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SMOOTHING ONE-DIMENSIONAL FOLIATIONS ON $S^1 \times S^1$

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Let $f: S^1 \to S^1$ be an orientation preserving C^1-diffeomorphism. Denote by $\sum(f)$ the flow on $S^1 \times S^1$ which is the suspension of f (see Smale [5]).

We consider the problem of approximating $\sum(f)$ by a smoother foliation.

THEOREM. (a) *If f is C^1 and structurally stable, or*

(b) *If f is C^1, has a finite* (>0) *number of periodic points of period p, and f^p has derivative $\neq 1$ at the periodic points of f, or*

(c) *If f is C^2,*

then $\sum(f)$ can be C^0-approximated (i.e. pointwise) by a C^∞ foliation C^0-conjugate to it.

(d) *There exist examples of C^1 maps f such that no foliation on $S^1 \times S^1$ of class C^2 is C^0-conjugate to $\sum(f)$.*

For the definition of approximation of foliations see Cohen [1].

Proof. Since the C^0-conjugacy class of $\sum(f)$ is determined by the C^0-conjugacy class of f, and since if f and f' are close then $\sum(f)$ and $\sum(f')$ are close (see Smale [5] and Denjoy [2]), (a) and (b) are immediate.

In [2] Denjoy constructs a C^1, orientation preserving diffeomorphism $f: S^1 \to S^1$ which has a minimal invariant closed set which is a Cantor set. The suspension $\sum(f)$ then has an exceptional leaf. On the other hand, Schwartz shows in [4] that C^2 foliations of codimension one on compact two-dimensional manifolds which come from vector fields (as $\sum(f)$ does) have no exceptional leaves. Hence $\sum(f)$ cannot be C^0-conjugate to a foliation of class C^2, which shows (d).

The proof of (c) is immediate using the following:

PROPOSITION. *Let $f: S^1 \to S^1$ be an orientation preserving C^2 diffeomorphism. Then f is C^0-conjugate to a C^∞ diffeomorphism $f': S^1 \to S^1$, which is C^0-close (i.e. pointwise) to f.*

Proof. By [2], a minimal closed invariant set for f is either S^1 or a finite set of points. Hence either very orbit is dense or f has periodic points with common period p. In the first case f is conjugate to a rotation by an irrational angle

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$R:S^1 \to S^1$ by a homeomorphism $h_0:S^1 \to S^1$ (see Van Kampen [6]). Approximate h_0 by a C^0-close C^∞ diffeomorphism $g:S^1 \to S^1$. We have

$$h_0 f h_0^{-1} = R, \qquad g^{-1} h_0 f h_0^{-1} g = g^{-1} R g.$$

Put $f' = g^{-1}Rg$ and $h = g^{-1}h_0$. Then f' is a C^∞ diffeomorphism and h is a homeomorphism C^0-close to the identity with $hfh^{-1} = f'$. In the second case, we may assume that the periodic points are fixed points. The only minimal invariant closed sets of f are the fixed points. It will suffice to show that $f \mid [x_0, x_1]$ is conjugate to a C^0-close C^∞ diffeomorphism $f_{[x_0, x_1]}$ of $[x_0, x_1]$ with $Df_{[x_0,x_1]} = Df$ at x_0 and x_1 and prescribed values for the higher derivatives of $f_{[x_0,x_1]}$ at x_0 and x_1, where $[x_0, x_1] \subset S^1$, x_0 and x_1 fixed points of f, has one of the following properties:

(a) x_0 and x_1 are the only fixed points of f in $[x_0, x_1]$,

(b) there is a sequence $\{y_n\}$ of fixed points of f, $y_0 = x_0$, $y_i < y_{i+1}$, $\lim_{n\to\infty} y_n = x_1$ (or $y_i > y_{i+1}$, $\lim_{n\to\infty} y_n = x_0$),

(c) there is a Cantor set of fixed points of f in $[x_0, x_1]$.

Case (a) is considered in the lemma below. Case (b) follows by applying (a) successively to intervals $[y_i, y_{i+1}]$ and case (c) follows by applying (a) simultaneously to closures of intervals of length greater than or equal to $\frac{1}{2^k}$ in $[x_0, x_1] - C$, where C is the Cantor set in question, for each $k = 1, 2, \ldots$.

LEMMA. *Let $f: [0, 1] \to [0, 1]$ be a C^1 diffeomorphism, $f(0) = 0$, $f(1) = 1$ and such that f has no fixed points in $(0, 1)$. Then f is C^0-conjugate to a C^∞ diffeomorphism f' by a homeomorphism which is C^0-close to the identity, with*

$$Df'|_0 = Df|_0, \qquad Df'|_1 = Df|_1$$

and with prescribed higher derivatives at 0 *and* 1.

Proof. If $Df|_0 \neq 1$, $Df|_1 \neq 1$, any f' C^1-close to f with $Df'|_0 = Df|_0$, $Df'|_1 = Df|_1$ and with higher derivatives equal to the prescribed values will do. If $Df|_0 = 1$ or $Df|_1 = 1$, we have in addition to choose f' such that the character of the point where the derivative equals one is preserved, i.e. remains an attractor or repeller.

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