Canad. Math. Bull. Vol. **28** (2), 1985

APPROXIMATION THEOREMS FOR MANIS VALUATIONS

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ABSTRACT. Throughout this paper rings are understood to be commutative with unity. In this paper we prove the general approximation theorem for valuations whose infinite ideals have large Jacobson radicals. We give an example in which it is shown that approximation theorems for Manis valuations do not hold in the general case. Also we prove that every valuation pair (R_v, P_v) of a total quotient ring $T(R)$ whose infinite ideal has large Jacobson radical is a Prüfer valuation pair.

Throughout this paper rings are understood to be commutative with unity. In this paper we will prove the general approximation theorem for valuations whose infinite ideals have large Jacobson radicals. We will give an example in which it is shown that approximation theorems for Manis valuations do not hold in the general case. Also we prove that every valuation pair (R_v, P_v) of a total quotient ring $T(R)$ whose infinite ideal has large Jacobson radical is a Prüfer valuation pair.

We consider Manis valuations of a commutative ring. Properties of Manis valuations of a commutative ring can be found in [7] and [6], Chapter X.

Let (v, G) and (w, Λ) be two valuations on a commutative ring R and $w = \varphi \cdot v$ where φ is an order homomorphism of the group G onto the group Λ. Then we say that w dominates v and we write $w \geq v$. Valuations v and v' are called dependent if there exists a valuation w with $w \geq v$ and $w \geq v'$ and $w(R) \neq \{w(1), w(0)\}$; and they are called independent otherwise. Note that $w \geq v$ implies that $v^{-1}(\infty) = w^{-1}(\infty)$. It is easy to show that $w \geq v$ if and only if $A_v \subseteq A_w$ and $v^{-1}(\infty) \subseteq P_w \subseteq P_v$, where A_v and A_w are valuation rings and P_v and P_w are positive ideals of v and w ([7], Proposition 4). Let (R, P) be a Prüfer valuation pair and let R_1 be an overring of R, i.e. let R_1 be a ring with $R \subseteq R_1 \subseteq T(R)$ where $T(R)$ is the total quotient ring of R. Then there exists a prime ideal P_1 of R such that $P_1 \subseteq P$ and (R_1, P_1) is a Prüfer valuation pair ([2], Theorem 2.5). Therefore, if v and w are Prüfer valuations of a total quotient ring $T(R)$, then $w \geq v$ if and only if $A_w \supseteq A_v$, where A_v and A_w are valuation rings of v and w.

Much of the notation and terminology of the three next paragraphs comes from [8], pages 126–127.

Let v_i, v_j be two incomparable valuations on a commutative ring R, V_i, V_j be the corresponding valuation rings, P_i, P_j be the corresponding positive ideals and let G_i, G_j

Received by the editors July 15, 1983 and, in revised form, January 19, 1984.
AMS Subject Classification (1980): Primary, 13A18; Secondary, 13F05.

be the corresponding value groups. Let $v_i^{-1}(\infty) = v_j^{-1}(\infty)$ and let P be the maximal prime ideal of V_i and V_j such that $P \subseteq P_i$ and $P \subseteq P_j$. Certainly, $P \supseteq v_i^{-1}(\infty) = v_j^{-1}(\infty)$ and $P = v_i^{-1}(\infty) = v_j^{-1}(\infty)$ if and only if the valuations v_i and v_j are independent, i.e. the valuation $v_i \wedge v_j$ is trivial, where $v_i \wedge v_j$ is the valuation on R such that $v_i \wedge v_j \geq v_i$, $v_i \wedge v_j \geq v_j$ and there is no valuation v such that $v \geq v_i$, $v \geq v_j$ and $v < v_i \wedge v_j$. Since the valuations v_i and v_j are incomparable it follows that $P \neq P_i$ and $P \neq P_j$. Let Δ_{ij}, Δ_{ji} be the isolated subgroups of the groups G_i, G_j respectively corresponding to P. Then $\Delta_{ij} = G_i$, $\Delta_{ji} = G_j$ if and only if the valuations v_i and v_j are independent. If $v_i^{-1}(\infty) \neq v_j^{-1}(\infty)$, then the valuations v_i and v_j are independent and let again $\Delta_{ij} = G_i$, $\Delta_{ji} = G_j$. Let

$$\Theta_{ij}: G_i \to G_i/\Delta_{ij}, \qquad \Theta_{ji}: G_j \to G_j/\Delta_{ji}$$

be the natural homomorphisms. The groups G_i/Δ_{ij} and G_j/Δ_{ji} are ordered isomorphic with the value group of $v_i \wedge v_j$, and consequently they can be identified.

A pair $(\alpha_i, \alpha_j) \in G_i \times G_j$ is called compatible if, in the preceding identification, $\Theta_{ij}(\alpha_i) = \Theta_{ji}(\alpha_j)$. Let $v_1, \ldots, v_s$ $(s \geq 2)$ be pairwise incomparable valuations of R. Then $(\alpha_1, \alpha_2, \ldots, \alpha_s) \in G_1 \times G_2 \times \ldots \times G_s$ is called compatible if and only if every pair (α_i, α_j) $(i \neq j)$ is compatible. If $\alpha_i = v_i(x)$, $\alpha_j = v_j(x)$ $(x \in R)$, then the pair (α_i, α_j) is compatible since $\overline{v_i(x)} = w(x) = \overline{v_j(x)}$, where

$$w = v_i \wedge v_j, \qquad \overline{v_i(x)} = \Theta_{ij}(v_i(x)), \qquad \overline{v_j(x)} = \Theta_{ji}(v_j(x)).$$

If valuations $v_1, v_2, \ldots, v_s$ are pairwise independent, then every $(\alpha_1, \alpha_2, \ldots, \alpha_s) \in G_1 \times G_2 \times G_s$ is compatible.

Many approximation theorems for valuations of a field are known. For example, ([8], Theorem 3, page 136) is the general approximation theorem for valuations of a field. Approximation theorems for Manis valuations are proved in [7], [3], [4], and [5]. In [7], [3] and [4] approximation theorems for valuations with the inverse property are proved. For a definition of the concept of the inverse property see [7], page 196. In [5] an approximation theorem is proved for a Manis valuations of a ring R with large Jacobson radical. For the definition of the concept of rings with large Jacobson radicals see [5], page 423. In this paper we prove the general approximation theorem for valuations whose infinite ideals have large Jacobson radicals (Theorem 3).

The main result of this paper is Theorem 3. To prove Theorem 3 we use the following lemma and Theorem 2.

LEMMA 1. *Let $v_1, v_2, \ldots, v_n$ be valuations of a commutative ring R with value groups $G_1, G_2, \ldots, G_n$ respectively and let $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in G_1 \times G_2 \times \ldots \times G_n$. Then there exists $x \in R$ such that $v_i(x) \leq \alpha_i$ $(i = 1, 2, \ldots, n)$.*

PROOF. Let $n = 2$. Take $x', x'' \in R$ such that $v_1(x') \leq \alpha_1$, $v_2(x'') \leq \alpha_2$. If $v_2(x') \leq \alpha_2$ or $v_1(x'') \leq \alpha_1$ we may take $x = x'$ or $x = x''$. If $v_2(x') > \alpha_2$ and $v_1(x'') > \alpha_1$, we may take $x = x' + x''$ since then

$$v_1(x) = v_1(x') \leq \alpha_1, \; v_2(x) = v_2(x'') \leq \alpha_2.$$

https://doi.org/10.4153/CMB-1985-019-5 Published online by Cambridge University Press

Let now $n > 2$. We may suppose that $\alpha_i < 0$ ($i = 1, 2, \ldots, n$). Suppose that the statement is true for $n - 1$. Take $x', x'' \in R$ such that $v_i(x') \leq \alpha_i$ ($i = 1, 2, \ldots, n - 1$) and $v_i(x'') \leq \alpha_i$ ($i = 2, 3, \ldots, n$). We may assume that $v_i(x') \neq v_i(x'')$ ($i = 1, 2, \ldots, n$) Namely, we have $v_i(x'^m) \neq v_i(x'')$ for some nonnegative integer m ($i = 1, 2, \ldots, n$) and $v_i(x'^m) \leq \alpha_i$ ($i = 1, 2, \ldots, n - 1$). If $v_n(x') \leq \alpha_n$ or $v_1(x'') \leq \alpha_1$ we may take $x = x'$ or $x = x''$. If $v_n(x') > \alpha_n$ and $v_1(x'') > \alpha_1$, then for $x = x' + x''$ we have $v_i(x) \leq \alpha_i$ ($i = 1, 2, \ldots, n$).

Let v be a valuation of a domain D such that $v^{-1}(\infty) = 0$. By writing $\bar{v}(a/b) = v(a) - v(b)$, $a, b \in D$, $b \neq 0$, we obtain a valuation of the quotient field Q of D. $\bar{v}$ extends v and v and $\bar{v}$ have the same value group.

We now consider integral domains having Jacobson radical different from zero. For example, $J \neq 0$ for every semi-quasi-local domain.

THEOREM 2. *Let D be an integral domain and let $J \neq 0$, where J is the Jacobson radical of D. Let v_i ($i = 1, 2, \ldots, n$) be incomparable valuations of D with value groups G_i ($i = 1, 2, \ldots, n$) and let $v_i^{-1}(\infty) = 0$ ($i = 1, 2, \ldots, n$). Let $(\alpha_1, \ldots, \alpha_n) \in G_1 \times \ldots \times G_n$ be compatible, and let $a_1, \ldots, a_n \in D$. Then there exists $x \in D$ such that $v_i(x - a_i) = \alpha_i$ ($i = 1, 2, \ldots, n$) if and only if*

$$v_i(a_i - a_j) < \alpha_i \Rightarrow \alpha_i - v_i(a_i - a_j) \in \Delta_{ij}.$$

PROOF. Let Q be the quotient field of D. Then there exists $y \in Q$ such that $\bar{v}_i(y - a_i) = \alpha_i$ ($i = 1, 2, \ldots, n$) by [1], Theorem 2. We will prove that there exists $x \in D$ such that $v_i(x - a_i) = \alpha_i (i = 1, 2, \ldots, n)$. We have $y = a/b$; $a, b \in D$. Let $u \in J$, $u \neq 0$. Let $\delta_i \in G_i$ be such that

$$\delta_i < -|v_i(ub)| - |v_i(a_i)| - |\alpha_i| \ (i = 1, 2, \ldots, n).$$

By Lemma 1 there exists $v \in D$ such that $v_i(v) \leq \delta_i$ ($i = 1, 2, \ldots, n$). Clearly $uvb \in J$, $v_i(uvb) < 0$ and this implies $v_i(uvb + 1) = v_i(uvb)$; furthermore $\bar{v}_i(a_i/uvb) > \alpha_i$ ($i = 1, 2, \ldots, n$). Consequently

$$\bar{v}_i\left(\frac{uva}{uvb + 1} - a_i\right) = \bar{v}_i\left(\frac{uva - uvba_i - a_i}{uvb + 1}\right) = \bar{v}_i\left(\frac{uva - uvba_i - a_i}{uvb}\right)$$

$$= \bar{v}_i\left(\frac{a}{b} - a_i - \frac{a_i}{uvb}\right) = \alpha_i$$

since $\bar{v}_i((a/b) - a_i) = \alpha_i$ and $\bar{v}_i(-(a_i/uvb)) > \alpha_i$ ($i = 1, 2, \ldots, n$). Since $uvb \in J$ it follows that $uvb + 1$ is an invertible element of D, therefore $v_i(x - a_i) = \alpha_i$ ($i = 1, 2, \ldots, n$) for $x = (uva/(uvb + 1)) \in D$. For the converse of this theorem see [8], Theorem 3, page 136.

As a special case of the preceding theorem we have the approximation theorem for independent valuations and the approximation theorem in the neighbourhood of zero.

We now prove the general approximation theorem for valuations whose infinite ideals have large Jacobson radicals.

THEOREM 3. *Let R be a ring and let v_i $(i = 1, 2, \ldots, n)$ be incomparable valuations of R whose infinite ideals have large Jacobson radicals. Let G_i $(i = 1, 2, \ldots, n)$ be the value groups of v_i $(i = 1, 2, \ldots, n)$ respectively, let $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in G_1 \times G_2 \times \ldots \times G_n$ be compatible and let $a_1, a_2, \ldots, a_n \in R$. Then there exists $x \in R$* such *that $v_i(x - a_i) = \alpha_i$ $(i = 1, 2, \ldots, n)$ if and only if*

$$(1) \qquad v_i(a_i - a_j) < \alpha_i \Rightarrow \alpha_i - v_i(a_i - a_j) \in \Delta_{ij}.$$

PROOF. Let condition (1) be satisfied. We will prove that there exists $x \in R$ such that $v_i(x - a_i) = \alpha_i$ $(i = 1, 2, \ldots, n)$. Let $v_i^{-1}(\infty) = v_j^{-1}(\infty)$ $i, j \in \{1, 2, \ldots, n\}$. Then there exists $x \in R$ such that $v_i(x - a_i) = \alpha_i$ $(i = 1, 2, \ldots, n)$ by Theorem 2. See also [5], Proposition 3. Let now $v_i^{-1}(\infty) \neq v_j^{-1}(\infty)$ for some $i, j \in \{1, 2, \ldots, n\}$. We will prove by induction on n that there exists $x \in R$ such that $v_i(x - a_i) = \alpha_i$ $(i = 1, 2, \ldots, n)$. The theorem is true for $n = 1$. Let $n > 1$. Let the valuations v_i $(i = 1, 2, \ldots, n)$ be so ordered that $v_j^{-1}(\infty) \not\subseteq v_n^{-1}(\infty)$ for every $j \in \{1, 2, \ldots, k\}$ and

$$v_{k+i}^{-1}(\infty) = v_{k+2}^{-1}(\infty) = \ldots = v_n^{-1}(\infty).$$

By the induction hypothesis there exists $a \in R$ such that $v_i(a - a_i) = \alpha_i$ $(i = 1, 2, \ldots, k)$ and by Theorem 2 there exists $b \in R$ such that

$$v_i(b + a - a_i) = \alpha_i \quad (i = k + 1, k + 2, \ldots, n).$$

Take

$$u \in J(v_n^{-1}(\infty) \cap v_1^{-1}(\infty) \cap v_2^{-1}(\infty) \cap \ldots \cap v_k^{-1}(\infty)$$

such that

$$v_i(u) < -|\alpha_i| - |v_i(b)| \quad (i = k + 1, k + 2, \ldots, n) \text{ (Lemma 1).}$$

Then $v_i(1 + u) = v_i(u)$ and there exists $v \in R$ such that $(1 + u)v - 1 \in v_n^{-1}(\infty)$. Therefore, we have

$$v_i((1 + u)vb + a - a_i) = \alpha_i \quad (i = k + 1, k + 2, \ldots, n),$$

i.e., since $v_i(vb) > \alpha_i$ $(i = k + 1, k + 2, \ldots, n)$ it follows that

$$v_i(uvb + a - a_i) = \alpha_i \quad (i = k + 1, k + 2, \ldots, n).$$

If $x = uvb + a$, then since $uvb \in v_1^{-1}(\infty) \cap \ldots \cap v_k^{-1}(\infty)$, we have $v_i(x - a_i) = \alpha_i$ $(i = 1, 2, \ldots, n)$. For the converse of this theorem see [8], Theorem 3, page 136.

Approximation theorems for Manis valuations do not hold in the general case. We show this in the following example where two incomparable valuations of a domain are given for which the approximation theorem in the neighbourhood of zero does not hold.

EXAMPLE. Let V be a valuation domain whose value group is isomorphic to the direct sum $Z \oplus Z$ of integral numbers with the lexicographic order and let K be the quotient field of V. Let M be the maximal ideal of V and let P be the prime ideal of V with $(0) \subset P \subset M$. The value group of V_P is isomorphic to the group $Z \oplus Z/0 \oplus Z$. We denote $(1 \oplus 0) + (0 \oplus Z) = \bar{1}$ and $0 \oplus 1 = \bar{\bar{1}}$. We consider the polynomial ring $K[x]$

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and let $f \sum_{k=0}^{n} a_k x^k \in K[x]$. $\bar{v}(f) = \min\{v(a_k) + k \cdot \bar{\bar{1}}\}$ and $\bar{v}_P(f) = \min\{v_P(a_k) - k \cdot \bar{1}\}$ define two Manis valuations on $K[x]$ respectively. The valuation rings $\bar{V}$, $\bar{V}_P$ of these valuations are incomparable. Namely, for some $a \in M \backslash P$ we have $\bar{v}(1/a) < 0$, $\bar{v}_P(1/a) = 0$ and therefore $1/a \in \bar{V}_P$, $1/a \notin \bar{V}$; and $\bar{v}(x) = \bar{\bar{1}} > 0$, $\bar{v}_P(x) = -\bar{1}$ and therefore $x \in \bar{V}$, $x \notin \bar{V}_P$. The approximation theorem in the neighbourhood of zero does not hold for the valuations $\bar{v}$ and $\bar{v}_P$. Namely, we easily see that there does not exist an element g such that $\bar{v}(g) = 0$, $\bar{v}_P(g) > 0$.

THEOREM 4. *Let $T(R)$ be a total quotient ring. If (R_v, P_v) is a valuation pair of $T(R)$ whose infinite ideal has large Jacobson radical, then (R_v, P_v) is a Prüfer valuation pair.*

PROOF. Let A be a regular ideal of R_v and $A \not\subseteq P_v$. Take $x \in A$ such that $v(x) = 0$ and $y \in J(v^{-1}(\infty))$ such that $v(y) = 0$. Then $xy \in A \cap J(v^{-1}(\infty))$ and $v(xy) = 0$. Since A is a regular ideal of R_v there exists $r \in A$ such that r is a regular element of R_v. Then $v(xy + r) = 0$, $xy + r \in A$ and there exists $z \in R_v$ such that $(xy + r)z - 1 \in v^{-1}(\infty)$, therefore $1 \in A$. Therefore, if A is a regular ideal of R_v, then $A \subseteq P_v$ and consequently (R_v, P_v) is a Prüfer valuation pair by [2], Theorem 2.3.

COROLLARY. *If the infinite ideal $v^{-1}(\infty)$ of a valuation v of a total quotient ring $T(R)$ is contained in only finitely many maximal ideals of $T(R)$, then (R_v, P_v) is a Prüfer valuation pair.*

Let R be a ring and let v_i $(i = 1, 2, \ldots, n)$ be valuations on R. Let P be a prime ideal of R such that $P \not\subseteq \cup v_i^{-1}(\infty)$ and let $x \in P \backslash \cup v_i^{-1}(\infty)$. If the valuations v_i $(i = 1, 2, \ldots, n)$ have the inverse property then there exists $y \in R$ such that $v_i(xy) = 0$ $(i = 1, 2, \ldots, n)$. Therefore, $xy \in P$ and $v_i(xy) = 0$ $(i = 1, 2, \ldots, n)$. In the following theorem we claim only that for every prime ideal P of R with $P \not\subseteq \cup v_i^{-1}(\infty)$ there exists $x_P \in P$ such that $v_i(x_P) = 0$ $(i = 1, 2, \ldots, n)$. Therefore the following theorem generalizes Proposition 15 of [7].

THEOREM 5. *Let v_i $(i = 1, 2, \ldots, n)$ be incomparable valuations of a ring R with the value groups G_i $(i = 1, 2, \ldots, n)$ and with the property: if P is a prime ideal of R such that $P \not\subseteq \cup v_i^{-1}(\infty)$, then $v_i(x_P) = 0$ $(i = 1, 2, \ldots, n)$ for some $x_P \in P$. If $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in G_1 \times G_2 \times \ldots \times G_n$ is compatible then there exists an $a \in R$ such that $v_i(a) = \alpha_i$ $(i = 1, 2, \ldots, n)$.*

PROOF. Let S be a multiplicative system of R generated by the set $\{x_P | P$ a prime ideal of R such that $P \not\subseteq \cup v_i^{-1}(\infty)\}$. Then the ring R_S is a semi-quasi-local ring. Let $(v_i)_S$ $(i = 1, 2, \ldots, n)$ be the valuations on R_S corresponding in the natural way to the valuations v_i $(i = 1, 2, \ldots, n)$. Since R_S is a semi-quasi-local ring it follows that $(v_i)_S(a/s) = \alpha_i$ $(i = 1, 2, \ldots, n)$ for some $a/s \in R_S$. For $a \in R$ we have $v_i(a) = \alpha_i$ $(i = 1, 2, \ldots, n)$.

I want to thank the referee for his careful reading of the manuscript and several valuable suggestions for improvement of this paper.

https://doi.org/10.4153/CMB-1985-019-5 Published online by Cambridge University Press

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