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QUASI-UNIFORM SPACES AND TOPOLOGICAL HOMEOMORPHISM GROUPS(1)

BY
MASSOOD SEYEDIN

Let X be a topological space and G a subgroup of the homeomorphism group $H(X)$ with the topology of point-wise convergence. It is well-known that if X is uniformizable and G is equicontinuous with respect to a compatible uniformity then G is a topological group. In this paper we show that essentially this same result applies when X is only an R_0-space (and hence in particular if X is T_1 or regular). A corresponding result for regular spaces has been proved [2].

A quasi-uniformity on a set X is a filter $\mathscr{U}$ on $X \times X$ such that

(i) $\Delta \subset U$ for each $U \in \mathscr{U}$

(ii) For each $U \in \mathscr{U}$ there is $V \in \mathscr{U}$ such that $V \circ V \subset U$.

A topological space X is R_0 (also called *essentially* T_1) provided that for $x, y \in X$ either $\{\bar{x}\}=\{\bar{y}\}$ or $\{\bar{x}\} \cap \{\bar{y}\}=\phi$ [1]. A quasi-uniformity $\mathscr{U}$ on a set X is *locally symmetric* provided that for each $x \in X$ and each $U \in \mathscr{U}$ there is a symmetric entourage $V \in \mathscr{U}$ such that $V(x) \subset U(x)$ [3]. It is known that a topological space is R_0 if it is either regular or T_1 and that a topological space admits a compatible locally symmetric quasi-uniformity if and only if it is an R_0 space [3, Theorem 3.6].

DEFINITION. Let F be a collection of maps from a topological space X to a quasi-uniform space $(Y, \mathscr{V})$. The *quasi-uniformity of point-wise convergence*, $\mathscr{V}$, on F has a subbase all sets of the form $W(x, V)=\{(f, g) \in F \times F : (f(x), g(x)) \in V\}$ where $x \in X$ and $V \in \mathscr{V}$. The collection F is *quasi-equicontinuous* if for every $x \in X$ and $V \in \mathscr{V}$ there is a neighborhood N of x so that for each $f \in F$, $f(N) \subset V(f(x))$.

THEOREM 1. *Let (X, τ) be an R_0 topological space, let μ be a compatible locally symmetric quasi-uniformity on X and let G be a quasi-equicontinuous group of homeomorphisms of X onto X. Let $\Psi: G \to G$ be defined by $\Psi(g)=g^{-1}$. Then $\Psi: G \to G$ is a continuous function with respect to the topology of point-wise convergence.*

Proof. For each $x \in X$ let P_x be the xth projection map and define $\phi_x: G^- X$ by $\phi_x(g)=g^{-1}(x)$. It is clear that for each $x \in X$, $\phi_x=P_x \circ \Psi$. Thus in order to show that Ψ is continuous it suffices to show that for each $x \in X$, ϕ_x is continuous. Let $f \in G$,

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let $x \in X$ and let $V \in \mathscr{U}$. There is a symmetric $U \in \mathscr{U}$ such that $U(f^{-1}(x)) \subset V(f^{-1}(x))$. Since G is a quasi-equicontinuous collection, there exists $A \in \mathscr{U}$ such that for $g \in G$ $g(A(x)) \subset U(g(x))$. Thus for $g \in G$, $(g^{-1}(y), g^{-1}(x)) \in U$ whenever $(x, y) \in A$. Suppose that $g \in W(f^{-1}(x), A)(f)$. Then $(f(f^{-1}(x), g(f^{-1}(x)) \in A$ so that $(g^{-1}(x), f^{-1}(x)) \in U$. Since U is symmetric $g^{-1}(x) \in U(f^{-1}(x)) \subset V(f^{-1}(x))$. Hence ϕ_x is continuous.

THEOREM 2. *Let (X, τ) be a topological space, let μ be a compatible quasi-uniformity on X, and let G be a group of homeomorphisms of X onto X which is quasi-equicontinuous with respect to $\mathscr{U}$. Then G is a topological semigroup under the topology of point-wise convergence.*

Proof. Throughout the proof, if $p \in X$ and $U \in \tau$, then $S(p, U)$ denotes $\{g \in G : g(p) \in U\}$. Let $g_1, g_2 \in G$ and let $x \in X$ and $B \in \tau$ such that $S(x, B)$ is a neighborhood of $g_1 \circ g_2$. Then $g_1 \circ g_2(x) \in B$. Let $y = g_1 \circ g_2(x)$ and let $U \in \mathscr{U}$ such that $U(y) \subset B$. Then $S(x, U(y))$ is a neighborhood of $g_1 \circ g_2$ which is contained in $S(x, B)$. Let $V \in \mathscr{U}$ such that $V \circ V \subset U$. Since G is quasi-equicontinuous with respect to $\mathscr{U}$, there exists $Z \in \mathscr{U}$ such that for each $g \in G$, $g(Z(g_2(x))) \subset V(g(g_2(x)))$. If $g \in G$ and $z \in Z(g_2(x))$, then $(g(g_2(x)), g(z)) \in V$. Let $C = S(g_2(x), V(y))$ and let $D = S(x, Z(g_2(x)))$. Then C and D are neighborhoods of g_1 and g_2 respectively. Let $g \in C$ and let $h \in D$. Then $(y, g(g_2(x))) \in V$ and $(g_2(x), h(x)) \in Z$. Since $(g_2(x), h(x)) \in Z$, $(g(g_2(x)), g(h(x))) \in V$. Since $(y, g(g_2(x)))$ and $(g(g_2(x)), g(h(x))) \in V$, $(y, g(h(x))) \in V \circ V \subset U$. It follows that $g \circ h(x) \in U(y) \subset B$.

THEOREM 3. *Let (X, τ) be an R_0 space, let μ be a compatible locally symmetric quasi-uniformity and let G be a group of homeomorphisms of X onto X such that G is quasi-equicontinuous with respect to $\mathscr{U}$. Then G is a topological group under the topology of point-wise convergence.*

In the case that (X, τ) is a regular space (and hence R_0), Theorem 3 may be obtained as a consequence of [2, Theorem 6].

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THE NATIONAL UNIVERSITY OF IRAN,
EVEEN, TEHRAN, IRAN

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