & r \\ p & p & q \end{pmatrix}$. No further non-idempotent transformation can be in $\rho(S)$, since multiplying ρ_a from the right by $\rho_b = \begin{pmatrix} p & q & r \\ p & r & p \end{pmatrix}$, $\rho_c = \begin{pmatrix} p & q & r \\ r & p & r \end{pmatrix}$, $\begin{pmatrix} p & q & r \\ r & q & q \end{pmatrix}$ or $\begin{pmatrix} p & q & r \\ q & r & r \end{pmatrix}$ yields a new transformation with the same partition as ρ_a.

If there is only one idempotent in $\rho(S)$ different from ρ_p, ρ_q and ρ_r, say $\rho_e = \begin{pmatrix} p & q & r \\ p & p & r \end{pmatrix}$, then either $\rho(S) = \{\rho_p, \rho_q, \rho_r, \rho_e\}$ or one of $\rho_b = \begin{pmatrix} p & q & r \\ p & r & p \end{pmatrix}$, $\rho_c = \begin{pmatrix} p & q & r \\ r & p & r \end{pmatrix}$, $\begin{pmatrix} p & q & r \\ r & q & q \end{pmatrix}$ and $\begin{pmatrix} p & q & r \\ q & r & r \end{pmatrix}$ is also in $\rho(S)$. By the same reason as before, only $\rho(S) = \{\rho_p, \rho_q, \rho_r, \rho_e, \rho_b\}$ and $\rho(S) = \{\rho_p, \rho_q, \rho_r, \rho_e, \rho_c\}$ are possible.

If ρ_e is not the only idempotent in $\rho(S)$ different from ρ_p, ρ_q and ρ_r, then $\rho_f = \begin{pmatrix} p & q & r \\ p & q & p \end{pmatrix}$,

$\begin{pmatrix} p & q & r \\ r & q & r \end{pmatrix}$, $\rho_g = \begin{pmatrix} p & q & r \\ p & r & r \end{pmatrix}$ and $\begin{pmatrix} p & q & r \\ p & q & q \end{pmatrix}$ have to be considered. Since multiplying ρ_e from the left by $\begin{pmatrix} p & q & r \\ r & q & r \end{pmatrix}$ and from the right by $\begin{pmatrix} p & q & r \\ p & q & q \end{pmatrix}$ yields new transformations with the same partition as $\begin{pmatrix} p & q & r \\ r & q & r \end{pmatrix}$ and ρ_e respectively, only ρ_f and ρ_g qualify. Since ρ_e and $\begin{pmatrix} p & q & r \\ r & q & q \end{pmatrix}$ as well as ρ_e and $\begin{pmatrix} p & q & r \\ q & r & r \end{pmatrix}$ cannot occur together, it is easily seen that the only possible cases are

$$\rho(S) = \{\rho_p, \rho_q, \rho_r, \rho_e, \rho_f\},$$
$$\rho(S) = \{\rho_p, \rho_q, \rho_r, \rho_e, \rho_g\},$$
$$\rho(S) = \{\rho_p, \rho_q, \rho_r, \rho_e, \rho_g, \rho_b\},$$

and

$$\rho(S) = \{\rho_p, \rho_q, \rho_r, \rho_e, \rho_g, \rho_c\}.$$

However, this last case may be omitted as it arises from the last but one if p and r are interchanged.

By replacing $\rho(S)$ by S, one obtains the semigroups appearing in the theorem except the second and the fourth. An identity element can only be adjoined to those not containing e, since in the other cases $\{e, 1\}$ would be a nontrivial left congruence class.

By looking over each pair of columns in the multiplication tables, one notices that every nontrivial left congruence class in each semigroup contains at least two of the right zeros p, q, and r. Therefore these semigroups have property (A). This concludes the proof of Theorem 3.

The semigroups listed are pairwise non-isomorphic. This is true since

$$\{x \in R(S) \mid xs = x \text{ for all } s \notin R(S)\}$$

contains one element if S is the 6th or the 8th semigroup, but two elements if S is the 7th or the 9th.

In a left zero semigroup, every equivalence relation is a congruence. Therefore such a semigroup has property (B) if and only if it contains at most two elements. It is easy to see that a group has property (B) if and only if it has only the trivial subgroups. Clearly left zero semigroups and groups containing more than one element have no right zeros.

THEOREM 4. *Let S be a nonempty semigroup without right zeros. Then S has property* (B) *if and only if it is a two element left zero semigroup or a cyclic group of prime order.*

Proof. It has to be shown that if S has property (B) then it is a left zero semigroup or a group. So let S have property (B). For any $s \in S$, the left congruence $\{(x, y) \mid xs = ys\}$ is the identity relation, because otherwise there would be a left ideal L such that $|Ls| = 1$, and such a left ideal could not exist, since S has no right zeros. Thus S is right cancellative.

Now every idempotent is a right identity, and the set of idempotents is a class of the left congruence $\{(u, v) \mid u = v \text{ or } u, v \text{ are right identities}\}$. If this class contains more than one element, then it is a left ideal, from which it follows that S is a left zero semigroup.

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Thus it may be assumed that S has either no idempotent or just one idempotent e. In the latter case, the congruence

$$\{(u, v) \mid su = sv \text{ for all } s \in S\} \text{ is equal to } \{(u, v) \mid eu = ev\},$$

because e is a right identity. Therefore an element x such that $x \neq ex$ can exist only if there is a left ideal K such that $x \in K$ and $|eK| = 1$. It follows from this that $eSx = \{ex\}$ and moreover, because of right cancellativity, $eS = \{e\}$. Hence e is both a right identity and a left zero, and S is a left zero semigroup. But this contradicts the fact that e is the only idempotent of S. Thus e is the identity element of S.

In order to show that S is left simple, which means that $S^1t = S$ for every $t \in S$, we consider, for every t such that $t \neq t^2$, the relation

$$\{(x, xt^m) \mid m = 0, 1, 2, \ldots\} \cup \{(xt^n, x) \mid n = 0, 1, 2, \ldots\},$$

which is a left congruence because of right cancellativity. The t-class of this left congruence i.e. the set $\{x \mid xt^m = t \text{ or } x = t^n, m \geqq 0, n \geqq 1\}$, is nontrivial and therefore a left ideal, so that it includes S^1t. If $s \in S$ is not in S^1t, then $stt^m = t$ or $st = t^n$. In the first case, because of right cancellativity, $m = 0$, for otherwise $st^m = e \in S^1t$, e being the identity element of S. Thus $st = t$, and the same is true in the second case. Therefore $s = e$ and $S = S^1t \cup \{e\}$. But then S^1t is a semigroup without idempotents and is easily seen to fulfil condition (B). Repeating the above considerations with S^1t instead of S, one concludes that S^1t is left simple and right cancellative. Such a semigroup, however, if nonempty, does contain an idempotent (cf. Clifford & Preston [**1**], §1.11). Thus the assumption that $S^1t \neq S$ for some $t \in S$ leads to a contradiction, and therefore S is left simple.

It is well known that a nonempty left simple right cancellative semigroup containing at most one idempotent is a group.

As is shown in [**2**, (3.16)], a semigroup with zero which has only Rees left and right congruences is commutative. By Theorem 4, this holds more generally for any semigroup in which the nontrivial left and right congruence classes are respectively left and right ideals, with the single exception of the two noncommutative two-element semigroups.

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