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WEAK PROPER DISTRIBUTION OF VALUES OF MULTIPLICATIVE FUNCTIONS IN RESIDUE CLASSES

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Abstract

For a class of multiplicative integer-valued functions f the distribution of the sequence $f(n)$ in restricted residue classes modulo N is studied. We consider a property weaker than weak uniform distribution and study it for polynomial-like multiplicative functions, in particular for $\varphi(n)$ and $\sigma(n)$.

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1. Introduction

Let X be a set partitioned into finitely many disjoint classes, say $X = \bigcup_{j=1}^{N} X_j$, let $A:\ a_1, a_2, \ldots$ be an infinite sequence of elements of X, and put

$$F_j(x) = |\{n \le x :\ a_n \in X_j\}|.$$

The sequence A is said to be uniformly distributed in classes X_j, provided

$$\lim_{x\to\infty} \frac{F_j(x)}{x} = \frac{1}{N}$$

holds for $j = 1, 2, \ldots, N$. If this happens, then the ratios

$$\frac{F_{j_1}(x)}{F_{j_2}(x)} \tag{1.1}$$

tend to unity. We shall consider a weaker condition, requiring only that each ratio (1.1) tends to a positive limit. If this holds, then we shall say that the sequence A is *properly distributed* in classes X_j.

In this paper we shall deal with the proper distribution of values of arithmetical functions in residue classes j modulo N satisfying $(j, N) = 1$ (restricted residue classes modulo N). This is interesting only for functions f for which the set $\{n : (f(n), N) = 1\}$ is infinite.

A necessary and sufficient condition for uniform distribution of the sequence $f(n) \bmod N$ in restricted residue classes modulo N (*weak uniform distribution*) has been given in [3] (see also [5]). It implies in particular that the values of the Euler function $\varphi(n)$ are weakly uniformly distributed in restricted residue classes modulo N if and only if $(N, 6) = 1$. This criterion has been applied for the sum of divisors $\sigma(n)$ in [9] and for $\sigma_k(n)$ in [4, 6, 7].

Some time ago Dence and Pomerance [2] considered the Euler function $\varphi(n)$ modulo 3 and showed that the ratio

$$\frac{|\{n \le x : \varphi(n) \equiv 1 \bmod 3\}|}{|\{n \le x : \varphi(n) \equiv 2 \bmod 3\}|}$$

tends to a positive value, thus $\varphi(n)$ has a weak proper distribution modulo 3.

We shall show that the method used in [3, 5] can be applied to obtain criteria for this property to hold for a large class of polynomial-like multiplicative functions and arbitrary moduli. We shall consider integer-valued multiplicative functions f which are polynomial-like, that is, for primes p satisfy the condition

$$f(p^k) = V_k(p), \tag{1.2}$$

where $k = 1, 2, \ldots,$ with $V_k(T) \in \mathbb{Z}[T]$.

For an integer $N \ge 3$ and $(k, N) = 1$ let $F_f(N, k; x)$ denote the number of integers $n \le x$ satisfying

$$f(n) \equiv k \mod N,$$

and let $F_f(N; x)$ be the number of $n \le x$ with $(f(n), N) = 1$. We assume that the last condition is satisfied for infinitely many n. Moreover, let

$$\varrho_f(N, k) = \lim_{x\to\infty} \frac{F_f(N, k; x)}{F_f(N; x)}$$

be the 'probability' of an integer n with $(f(n), N) = 1$ having $f(n)$ in the residue class $k \bmod N$, provided this limit exists. We shall say that the function f is *weakly properly distributed* modulo N if there are infinitely many n with $(f(n), N) = 1$, and for each k prime to N the number $\varrho_f(N, k)$ is positive. We shall establish the existence of $\varrho_f(N, k)$ for a large class of integer-valued multiplicative functions and give a criterion for weak proper distribution. We shall also obtain a formula permitting one to evaluate $\varrho_f(N, k)$. It will turn out in particular that the Euler function $\varphi(n)$ is weakly properly distributed modulo N for every odd N, the sum of divisors $\sigma(n)$ has this property for every $N \ge 3$, but the function $\mu_3(n)\sigma(n)$, where $\mu_3(n)$ denotes the characteristic function of cube-free integers, is weakly properly distributed modulo N only in the case where it is weakly uniformly distributed modulo N, which happens if and only if $6 \nmid N$.

2. Notation

We shall utilize in the case $(N, k) = 1$ the function

$$g(N, k; s) = \sum_{p \equiv k \bmod N} \frac{1}{p^s} - \frac{1}{\varphi(N)} \log \frac{1}{s-1},$$

which can be continued to a function regular in Re $s \geq 1$. Its value at $s = 1$, which will appear later in certain formulas, has the explicit form

$$g(N, k; 1) = \frac{1}{\varphi(N)} \sum_{\chi \neq \chi_0} \overline{\chi(k)} \log L(1, \chi) - \frac{\alpha(N)}{\varphi(N)} - \beta(N, k), \tag{2.1}$$

where

$$\alpha(N) = \log \frac{N}{\varphi(N)}$$

and

$$\beta(N, k) = \sum_{j=2}^{\infty} \frac{1}{j} \sum_{p^j \equiv k \bmod N} \frac{1}{p^j}.$$

For $m \geq 2$ and $(k, N) = 1$ we shall need also the equality

$$\sum_{p \equiv k \bmod N} \frac{1}{p^{sm}} - \frac{1}{\varphi(N)} \log \frac{1}{s - 1/m} = g(N, k; ms) - \frac{\log m}{\varphi(N)} \tag{2.2}$$

valid for Re $s > 1/m$.

By $\mu_k(n)$ $(k \geq 2)$ we shall denote the characteristic function of the set of k-free integers, so $\mu_2(n) = \mu^2(n)$.

The group of restricted residue classes mod N will be denoted by $G(N)$, by χ we shall denote Dirichlet characters modulo N, and χ_0 will be the principal character. We shall consider integer-valued multiplicative function f satisfying the condition (1.1). For $j = 1, 2, \ldots$ put

$$R_j(f, N) = \{V_j(x) \bmod N : (xV_j(x), N) = 1\}$$

and denote by $r_f(N)$ the smallest value of j for which $R_j(f, N)$ is nonempty, provided it exists. If all sets $R_j(f, N)$ are empty, then put $r_f(N) = \infty$.

If $r_f(N) = \infty$, then the condition $(f(p^j), N) = 1$ for some $j \geq 1$ and prime p implies that $p \mid N$, hence in this case the condition $(f(n), N) = 1$ can be satisfied only if all prime factors of n divide N, and this implies that

$$F_f(N; x) = O(\log^{\omega(n)} x),$$

$\omega(n)$ denoting the number of distinct prime divisors of N. We shall always assume that $r = r_f(N)$ is finite. Moreover, put

$$M_f(N) = \{x \bmod N : (xV_r(x), N) = 1\},$$

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and denote by $m_f(N)$ the ratio $|M_f(N)|/\varphi(N)$. By $\Lambda_f(N)$ we shall denote the subgroup of $G(N)$ generated by $R_r(f)$. The letter p will be restricted to prime numbers.

Note that if $r = r_f(N)$ is finite, then

$$F_f(N; x) = (c(f, N) + o(1))\frac{x^{1/r}}{\log^{1-m} x}$$

with some $c(f, N) > 0$ and $m = m_f(N)$. This follows from Delange's tauberian theorem [1] and the equality

$$\sum_{n=1}^{\infty} \frac{\chi_0(f(n))}{n^s} = g_f(N; s) \exp\Big(\sum_{\substack{p\nmid N\\ (V_r(p),N)=1}} \frac{1}{p^{rs}}\Big) = \frac{h_f(N; s)}{(s-1/r)^m},$$

valid for $\mathrm{Re}\, s > 1/r$, with $g_f(N; s)$, $h_f(N; s)$ regular for $\mathrm{Re}\, s \ge 1/r$ and not vanishing at $s = 1/r$.

3. Main result

We shall establish the following theorem.

THEOREM 3.1. *Let N be a fixed integer and let f be an integral-valued multiplicative function satisfying (1.2). Assume that $r = r_f(N) < \infty$ and denote by Ω the set of characters modulo N which are equal to 1 on the group $\Lambda = \Lambda_f(N)$. For $j \in R = R_r(f)$ let U_j be the set of solutions of the congruence*

$$V_r(x) \equiv j \mod N,$$

so that

$$\bigcup_{j\in R} U_j = M_f(N),$$

and put $m = m_f(N)$. Finally, put

$$\alpha_\chi(s) = \prod_{p|N}\Big(1 + \sum_{j=1}^{\infty} \frac{\chi(f(p^j))}{p^{js}}\Big) \cdot \exp\Big(\sum_{p\nmid N} \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j}\frac{\chi^j(f(p^r))}{p^{jrs}}\Big) \cdot \exp\Big(\sum_{j\in R}\chi(j) \sum_{i\in U_j}\Big(g(N, i, rs) - \frac{\log r}{\varphi(N)}\Big)\Big).$$

(i) *If $(k, N) = 1$, then for* $\mathrm{Re}\, s > 1/r$ *one has, with some integer t,*

$$\Phi_k(N; s) := \sum_{\substack{n\\ f(n)\equiv k \bmod N}} \frac{1}{n^s} = \frac{1}{\varphi(N)}\frac{c_k(s)}{(s-1/r)^m} + \sum_{j=1}^{t} \frac{\lambda_j(s)}{(s-1)^{\mu_j}},$$

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where

$$c_k(s) = \sum_{\chi\in\Omega} \overline{\chi(k)}\alpha_\chi(s),$$

$\lambda_1(s), \ldots, \lambda_t(s)$ *are regular for* $\mathrm{Re}\, s \geq 1/r$, *and* μ_j *are complex numbers satisfying* $\mathrm{Re}\,\mu_j < m$.

(ii) *If* $c_k(1/r) \neq 0$, *then*

$$F_f(N, k; x) = \left(\frac{rc_k(1/r)}{\varphi(N)\Gamma(m)} + o(1)\right)\frac{x^{1/r}}{\log^{1-m} x}.$$

If $c_k(1/r) = 0$ *but* $c_k(s)$ *does not vanish identically, then, with a certain u,*

$$c_k(s) = (s - 1/r)^u c_k'(s)$$

with $c_k'(s)$ *regular for* $\mathrm{Re}\, s \geq 1/r$, $c_k'(1/r) \neq 0$, *and*

$$F_f(N, k; x) = \left(\frac{rc_k'(1/r)}{\varphi(N)\Gamma(m-u)} + o(1)\right)\frac{x^{1/r}}{\log^{1+u-m} x}.$$

(iii) *The ratio* $\varrho_f(N, k)$ *exists for each k prime to N, is equal to*

$$\varrho_f(N, k) = \frac{1}{\varphi(N)}\frac{c_k(1/r)}{\alpha_{\chi_0}(1/r)},$$

and depends only on the coset $k\Lambda$.

(iv) *The function f is weakly properly distributed modulo N if and only if, for each k prime to N, one has* $c_k(1/r) \neq 0$.

4. Proof of Theorem 3.1

Proof. Our starting point is the equality

$$\Phi_k(N; s) = \frac{1}{\varphi(N)}\sum_{\chi} \overline{\chi(k)} F_\chi(s), \tag{4.1}$$

with

$$F_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(f(n))}{n^s} = \prod_p \left(1 + \sum_{j=1}^{\infty}\frac{\chi(f(p^j))}{p^{js}}\right),$$

the series and the product being absolutely convergent for $\mathrm{Re}\, s > 1/r$ in view of the definition of r.

The behavior of $F_\chi(s)$ is determined in the following lemma.

Lemma 4.1. *For* $\mathrm{Re}\, s > 1/r$,

$$F_\chi(s) = \frac{\alpha_\chi(s)}{(s - 1/r)^{m(\chi)}},$$

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where

$$m(\chi) = \frac{1}{\varphi(N)} \sum_{j\in R} |U_j|\chi(j).$$

The function $\alpha_\chi(s)$ *is regular for* $\mathrm{Re}\, s \ge 1/r$, *and vanishes at* $s = 1/r$ *if and only if there is a prime* p *dividing* N *and satisfying* $p \le 2^r$ *with*

$$\sum_{j=1}^{\infty} \frac{\chi(f(p^j))}{p^{j/r}} = -1.$$

In the case $r = 1$ *this is possible only if, for* $j = 1, 2, \ldots, \chi(f(2^j)) = -1$.

Explicitly,

$$\alpha_\chi(s) = B_\chi(s)C_\chi(s)\exp\Big(h_\chi(s) + \sum_{j\in R}\chi(j)\sum_{i\in U_j}\Big(g(N, i, rs) - \frac{\log r}{\varphi(N)}\Big)\Big),$$

with

$$B_\chi(s) = \prod_{p\mid N}\Big(1 + \sum_{j=1}^{\infty}\frac{\chi(f(p^j))}{p^{js}}\Big), \tag{4.2}$$

$$C_\chi(s) = \prod_{p\nmid N}\frac{1 + \sum_{j=r}^{\infty}\chi(f(p^j))p^{-js}}{1 + \chi(f(p^r))p^{-rs}}, \tag{4.3}$$

$$h_\chi(s) = \sum_{p\nmid N}\sum_{j=2}^{\infty}\frac{(-1)^{j+1}}{j}\frac{\chi^j(f(p^r))}{p^{jrs}}.$$

If $\chi \in \Omega$, *then neither* $h_\chi(s)$ *nor the sum*

$$\sum_{j\in R}\chi(j)\sum_{i\in U_j}\Big(g(N, i, rs) - \frac{\log r}{\varphi(N)}\Big)$$

depend on χ, *hence in this case one can write*

$$\alpha_\chi(s) = D_f(N; s)B_\chi(s)C_\chi(s),$$

with $D_f(N; s)$ *regular for* $\mathrm{Re}\, s \ge 1/r$ *and nonvanishing at* $s = 1/r$.

Proof. Observe first that for $j \le r - 1$ one can have $\chi(f(p^j)) \ne 0$ only for p dividing N. Therefore we can write

$$F_\chi(s) = A_\chi(s)B_\chi(s)C_\chi(s)$$

with

$$A_\chi(s) = \prod_{p\nmid N}\Big(1 + \frac{\chi(f(p^r))}{p^{rs}}\Big).$$

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In view of
$$\left|1+\frac{\chi(f(p^r))}{p^{rs}}\right| \geq 1-\frac{1}{p^{r\,\mathrm{Re}\,s}} \geq \frac{1}{2}$$
$A_\chi(s)$ does not vanish in $\mathrm{Re}\, s > 1/r$, hence we can write
$$A_\chi(s) = \exp\biggl(\sum_{p\nmid N} \frac{\chi(f(p^r))}{p^{rs}} + h_\chi(s)\biggr);$$
note that by virtue of
$$\sum_{p\nmid N} \frac{\chi(f(p^r))}{p^{rs}} = \sum_{p\nmid N} \frac{\chi(V_r(p))}{p^{rs}} = \sum_{j\in R} \chi(j) \sum_{\substack{p \\ V_r(p)\equiv j \bmod N}} \frac{1}{p^{rs}}$$
and (2.2) we obtain
$$\sum_{p\nmid N} \frac{\chi(f(p^r))}{p^{rs}} = m(\chi)\log\frac{1}{s-1/r} + \sum_{j\in R}\chi(j)\sum_{i\in U_j}\biggl(g(N,i,rs) - \frac{\log r}{\varphi(N)}\biggr).$$
Thus
$$A_\chi(s) = \frac{a_\chi(s)}{(s-1/r)^{m(\chi)}},$$
with
$$a_\chi(s) = \exp\biggl(h_\chi(s) + \sum_{j\in R}\chi(j)\sum_{i\in U_j} g(N,i,rs)\biggr).$$

Note that if χ lies in Ω, then $a_\chi(s)$ does not depend on χ. Indeed, in this case, for $p \nmid N$,
$$\chi(f(p^r)) = \begin{cases} 1 & \text{if } (V_r(p), N) = 1, \\ 0 & \text{otherwise,} \end{cases}$$
and
$$\sum_{j\in R}\chi(j)\sum_{i\in U_j}\biggl(g(N,i,rs) - \frac{\log r}{\varphi(N)}\biggr) = \sum_{i\in M} g(N,i,rs) - \frac{m\log r}{\varphi(N)}.$$

The functions $B_\chi(s)$ and $C_\chi(s)$ are both regular for $\mathrm{Re}\, s \geq 1/r$, and we have $C_\chi(1/r) \neq 0$. The function $B_\chi(s)$ may vanish at $s = 1/r$, and this happens if, for some prime p,
$$\sum_{j=1}^{\infty} \frac{\chi(f(p^j))}{p^{j/r}} = -1,$$
forcing $p \leq 2^r$. In the case $r = 1$ this can happen only if, for every $j \geq 1$,
$$\chi(f(2^j)) = -1.$$

It would be convenient to present the product $B_\chi(s)$ in another form. If $d = \prod_{j=1}^{k} p_j$ is a square-free divisor of N and S_d is the set of integers whose prime divisors divide d,

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then

$$B_\chi(s) = \sum_{d|N} \mu^2(d) \sum_{m\in S_d} \frac{\chi(f(m))}{m^s}.$$

Indeed, it suffices to observe that if $W_\chi(p) = \sum_{j=1}^{\infty} \chi(f(p^j))p^{-s}$, then

$$B_\chi(s) = \sum_{d|N} \mu^2(d) \prod_{p|d} W_\chi(p).$$

Putting

$$\alpha_\chi(s) = B_\chi(s)C_\chi(s) \exp\Big(h_\chi(s) + \sum_{j\in R} \chi(j) \sum_{i\in U_j}\Big(g(N, i, rs) - \frac{\log r}{\varphi(N)}\Big)\Big),$$

we get the assertion of the lemma. □

Using (4.1) and Lemma 4.1,

$$\Phi_k(N; s) = \frac{1}{\varphi(N)} \sum_\chi \overline{\chi(k)} \frac{\alpha_\chi(s)}{(s-1/r)^{m(\chi)}}. \tag{4.4}$$

Observe now that we have $\mathrm{Re}(m(\chi)) \le \mathrm{Re}(m(\chi_0)) = m$, with equality occurring only if for $j \in R$ one has $\chi(j) = 1$, that is, $\chi \in \Omega$, and therefore we may write, with some t,

$$\Phi_k(N; s) = \frac{1}{\varphi(N)} \frac{\sum_{\chi\in\Omega} \overline{\chi(k)}\alpha_\chi(s)}{(s-1/r)^m} + \sum_{j=1}^{t} \frac{\lambda_j(s)}{(s-1)^{\mu_j}},$$

where $\lambda_j(s)$ are regular for $\mathrm{Re}\, s \ge 1/r$ and μ_j are complex numbers satisfying $\mathrm{Re}\,\mu_j < r$. This establishes (i), and (ii) follows immediately by the tauberian theorem of Delange.

We now prove (iii) and write $\varrho_k = \varrho_f(N, k)$ for short. If the sum $c_k(s)$ does not vanish at $s = 1/r$, then in view of

$$\sum_{(k,N)=1} c_k(s) = \sum_{\chi\in\Omega} \alpha_\chi(s) \sum_{(k,N)=1} \overline{\chi(k)} = \varphi(N)\alpha_{\chi_0}(s)$$

and

$$\alpha_{\chi_0}(1/r) > 0$$

the application of Delange's tauberian theorem gives

$$\varrho_k = \frac{c_k(1/r)}{\varphi(N)\alpha_{\chi_0}(1/r)}.$$

If $c_k(1/r) = 0$, but $c_k(s)$ does not vanish identically, then with some $t \ge 1$ we can write

$$c_k(s) = (s-1/r)^t H(s),$$

where $H(s)$ is regular for $\mathrm{Re}\, s \ge 1$ and $H(1/r) \ne 0$. Delange's theorem now gives $\varrho_k = 0$.

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If $c_k(s)$ vanishes identically, then $\varrho_k = 0$. This is a simple corollary of Delange's theorem (see, for example, [3, Lemma 2]).

Because $c_k(s)$ depends only on the coset $k\Lambda$, so does ϱ_k.

The assertion (iv) follows immediately from (ii). □

Remark 4.2. To obtain a more explicit formula for $c_k(1/R)$ one may utilize (2.2).

Corollary 4.3. *If Λ is of index 2 in $G(N)$, then $\Omega = \{\chi_0, \chi\}$, where χ is a real character modulo N, and f is weakly properly distributed modulo N if and only if*

$$\alpha_{\chi_0}(1/r) \neq \pm\alpha_\chi(1/r). \tag{4.5}$$

Proof. In this case

$$c_k(s) = \begin{cases} \alpha_{\chi_0}(s) + \alpha_\chi(s) & \text{if } k \in \Lambda, \\ \alpha_{\chi_0}(s) - \alpha_\chi(s) & \text{otherwise,} \end{cases}$$

hence (4.5) is equivalent to $c_k(1/r) \neq 0$. It remains to apply part (iv) of Theorem 3.1. □

5. Some special cases

Checking the conditions for weak proper distribution given in Theorem 3.1 may sometimes be awkward. The next theorem gives a simpler criterion in the case of polynomial-like multiplicative functions f with $r_f(N) < \infty$ and $f(p^n) = 0$ for $n \geq r+1$.

Theorem 5.1. *Let $N \geq 3$, let f be an integer-valued polynomial-like multiplicative function satisfying $r = r_f(N) < \infty$ and denote by $V(T)$ the polynomial satisfying $f(p^r) = V(p)$ for prime p. Assume, moreover, that for $n \geq r+1$ and all primes p one has $f(p^n) = 0$.*

The function f is weakly properly distributed modulo N if and only if for every k prime to N there exists an $(r+1)$-free integer m all of whose prime factors divide N and which satisfies $f(m) \in k\Lambda$, Λ being the subgroup of $G(N)$ generated by the set $R = \{V(x) \bmod N : (xV(x), N) = 1\}$. For $k \in \Lambda$ this condition is satisfied with $m = 1$.

Proof. Since $f(p^n)$ vanishes for $n \geq r+1$ we use (4.2), (4.3) and (4.4) to obtain for $\chi \in \Omega$ the equalities

$$C_\chi(1/r) = 1$$

and

$$B_\chi(1/r) = \prod_{p \mid N}\Big(1 + \sum_{j=1}^{r} \frac{\chi(f(p^j))}{p^{j/r}}\Big).$$

For a square-free divisor $d = p_1 p_2 \cdots p_k$ of N denote by S_d the set of all integers of the form $\prod_{j=1}^{k} p_j^{a_j}$ with $0 \leq a_j \leq r$.

Lemma 4.1 shows now that we can write

$$\alpha_\chi(1/r) = D_f(N) \prod_{p \mid N}\Big(1 + \sum_{j=1}^{r} \frac{\chi(f(p^j))}{p^{j/r}}\Big),$$

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with a positive constant $D_f(N)$ depending only on f and N. Therefore

$$\frac{c_k(1/r)}{D_f(N)} = \sum_{\chi\in\Omega} \overline{\chi(k)}\alpha_\chi(1/r) = \sum_{d|N} \mu^2(d) \sum_{m\in S_d} \frac{\chi(f(m))}{m^{1/r}}.$$

Since

$$\sum_{\chi\in\Omega} \chi(f(m))\overline{\chi(k)} = \begin{cases} |\Omega| & \text{if } f(m) \in k\Lambda, \\ 0 & \text{otherwise,} \end{cases}$$

one obtains that $c_k(1/r)$ does not vanish if and only if there exists an $(r + 1)$-free integer m all of whose prime factors divide N and which satisfies $f(m) \in k\Lambda$. Now apply Theorem 3.1. □

Corollary 5.2. *Let $N = q^k$ be a prime power, and let f be a polynomial-like multiplicative function with $r = r_f(N) < \infty$. Moreover, denote by q_n the sequence of $(r + 1)$-free integers.*

(i) *If the index of Λ in $G(N)$ exceeds 2, then the sequence $f(q_n)$ is not weakly properly distributed modulo N.*

(ii) *If the index of Λ is equal to 2, then the sequence will be weakly properly distributed modulo N if and only if for some $j \le r$ one has $(f(q^j), N) = 1$ and $f(q^j) \notin \Lambda$.*

Proof. (i) Apply Theorem 5.1 to the function $g(n) = \mu_{r+1}(n)f(n)$, note that $r_f(N) = r_g(N)$ and observe that the only $(r + 1)$-free divisors of N are $1, q, \ldots, q^r$, hence the condition of the theorem can be satisfied only by k lying in at most two different cosets with respect to Λ.

(ii) Immediate by Theorem 5.1. □

The following corollary can sometimes be used to simplify the proof that a particular function is weakly properly distributed modulo N.

Corollary 5.3. *Let $N \ge 3$, let f be an integer-valued polynomial-like multiplicative function with $r = r_f(N) < \infty$ and $f(p^r) = V(p)$ for a polynomial $V(T)$ and put $g(n) = \mu_{r+1}(n)f(n)$. If $g(n)$ is weakly properly distributed modulo N, so is $f(n)$.*

Proof. The function g is polynomial-like, and since for $i \le r$ one has $g(p^i) = f(p^i)$ the equality $g(p^r) = V(p)$ follows, hence the sets $R_r(f)$ and $R_r(g)$ coincide, thus $r_g(N) = r$ and $m_f(N) = m_g(N) = m$, say. Equality (2.1) leads to

$$F_f(N; x) = (c_1 + o(1))\frac{x^{1/r}}{\log^{1-m} x}, \quad F_g(N; x) = (c_2 + o(1))\frac{x^{1/r}}{\log^{1-m} x}$$

with positive c_1, c_2. If g is weakly properly distributed modulo N, then, for $(k, N) = 1$,

$$F_g(N, k; x) = (c(k) + o(1))\frac{x^{1/r}}{\log^{1-m} x}$$

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with $c(k) > 0$, and in view of

$$F_g(N, k; x) \le F_f(N, k; x)$$

and part (iii) of Theorem 3.1 we obtain that f is weakly properly distributed mod N. □

Note that the converse implication may fail. Indeed, we shall see in Theorem 6.2 that although $\sigma(n)$ is for every N weakly properly distributed modulo N, the function $\mu_3(n)\sigma(n)$ does not share this property.

6. Applications

6.1. Euler function. We now utilize Corollary 5.3 to deal with the Euler function. It suffices to consider only odd moduli, because if N is even, then $(\varphi(n), N) = 1$ holds only for $n = 1$.

THEOREM 6.1. *Euler's function $\varphi(n)$ is weakly properly distributed modulo N for every odd integer N.*

PROOF. Let $N \ge 3$ be an odd integer. If $3 \nmid N$, then $\varphi(n)$ is weakly uniformly distributed modulo N by [9], hence we may henceforth assume that $3 \mid N$. In this case $1 \in R_1 \ne \emptyset$ holds, hence $r_\varphi(N) = 1$, and the set $R_1(N)$ consists of all a modulo N satisfying $(a, N) = 1$ and $a \not\equiv -1 \bmod p$ for every prime divisor of N, thus

$$m = m_\varphi(N) = \prod_{p\mid N}\Big(1 + \frac{1}{p-1}\Big).$$

Lemma 5.3 shows that it suffices to prove weak proper distribution modulo N for the function $f(n) = \mu^2(n)\varphi(n)$.

Let Λ denote the subgroup of $G(N)$ generated by R, and let Ω be the family of characters attaining the value 1 in Λ. Denote by H the subgroup $\{a \bmod N : a \equiv 1 \bmod 3\}$ of $G(N)$. Since $3 \mid N$ every element of $a \in R$ lies in H, thus $\Lambda \subset H$. We will show that $\Lambda = H$. Write $N = \prod_{i=1}^k p_i^{a_i}$ with $p_1 = 3$ and note that every element $x \in \Lambda$ can be considered as a vector

$$x = [x_1, x_2, \dots, x_k]$$

with $x_i \in G(p_i^{a_i})$, $x \equiv x_i \bmod p_i^{a_i}$ and $x_1 \equiv 1 \bmod 3$. Given $x \in \Lambda$ in this form choose for $i = 2, 3, \dots, k$ an element $c_i \in G(p_i^{a_i})$ with

$$c_i \not\equiv -1 \mod p_i, \quad c_i \not\equiv -x_i \mod p_i,$$

and put

$$y_i = \begin{cases} c_i & \text{if } x_i \equiv -1 \bmod p_i, \\ x_i & \text{otherwise,} \end{cases}$$

$$z_i = \begin{cases} c_i^{-1} & \text{if } x_i \equiv -1 \bmod p_i, \\ 1 & \text{otherwise,} \end{cases}$$

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and

$$y = [1, y_2, \ldots, y_k], \quad z = [x_1, z_2, \ldots, z_k].$$

Since $y, z \in R$ and $x = yz$, we obtain $x \in \Lambda$. Since Λ is of index 2 in $G(N)$ and $2 \notin \Lambda$, the cosets of $G(N)$ with respect to Λ are Λ and 2Λ. Since $3 \mid N$ and $\varphi(3) = 2 \in 2\Lambda$, the assertion follows from Theorem 5.1. □

6.2. Sum of divisors. We now consider $\sigma(n)$, the sum of divisors.

THEOREM 6.2.

(i) *The function $\sigma(n)$ is weakly properly distributed modulo N for every $N \geq 3$.*

(ii) *The function $f(n) = \mu_3(n)\sigma(n)$ is weakly properly distributed modulo N if and only if it is weakly uniformly distributed modulo N, that is, $6 \nmid N$.*

PROOF. (i) If $6 \nmid N$, then $\sigma(n)$ is weakly uniformly distributed modulo 6 by [9], so we may assume that $6 \mid N$. Let $N = \prod_{p \mid N} p^{a_p}$ with $a_2, a_3 \geq 1$. In this case we have $V_1(T) = T + 1$, $V_2(T) = T^2 + T + 1$, hence $R_1 = \emptyset$, and $1 \in R_2 \neq \emptyset$. We have

$$R_2 = \{1 + x + x^2 \bmod N : (x(1 + x + x^2), N) = 1\},$$

and since the congruence

$$1 + X + X^2 \equiv 0 \mod p \tag{6.1}$$

has one solution for $p = 3$, two solutions for $p \equiv 1, 7 \bmod 12$, and no solutions for other primes,

$$m = \frac{1}{2} \prod_{p \equiv 1,7 \bmod 12} \Big(1 - \frac{1}{p-1}\Big).$$

Since all elements of R_2 are congruent to 1 mod 6,

$$\Lambda \subset H = \{x \bmod N : x \equiv 1 \bmod 6\}.$$

Observe now that in fact there is equality here. Indeed, let $x = \langle x_p \rangle_p \in H$, with p ranging over prime divisors of N, and $x_p \in G(p^{a_p})$, $x_p \equiv x \bmod p^{a_p}$. For primes $p \mid N$ congruent to 1 or 7 modulo 12 denote by u_p, v_p the solutions of the congruence (6.1) and choose $c_p \in G(p^{a_p})$ with $c_p \not\equiv u_p, v_p, -x_p \bmod p$. For these primes put

$$y_p = \begin{cases} c_p & \text{if } x_p \equiv u_p, v_p \bmod p, \\ x_p & \text{otherwise,} \end{cases}$$

$$z_p = \begin{cases} x_p c_p^{-1} & \text{if } x_p \equiv u_p, v_p \bmod p, \\ 1 & \text{otherwise,} \end{cases}$$

and for the remaining $p \mid N$ put

$$y_p = \begin{cases} x_p & \text{if } p \nmid 6, \\ 1 & \text{if } p \mid 6, \end{cases}$$

and $z_p = 1$. Then $y = \langle y_p \rangle_p$ and $z = \langle z_p \rangle_p$ lie in R_2, hence $x = yz \in \Lambda$. This shows that $\Lambda = H$ and it follows that the index of Λ in $G(N)$ is equal to 2. Thus $\Omega = \{\chi_0, \chi_3\}$,

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where χ_3 is the character mod N induced by the quadratic character modulo 3. If $p \equiv 1 \bmod 3$ and $(\sigma(p^j), N) = 1$, then

$$\chi_0(\sigma(p^j)) = \begin{cases} 1 & \text{if } j \equiv 0, 1 \bmod 3, \\ 0 & \text{if } j \equiv 2 \bmod 3, \end{cases}$$

and

$$\chi_3(\sigma(p^j)) = \begin{cases} 1 & \text{if } j \equiv 0 \bmod 3, \\ -1 & \text{if } j \equiv 1 \bmod 3, \\ 0 & \text{if } j \equiv 2 \bmod 3. \end{cases}$$

If $p \equiv 2 \bmod 3$ and $(\sigma(p^j), N) = 1$, then

$$\chi_0(\sigma(p^j)) = \chi_3(\sigma(p^j)) = \begin{cases} 1 & \text{if } 2 \mid j, \\ 0 & \text{if } 2 \nmid j. \end{cases}$$

Since moreover, $\chi_0(3^j) = \chi_1(3^j) = 1$, we get, utilizing the notation used in Lemma 4.1,

$$A_{\chi_0}(s) = A_{\chi_3}(s) = \prod_{\substack{p\nmid N, p\equiv 2 \bmod 3 \\ (1+p+p^2,N)=1}} \Big(1 + \frac{1}{p^{2s}}\Big),$$

$$B_{\chi_0(s)} = B(N; s) \prod_{\substack{p\mid N \\ p\equiv 1 \bmod 3}} \Big(1 + \sum_{\substack{3\le j \equiv 0,1 \bmod 3 \\ (\sigma(p^j),N)=1}} \frac{1}{p^{js}}\Big),$$

$$B_{\chi_3(s)} = B(N; s) \prod_{\substack{p\mid N \\ p\equiv 1 \bmod 3}} \Big(1 + \sum_{\substack{3\le j \equiv 0 \bmod 3 \\ (\sigma(p^j),N)=1}} \frac{1}{p^{js}} - \sum_{\substack{3\le j \equiv 1 \bmod 3 \\ (\sigma(p^j),N)=1}} \frac{1}{p^{js}}\Big),$$

where $B(N; s)$ is a function regular for $\mathrm{Re} \ge 1/2$ and not vanishing at $1/2$.

Finally,

$$C_{\chi_0}(s) = C(N; s) \prod_{\substack{p\nmid N \\ p\equiv 1 \bmod 3}} \Big(1 + \sum_{\substack{2\le j \equiv 0,1 \bmod 3 \\ (\sigma(p^j),N)=1}} \frac{1}{p^{js}}\Big)$$

and

$$C_{\chi_3}(s) = C(N; s) \prod_{\substack{p\nmid N \\ p\equiv 1 \bmod 3}} \Big(1 + \sum_{\substack{3\le j \equiv 0 \bmod 3 \\ (\sigma(p^j),N)=1}} \frac{1}{p^{js}} - \sum_{\substack{3\le j \equiv 1 \bmod 3 \\ (\sigma(p^j),N)=1}} \frac{1}{p^{js}}\Big),$$

with $C(N; s)$ regular for $\mathrm{Re} \ge 1/2$ and not vanishing at $1/2$.

Since $A_{\chi_0}(s) = A_{\chi_3}(s) = g(s)(s - 1/2)^{-m}$ with $g(s)$ regular for $\mathrm{Re}\, s \ge 1/r$ and nonvanishing at $s = 1/r$, we obtain

$$\alpha_{\chi_0}(1) \ne \pm\alpha_{\chi_3}(1),$$

and by Corollary 4.3 assertion (i) follows.

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(ii) Since, for 3-free n, $f(n)$ coincides with $\sigma(n)$,

$$r_f(N) = r_\sigma(N) = \begin{cases} 1 & \text{if } 6 \nmid N, \\ 2 & \text{if } 6 \mid N. \end{cases}$$

If $6 \nmid N$, then

$$R = R_1(f, N) = \{x \bmod N : p \nmid x(x-1) \text{ for } p \mid N\}$$

and the argument used in the proof of (i) leads to $\Lambda = G(N)$, hence f is weakly uniformly distributed modulo N.

Now assume that $6 \mid N$. From the proof of (i) one infers the equality

$$\Lambda = \{a \in G(N) : x \equiv 1 \bmod 6\},$$

hence the index of Λ is equal to 2. Were f weakly properly distributed modulo N, then according to Theorem 5.1 there would exist an integer

$$d = p_1 \cdots p_k (q_1 \cdots q_l)^2$$

with primes p_i, q_j dividing N, satisfying $(\sigma(d^2), N) = 1$ and

$$\sigma(d^2) = f(d^2) \equiv 5 \mod N.$$

Since for every prime p one has $(1 + p, N) > 1$, as N is divisible by 6, therefore $k = 0$, and there exists a prime q dividing d with $(1 + q + q^2, N) = 1$ and $1 + q + q^2 \equiv 5 \bmod 6$, thus $q^2 + q \equiv 4 \bmod 6$. This is obviously impossible, hence $f(n)$ is not properly weakly distributed modulo N. □

6.3. Ramanujan τ-function. Our last example deals with the Ramanujan τ-function, defined by

$$\sum_{n=1}^{\infty} \tau(n) X^n = X \prod_{j=1}^{\infty} (1 - X^j)^{24}.$$

It has been shown by Serre [8] (see also [5, Theorem 5.18]) that $\tau(n)$ is weakly uniformly distributed modulo N if and only if either N is odd and not divisible by 7, or N is even and $(N, 7 \cdot 23) = 1$. In particular, $\tau(n)$ is weakly uniformly distributed modulo p for every prime $p \neq 7$. Nevertheless, it turns out that its distribution modulo 7 is not too bad.

Theorem 6.3. *The function $\tau(n)$ is weakly properly distributed modulo* 7.

Proof. In 1931, Wilton [10] established the congruence

$$\tau(n) \equiv n\sigma_3(n) \mod 7,$$

where

$$\sigma_3(n) = \sum_{d \mid n} d^3,$$

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hence it suffices to show that the function $f(n) = n\sigma_3(n)$ is weakly properly distributed modulo 7.

For this function we obtain $V_1(X) = X^4 + X$, thus $R_1 = \{1, 2, 4\}$, hence $r = 1$ and $\Lambda = R_1$ is of index 2. Thus $\Omega = \{\chi_0, \chi_7\}$, χ_7 being the quadratic character modulo 7. Denote by P the set of primes p with $p \bmod 7 \in \Lambda$.

In view of $7 \mid f(7^j)$ for $j \geq 1$ we get $B_{\chi_0} = B_{\chi_7} = 1$. Moreover, for both characters $\chi \in \Omega$,

$$1 + \frac{\chi(f(p))}{p} = \begin{cases} 1 + 1/p & \text{if } p \in P, \\ 1 & \text{otherwise,} \end{cases}$$

hence

$$C_{\chi_0}(1) = \prod_{p \notin P} \Big(1 + \sum_{\substack{j \geq 2 \\ 7 \nmid f(p^j)}} \frac{1}{p^j}\Big) \prod_{p \in P} \Big(\Big(1 + \sum_{\substack{j \geq 2 \\ 7 \nmid f(p^j)}} \frac{1}{p^j}\Big) \frac{p}{p+1}\Big),$$

and

$$C_{\chi_7}(1) = \prod_{p \notin P} \Big(1 + \sum_{\substack{j \geq 2 \\ 7 \nmid f(p^j)}} \frac{\chi_7(f(p^j))}{p^j}\Big) \prod_{p \in P} \Big(\Big(1 + \sum_{\substack{j \geq 2 \\ 7 \nmid f(p^j)}} \frac{\chi_7(f(p^j))}{p^j}\Big) \frac{p}{p+1}\Big).$$

Since the character χ_7 is real and $\chi_7(f(29^2)) = \chi_7(3) = -1$,

$$C_{\chi_7}(1) < C_{\chi_0}(1), \tag{6.2}$$

and the observation that $7 \nmid f(p)$ implies $\chi_7(f(p)) = 1$ leads to the equality

$$h_{\chi_0)}(1) = h_{\chi_7)}(1). \tag{6.3}$$

Noting, finally, that the sum

$$\sum_{j \in R} \chi(j) \sum_{i \in \Lambda_j} g(N, i, 1)$$

does not depend on χ, as for $j \in R$ we have $\chi_0(j) = \chi_7(j) = 1$, and using (6.2) and (6.3) we arrive at

$$\alpha_{\chi_0} > \alpha_{\chi_7}(1),$$

and the assertion follows from Corollary 4.3. □

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