Bull. Aust. Math. Soc. **94** (2016), 449–456
doi:10.1017/S0004972716000435

A TORSION-FREE ABELIAN GROUP EXISTS WHOSE QUOTIENT GROUP MODULO THE SQUARE SUBGROUP IS NOT A NIL-GROUP

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(Received 12 April 2016; accepted 30 April 2016; first published online 30 August 2016)

Abstract

The first example of a torsion-free abelian group $(A, +, 0)$ such that the quotient group of A modulo the square subgroup is not a nil-group is indicated (for both associative and general rings). In particular, the answer to the question posed by Stratton and Webb ['Abelian groups, nil modulo a subgroup, need not have nil quotient group', *Publ. Math. Debrecen* **27** (1980), 127–130] is given for torsion-free groups. A new method of constructing indecomposable nil-groups of any rank from 2 to $2^{\aleph_0}$ is presented. Ring multiplications on p-pure subgroups of the additive group of the ring of p-adic integers are investigated using only elementary methods.

2010 *Mathematics subject classification*: primary 20K99; secondary 13A99.

Keywords and phrases: nil-groups, square subgroups, abelian groups.

1. Introduction

In this paper, we continue the research on the square subgroup of an abelian group. It can be understood as follows. Given an abelian group $(A, +, 0)$, the square subgroup $\square A$ of A is the smallest subgroup B of A satisfying the condition that if R is any ring (not necessarily associative) with the additive group A, then $R^2 \subseteq B$. The notion was partially investigated by Aghdam in [1] and it is closely connected with the paper [16] by Stratton and Webb. Aghdam continued his research on the square subgroup together with Najafizadeh in [2–4]. Nevertheless, the basic question related to the topic remained unanswered. Namely, it was not known whether the quotient group of any abelian group A modulo the square subgroup $\square A$ is a nil-group (see [1, 16]). The first (negative) answer with an example of a mixed abelian group was given recently by Najafizadeh in [15]. Previously, it was known that the answer is positive for torsion abelian groups (see [16, Theorem 2.4]). In his proof, Najafizadeh used advanced tools such as the tensor product of abelian groups and theorems for splitting modules. Therefore he could not assume the associativity of rings, which is important for many algebraists. Our much more elementary proof in [6] allows the conclusion that Najafizadeh's result remains true also for the case of associative rings. It is a well-known fact that there exists a torsion-free nil-group A such that A/nA is not a nil-group for some positive integer n and, consequently, any ring R defined on A satisfies $R^2 \subseteq nA$ (see [16]). However, the square subgroup of a torsion-free abelian group was investigated only in some special cases: for example, Aghdam and Najafizadeh have indicated some classes of torsion-free abelian groups A of rank two for which $A/\square A$ is a nil-group (see [4]).

The main result of this paper is a construction of a torsion-free abelian group A such that $A/\square A$ is not a nil-group in both cases of associative and general rings. Moreover, we give a new method of constructing indecomposable nil-groups of any rank from 2 to $2^{\aleph_0}$. We also present various effects concerning ring multiplications on p-pure subgroups of the additive group of the ring of p-adic integers. In particular, we show, using only elementary methods, that any ring multiplication on a p-pure subgroup of the additive group of the ring of p-adic integers is associative and commutative. Furthermore, we characterise, in an elementary way, subgroups of the additive group of that ring which are not nil.

This topic has a long history in algebra and is generating renewed interest. In addition to the work cited above, there are developments in recent papers of Feigelstock [9], Pham Thi Thu Thuy [17, 18] and Kompantseva [13, 14].

2. Preliminaries

Throughout the paper, the letter p stands for an arbitrary fixed prime. Symbols $\mathbb{Q}$, Q_p, $\mathbb{Z}_p$, $\mathbb{Z}$, Z_p, $\mathbb{P}$, $\mathbb{N}$, $\mathbb{N}_0$ stand for the fields of rationals, p-adic numbers and integers modulo p, the rings of integers and p-adic integers, and the sets of all prime numbers, positive integers and nonnegative integers, respectively. In this paper, only abelian groups with the traditional additive notation will be considered. Every abelian group $(A, +, 0)$ can be provided with a ring structure in a trivial way by defining $a \cdot b = 0$ for all $a, b \in A$. An abelian group A is called a nil-group (nil$_a$-group) if, on A, there does not exist any nonzero (associative) ring multiplication. It follows, from [5, Remark 2.6] and [8, Conjecture 2.1.4], that if the concepts of nil$_a$-group and nil-group are not equivalent, then there exists a torsion-free nil$_a$-group of rank more than one which is not a nil-group. Obviously, A is a nil-group exactly if $\square A = \{0\}$, so the notion of square subgroup generalises the concept of nil-group. The following formula greatly simplifies the considerations related to the topic: that is,

$$\square A = \sum_{*\in \mathrm{Mult}(A)} A * A,$$

where Mult(A) means the set of all ring multiplications on the group A. If we restrict our consideration to associative rings R with the additive group A, then the square subgroup of A is denoted by $\square_a A$. It follows, from [6, Corollary 2.6], that if there exists an abelian group A which satisfies $\square_a A \subsetneq \square A$, then A is reduced and nontorsion. More basic information about square subgroups and their generalisations is available in [1, 3, 6].

https://doi.org/10.1017/S0004972716000435 Published online by Cambridge University Press

The additive group of a ring R is denoted by R^+. The notation $I \lhd R$ means that I is an ideal of R. If R is a unital ring, then its group of units is denoted by R^*.

It is a well-known fact that any p-adic integer α is determined by a sequence $(x_n)_{n=0}^{\infty}$ of integers satisfying $x_n \equiv x_{n-1} \pmod{p^n}$ for each $n \in \mathbb{N}$. We shall write the above expression as $(x_n)_{n=0}^{\infty} \to \alpha$. Moreover, $(x_n)_{n=0}^{\infty} \to \alpha$ and $(y_n)_{n=0}^{\infty} \to \alpha$ if and only if $x_n \equiv y_n \pmod{p^{n+1}}$ for each $n \in \mathbb{N}_0$. For more preliminary knowledge of p-adic integers we refer the reader to [7].

All other designations are consistent with generally accepted standards (see, for example, [11]).

3. A simple characterisation of ring multiplications on p-pure subgroups of Z_p^+

Lemma 3.1. *For every nontrivial subgroup A of Z_p^+ the following conditions are equivalent:*

(i) $A = M \cap Z_p$ *for some nontrivial* $\mathbb{Z}[p^{-1}]$*-submodule M of the field Q_p;*
(ii) *A is p-pure in Z_p^+; and*
(iiii) $A = \langle \varepsilon \rangle + pA$ *for some* $\varepsilon \in A \cap Z_p^*$.

Proof. (i) $\Rightarrow$ (ii). Take any $x \in Z_p$. If $px \in A$, then $px \in M$ and, consequently, $x = p^{-1} \circ (px) \in M$. Thus $x \in M \cap Z_p$, that is, $x \in A$. Moreover, Z_p^+ is a torsion-free group, so A is a p-pure subgroup of Z_p^+.

(ii) $\Rightarrow$ (iii). Take any $a \in A \backslash \{0\}$. Then $a = p^m \varepsilon$ for some uniquely determined $m \in \mathbb{N}_0$ and $\varepsilon \in Z_p^*$ (compare with [7, Theorem 2]). Hence, by the p-purity of A in Z_p^+, we obtain $\varepsilon \in A$. Moreover, $\varepsilon \notin pZ_p$, so $\varepsilon \in A \backslash pA$. Next, $(A + pZ_p) \cdot Z_p = A \cdot Z_p + pZ_p = A \cdot (\mathbb{Z} + pZ_p) + pZ_p = A \cdot \mathbb{Z} + pZ_p = A + pZ_p$, and hence $A + pZ_p \lhd Z_p$. But $\varepsilon \in A \cap Z_p^*$, so $1 \in A + pZ_p$ and, consequently, $A + pZ_p = Z_p$. As A is a p-pure subgroup of Z_p^+, $A \cap pZ_p^+ = pA$. Thus $A/pA = A/(A \cap pZ_p^+) \cong (A + pZ_p^+)/pZ_p^+ = (Z_p/pZ_p)^+ \cong \mathbb{Z}_p^+$. Furthermore, $\varepsilon + pA \neq pA$ in A/pA, so $A = \langle \varepsilon \rangle + pA$.

(iii) $\Rightarrow$ (i). An easy computation shows that $M = \{ap^{-n} : a \in A, n \in \mathbb{N}_0\}$ is a nontrivial $\mathbb{Z}[p^{-1}]$-submodule of Q_p. Directly from the definition of M, it follows that $A \subseteq M \cap Z_p$. To prove the opposite inclusion, take any $x \in M \cap Z_p$. Then $x = ap^{-s}$ for some $a \in A$ and $s \in \mathbb{N}_0$. Moreover, by a simple induction argument, $a = k\varepsilon + p^{s+1}c$ for some $k \in \mathbb{Z}$ and $c \in A$. Thus $p^s x = k\varepsilon + p^{s+1}c$, and hence $k\varepsilon = p^s(x - pc)$. It follows, from [7, Theorem 2], that $k = p^s k_s$ for some $k_s \in \mathbb{Z}$, so $k_s\varepsilon = x - pc$. Hence $x = k_s\varepsilon + pc \in A$. Thus $M \cap Z_p \subseteq A$ and, finally, $A = M \cap Z_p$. □

We get the following result directly from the proof of Lemma 3.1.

Corollary 3.2. *For every nontrivial p-pure subgroup A of Z_p^+, $A/pA \cong \mathbb{Z}_p^+$. In particular, $A = \langle a \rangle + p^n A$ for all $a \in A \backslash pA$ and $n \in \mathbb{N}$.*

Lemma 3.3. *If A and B are nontrivial p-pure subgroups of Z_p^+, then so is AB.*

Proof. Take any $x \in Z_p$. If $px \in AB$, then $px = \sum_{i=1}^{n} a_i b_i$ for some $n \in \mathbb{N}$, $a_1, \dots, a_n \in A$ and $b_1, \dots, b_n \in B$. First, suppose that $a_1, \dots, a_n \in pA$. Then, for each a_i, there

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exists $x_i \in A$ such that $a_i = px_i$. Hence $p(x - \sum_{i=1}^n x_ib_i) = 0$ and, consequently, $x = \sum_{i=1}^n x_ib_i \in AB$. If $b_1, \dots, b_n \in pB$, then we proceed analogously. Now suppose that $a_j \notin pA$ and $b_s \notin pB$ for some $j, s \in \{1, \dots, n\}$. The p-purity of A and B in Z_p^+, together with [7, Theorem 2], implies that $a_j, b_s \in Z_p^*$. Moreover, it follows, from Lemma 3.1, that $A = \langle a_j\rangle + pA$ and $B = \langle b_s\rangle + pB$. Thus, for each $i \in \{1, 2, \dots, n\}$, there exist $a_i' \in A$, $b_i' \in B$ and $k_i, l_i \in \mathbb{Z}$ such that $a_i = k_ia_j + pa_i'$ and $b_i = l_ib_s + pb_i'$. Hence $px = (\sum_{i=1}^n k_il_i)a_jb_s + p\sum_{i=1}^n k_ia_jb_i' + p\sum_{i=1}^n l_ia_i'b_s + p^2\sum_{i=1}^n a_i'b_i'$ and, consequently, we get $p \mid (\sum_{i=1}^n k_il_i)a_jb_s$. Since $a_j, b_s \in Z_p^*$, it follows, from [7, Theorem 2], that $\sum_{i=1}^n k_il_i = ph$ for some $h \in \mathbb{Z}$. Thus $x = ha_jb_s + \sum_{i=1}^n k_ia_jb_i' + \sum_{i=1}^n l_ia_i'b_s + p\sum_{i=1}^n a_i'b_i'$. Hence $x \in AB$ and, finally, AB is a p-pure subgroup of Z_p^+. □

Lemma 3.4. *If $*$ is a ring multiplication on a nontrivial p-pure subgroup A of Z_p^+, then there exists $c \in Z_p$ such that $a * b = a \cdot c \cdot b$ for all $a, b \in A$. In particular, every ring R with $R^+ = A$ is associative and commutative.*

Proof. It follows, from Lemma 3.1 and Corollary 3.2, that there exists $\varepsilon \in A \cap Z_p^*$ such that, for every $n \in \mathbb{N}$, $A = \langle\varepsilon\rangle + p^nA$. Take any $a, b \in A$, $n \in \mathbb{N}$ and define $e = \varepsilon * \varepsilon$. Then $a = k_n\varepsilon + p^na_n$, $b = l_n\varepsilon + p^nb_n$ and $a * b = (k_nl_n)e + p^nx_n$ for some $k_n, l_n \in \mathbb{Z}$, $a_n, b_n, x_n \in A$. Thus, for $c = \varepsilon^{-2} \cdot e$, we get $a \cdot c \cdot b = (k_nl_n)e + p^ny_n$, where y_n is some element of A. Hence $a * b - a \cdot c \cdot b \in p^nZ_p$. Since $\bigcap_{i=1}^\infty p^iZ_p = \{0\}$ (see [7, Theorem 2]), the arbitrary choice of n implies that $a * b = a \cdot c \cdot b$. Thus the multiplication $*$ is associative and commutative. □

Proposition 3.5. *Let A be a nontrivial p-pure subgroup of Z_p^+. Then A is not a nil-group if and only if A is isomorphic to the additive group of some subring of Z_p.*

Proof. Suppose that $\square A \neq \{0\}$. It follows, from Lemma 3.1, that $A = \langle\varepsilon\rangle + pA$ for some $\varepsilon \in A \cap Z_p^*$. Since Z_p is an integral domain, the function $x \mapsto x \cdot \varepsilon^{-1}$ is an automorphism of Z_p^+. Hence $B = A \cdot \varepsilon^{-1}$ is a subgroup of Z_p^+ such that $B \cong A$ and $1 \in B$. Thus B is a p-pure subgroup of Z_p^+ with $\square B \neq \{0\}$. Hence, by Lemma 3.1 and Corollary 3.2, we get $B = \langle 1\rangle + pB$. Moreover, Lemma 3.4 implies the existence of a nonzero element c of Z_p such that $a \cdot c \cdot b \in B$ for all $a, b \in B$. Therefore $c = 1 \cdot c \cdot 1 \in B$. Furthermore, $c = p^\alpha\eta$ for some uniquely determined $\alpha \in \mathbb{N}_0$ and $\eta \in Z_p^*$ (compare with [7, Theorem 2]). Thus $p^\alpha(a \cdot \eta \cdot b) \in B$ for all $a, b \in B$. Hence, by the p-purity of B in Z_p^+, we obtain $\eta \in B$ and $a \cdot \eta \cdot b \in B$ for all $a, b \in B$. Define $S = B \cdot \eta$. Then S is a subgroup of Z_p^+ satisfying $S \cong B$ and $S \cdot S = (B \cdot \eta \cdot B) \cdot \eta \subseteq S$. Consequently, S is a subring of Z_p with $S^+ \cong A$. The opposite implication is obvious. □

The next result follows directly from the proof of the above proposition.

Corollary 3.6. *Let A be a nontrivial p-pure subgroup of Z_p^+. Then A is not a nil-group exactly if $A = S \cdot \omega$ for some subring S of Z_p and $\omega \in Z_p^*$.*

Proposition 3.7. *For every subgroup A of Z_p^+, the following conditions are equivalent:*

(i) *$A = \langle a_0\rangle + pA$ for some $a_0 \in A \setminus pA$; and*

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(ii) $A = p^\alpha B$ *for some nonnegative integer* α *and nontrivial* p*-pure subgroup* B *of* Z_p^+.

Proof. (i) $\Rightarrow$ (ii). Since $a_0 \neq 0$, it follows, from [7, Theorem 2], that $a_0 = p^m\varepsilon$ for some uniquely determined $m \in \mathbb{N}_0$ and $\varepsilon \in Z_p^*$. If $m = 0$, then Lemma 3.1 implies that A is a p-pure subgroup of Z_p^+ and it is sufficient to put $\alpha = 0$. Now suppose that $m > 0$ and define $B = \langle\varepsilon\rangle + pA$. Then $A = \langle a_0\rangle + p^{m+1}A = \langle p^m\varepsilon\rangle + p^{m+1}A = p^m B$. Thus $B = \langle\varepsilon\rangle + pB$. We apply Lemma 3.1 again to infer that B is a p-pure subgroup of Z_p^+.

(ii) $\Rightarrow$ (i). It follows, from Lemma 3.1, that there exists $\varepsilon \in B \cap Z_p^*$ such that $B = \langle\varepsilon\rangle + pB$. Hence $A = p^\alpha B = \langle p^\alpha\varepsilon\rangle + p^{\alpha+1}B = \langle p^\alpha\varepsilon\rangle + pA$. Notice that $pA = \langle p^{\alpha+1}\varepsilon\rangle + p^2A = \langle p^{\alpha+1}\varepsilon\rangle + p^{\alpha+2}B \subseteq p^{\alpha+1}Z_p$. Therefore, if $p^\alpha\varepsilon \in pA$, then $\varepsilon \in pZ_p$ and, consequently, $\varepsilon \notin Z_p^*$, which is a contradiction. Thus it suffices to put $a_0 = p^\alpha\varepsilon$. □

Remark 3.8. It is a well-known fact that there exist indecomposable nil-groups of any rank up to $2^{\aleph_0}$ (see [10, page 292, Exercise 25]). For groups of rank one, the result is obvious. Notice that Lemmas 3.1 and 3.4 are useful for constructing an indecomposable nil-group of any rank r satisfying $1 < r \leq 2^{\aleph_0}$. There exists a subset Y of Z_p of cardinality $2^{\aleph_0}$ that is algebraically independent over $\mathbb{Q}$. Let X be a nonempty subset of Y. An easy computation shows that $M = \mathbb{Z}[p^{-1}] + \sum_{x\in X}\mathbb{Z}[p^{-1}]x$ is a $\mathbb{Z}[p^{-1}]$-submodule of Q_p. Hence, by Lemma 3.1, we infer that $A = M \cap Z_p$ is a p-pure subgroup of Z_p^+. Suppose, contrary to our claim, that $\square A \neq \{0\}$. It follows, from Lemma 3.4, that there exists $c \in Z_p \backslash \{0\}$ such that, for all $a, b \in A$, $a \cdot c \cdot b \in A$. As $1 \in A$, we obtain $c = 1 \cdot c \cdot 1 \in A$. Hence there exist $s \in \mathbb{N}$, $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{Z}[p^{-1}]$, not all equal to zero, and pairwise distinct $x_1, x_2, \ldots, x_s \in X$ such that $c = \alpha_0 + \sum_{i=1}^s \alpha_i x_i$. Thus $x_1^2 \cdot c \in A$ and, consequently, $x_1^2 \cdot (\alpha_0 + \sum_{i=1}^s \alpha_i x_i) \in M$, which contradicts the algebraic independence of X over $\mathbb{Q}$. Therefore A is a nil-group. The indecomposability of A follows from [12, Theorem 88.1]. If $1 < |X| < \aleph_0$, then A is a group of rank $|X| + 1$. If $\aleph_0 \leq |X| \leq 2^{\aleph_0}$, then A is a group of rank $|X|$.

4. Main results

Proposition 4.1. *If* M *and* N *are nontrivial* $\mathbb{Z}[p^{-1}]$*-submodules of the field* Q_p*, then* $(M \cap Z_p) \cdot (N \cap Z_p) = (MN) \cap Z_p$.

Proof. Since $M \cap Z_p \subseteq M$ and $N \cap Z_p \subseteq N$, we see that $(M \cap Z_p) \cdot (N \cap Z_p) \subseteq MN$. Moreover, $M \cap Z_p$, $N \cap Z_p \subseteq Z_p$, so $(M \cap Z_p) \cdot (N \cap Z_p) \subseteq (MN) \cap Z_p$. To prove the opposite inclusion, take any $x \in (MN) \cap Z_p$. Then $x = \sum_{i=1}^n a_i b_i$ for some $n \in \mathbb{N}$, $a_1, a_2, \ldots, a_n \in M$ and $b_1, b_2, \ldots, b_n \in N$. Furthermore, it follows, from [7, Theorem 4], that there exists $s \in \mathbb{N}$ such that, for each $i \in \{1, 2, \ldots, n\}$, $p^s a_i$, $p^s b_i \in Z_p$. Thus $p^{2s}x \in (M \cap Z_p) \cdot (N \cap Z_p)$. Hence, by Lemmas 3.1 and 3.3, $x \in (M \cap Z_p) \cdot (N \cap Z_p)$. □

Remark 4.2. Let α, β be elements of Z_p that are algebraically independent over $\mathbb{Q}$. Let R be the subring of Q_p generated by p^{-1}, α, β and let S be the subring of Z_p generated by α, β. The polynomial ring $(\mathbb{Z}[p^{-1}])[x, y]$ can be treated as a subring of the polynomial ring $Q_p[x, y]$. Similarly, the polynomial ring $\mathbb{Z}[x, y]$ can be treated

https://doi.org/10.1017/S0004972716000435 Published online by Cambridge University Press

as a subring of the polynomial ring $Z_p[x,y]$. Moreover, the algebraic independence implies that $R \cong (\mathbb{Z}[p^{-1}])[x,y]$, $S \cong \mathbb{Z}[x,y]$ and

$$R = \{f(\alpha,\beta) : f \in (\mathbb{Z}[p^{-1}])[x,y]\}, \quad S = \{g(\alpha,\beta) : g \in \mathbb{Z}[x,y]\}. \tag{4.1}$$

Let $(a_n)_{n=0}^{\infty} \to \alpha$ and $(b_n)_{n=0}^{\infty} \to \beta$. From the basic properties of p-adic integers (see [7]) it follows that, for all $\gamma,\delta \in Z_p$ and $g \in \mathbb{Z}[x,y]$,

$$((c_n)_{n=0}^{\infty} \to \gamma, (d_n)_{n=0}^{\infty} \to \delta) \Rightarrow (g(c_n,d_n))_{n=0}^{\infty} \to g(\gamma,\delta). \tag{4.2}$$

Furthermore, if $(c_n)_{n=0}^{\infty} \to \gamma$ and $k \in \mathbb{N}$, then it follows, from [7, Corollary 1 and (3.4)], that p^k divides γ in Z_p exactly if p^k divides c_{k-1} in $\mathbb{Z}$. Moreover, $Z_p \cap R = \{\omega p^{-k} : \omega \in S, k \in \mathbb{N}, p^k \mid \omega\}$, so (4.2) implies that

$$Z_p \cap R = \{f(\alpha,\beta)p^{-k} : f \in \mathbb{Z}[x,y], k \in \mathbb{N}, p^k \mid f(a_{k-1},b_{k-1})\}. \tag{4.3}$$

THEOREM 4.3. *There exists a torsion-free abelian group A such that $A/\Box A$ is not a nil$_a$-group and $\Box A = \Box_a A = A \circledast A$ for some $\circledast \in \mathrm{Mult}(A)$.*

PROOF. We retain all designations of Remark 4.2 under the additional assumption that $\alpha,\beta \in Z_p^*$. Define $I = \alpha\cdot R + \beta\cdot R$ and $A = Z_p \cap I$. Then $I \lhd R$ and I is a $\mathbb{Z}[p^{-1}]$-submodule of Q_p. Hence A is a subring of Z_p and Lemma 3.1 implies that A is a p-pure subgroup of Z_p^+.

Take any $* \in \mathrm{Mult}(A)$. It follows, from Lemma 3.4, that there exists $c \in Z_p$ such that $a * b = a\cdot c\cdot b$ for all $a,b \in A$. Define $s_1 = \alpha * \alpha$ and $s_2 = \beta * \beta$. Then $s_1 = c\cdot\alpha^2$ and $s_2 = c\cdot\beta^2$, and hence $s_1/\alpha^2 = s_2/\beta^2$. Thus $s_1\beta^2 = s_2\alpha^2$. Moreover, $\mathbb{Z}[p^{-1}]$ is a unique factorisation domain, and so is R, by Gauss's lemma and Remark 4.2. Furthermore, α and β are nonassociate prime elements of R, so $\alpha^2 \mid s_1$ in R. Hence $c = s_1/\alpha^2 \in R$ and, consequently, $c \in R \cap Z_p$. Since $A \subseteq I$, $c \in R$ and $I \lhd R$, we obtain $A\cdot c \subseteq I$. Moreover, $A\cdot c \subseteq Z_p$ because $A \subseteq Z_p$ and $c \in Z_p$. Thus $A\cdot c \subseteq A$, and hence $A * A = A\cdot c\cdot A \subseteq A\cdot A = A^2$. As $*$ has been chosen arbitrarily, we get $\Box A \subseteq A^2$. Obviously, $A^2 \subseteq \Box_a A \subseteq \Box A$ so $\Box A = \Box_a A = A^2$.

Notice that $I^2 = \alpha^2\cdot R + (\alpha\cdot\beta)\cdot R + \beta^2\cdot R$. Moreover, it follows, from Proposition 4.1, that $A^2 = Z_p \cap I^2$. Take any $\Psi \in A/A^2$. Then $\Psi = \alpha\cdot\xi + \beta\cdot\zeta + A^2$ for some $\xi,\zeta \in R$. Moreover, (4.1) implies the existence of $g,h \in \mathbb{Z}[x,y]$ and $k \in \mathbb{N}$ such that $\alpha\cdot\xi + \beta\cdot\zeta = (\alpha\cdot g(\alpha,\beta) + \beta\cdot h(\alpha,\beta))p^{-k}$. As $\alpha\cdot\xi + \beta\cdot\zeta \in Z_p$, $p^k \mid \alpha\cdot g(\alpha,\beta) + \beta\cdot h(\alpha,\beta)$. Let a and b denote the constant terms of polynomials g and h, respectively. We will show that there exists $c \in \mathbb{Z}$ for which $\Psi = (a\alpha + b\beta + c\alpha^2)p^{-k} + A^2$. This equation holds if and only if $((g(\alpha,\beta) - a)\alpha + (h(\alpha,\beta) - b)\beta - c\alpha^2)p^{-k} \in A^2$, which is equivalent to $((g(\alpha,\beta) - a)\alpha + (h(\alpha,\beta) - b)\beta - c\alpha^2)p^{-k} \in Z_p$. It is true exactly if $p^k \mid (g(\alpha,\beta) - a)\alpha + (h(\alpha,\beta) - b)\beta - c\alpha^2$. It follows, from (4.3), that it holds if and only if $p^k \mid (g(a_{k-1},b_{k-1}) - a)a_{k-1} + (h(a_{k-1},b_{k-1}) - b)b_{k-1} - ca_{k-1}^2$, which is equivalent to $ca_{k-1}^2 \equiv (g(a_{k-1},b_{k-1}) - a)a_{k-1} + (h(a_{k-1},b_{k-1}) - b)b_{k-1} \pmod{p^k}$. Since $\alpha \in Z_p^*$, it follows, from [7, Theorem 1 and (3.4)], that $p \nmid a_{k-1}$. Thus the claimed element c exists. Obviously, for any $a,b,c \in \mathbb{Z}$ and $k \in \mathbb{N}$ satisfying $p^k \mid a\alpha + b\beta + c\alpha^2$, $(a\alpha + b\beta + c\alpha^2)p^{-k} \in A$, so

$$A/A^2 = \{(a\alpha + b\beta + c\alpha^2)p^{-k} + A^2 : k \in \mathbb{N}, a,b,c \in \mathbb{Z}, p^k \mid aa_{k-1} + bb_{k-1} + ca_{k-1}^2\}.$$

Consider the function $\varphi : (A/A^2)^+ \to (\mathbb{Z}[p^{-1}] \times \mathbb{Z}[p^{-1}])^+$ given by

$$\varphi((a\alpha + b\beta + c\alpha^2)p^{-k} + A^2) = (ap^{-k}, bp^{-k}).$$

Take any $a, b, c, d, e, f \in \mathbb{Z}$, $k, l \in \mathbb{N}$ such that $p^k \mid a\alpha + b\beta + c\alpha^2$, $p^l \mid d\alpha + e\beta + f\alpha^2$ and $(a\alpha + b\beta + c\alpha^2)p^{-k} + A^2 = (d\alpha + e\beta + f\alpha^2)p^{-l} + A^2$. Then

$$\begin{aligned}((p^l a - p^k d)\alpha + (p^l b - p^k e)\beta + (p^l c - p^k f)\alpha^2)p^{-(k+l)} \\ = (a\alpha + b\beta + c\alpha^2)p^{-k} - (d\alpha + e\beta + f\alpha^2)p^{-l}\end{aligned}$$

is in A^2, so $(p^l a - p^k d)\alpha + (p^l b - p^k e)\beta + (p^l c - p^k f)\alpha^2 \in A^2$. Hence $(p^l a - p^k d)\alpha + (p^l b - p^k e)\beta \in A^2$. For abbreviation, define $U = p^l a - p^k d$ and $V = p^l b - p^k e$. Since $U\alpha + V\beta \in I^2$, (4.1) implies that there exist $f_1, f_2, f_3 \in (\mathbb{Z}[p^{-1}])[x, y]$ such that $U\alpha + V\beta = \alpha^2 f_1(\alpha, \beta) + \alpha\beta f_2(\alpha, \beta) + \beta^2 f_3(\alpha, \beta)$. We apply (4.1) again to obtain $Ux = x^2 f_1(x, 0)$ and, consequently, $U = 0$. Similarly, $V = 0$, and hence $p^l a = p^k d$ and $p^l b = p^k e$. Thus $ap^{-k} = dp^{-l}$ and $bp^{-k} = ep^{-l}$. Therefore the definition of φ is correct. A straightforward verification shows that φ is an additive homomorphism. If $(a\alpha + b\beta + c\alpha^2)p^{-k} + A^2 \in \ker\varphi$, then $(ap^{-k}, bp^{-k}) = (0, 0)$ and, consequently, $a = b = 0$. Hence $(a\alpha + b\beta + c\alpha^2)p^{-k} + A^2 = c\alpha^2 p^{-k} + A^2$. Moreover, $p^k \mid ca_{k-1}^2$, so $cp^{-k} \in \mathbb{Z}$. Therefore $p^k \mid c\alpha^2$, and hence $c\alpha^2 p^{-k} \in Z_p \cap I^2 = A^2$. Thus φ is a monomorphism. Take any $z \in (\mathbb{Z}[p^{-1}] \times \mathbb{Z}[p^{-1}])^+$. Then $z = (up^{-s}, vp^{-s})$ for some $u, v \in \mathbb{Z}$ and $s \in \mathbb{N}$. Since $p \nmid a_{s-1}$, there exists $r \in \mathbb{Z}$ satisfying $-ra_{s-1}^2 \equiv ua_{s-1} + vb_{s-1} \pmod{p^s}$. Hence, by Remark 4.2, we get $p^k \mid u\alpha + v\beta + r\alpha^2$. Thus $(u\alpha + v\beta + r\alpha^2)p^{-s} \in A$ and $z = \varphi((u\alpha + v\beta + r\alpha^2)p^{-s} + A^2)$. Therefore φ is an isomorphism and, consequently, $A/\square A \cong (\mathbb{Z}[p^{-1}] \times \mathbb{Z}[p^{-1}])^+$. Finally, $A/\square A$ is not a nil_a-group. □

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https://doi.org/10.1017/S0004972716000435 Published online by Cambridge University Press

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