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ON THE CODIMENSION OF A MINIMAL IMMERSION

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We show that a certain local pinching condition on a compact minimal submanifold in a sphere leads to a condition on the codimension of the immersion.

Let M^n be a n-dimensional compact connected smooth Riemannian manifold which can be isometrically immersed in the unit sphere S^{n+p} with p denoting the codimension. If all sectional curvatures of M are > 0 and < 1, then it is a well-known conjecture that we must have $p \geqslant n$ ([4] p.198). In the special cases that either M lies in an open hemisphere or M has constant sectional curvatures, the conjecture has been verified (see [4] p.198 and [2] respectively). Apart from these two special cases, it seems nothing else is known about the conjecture.

In this note we propose to study the case when the immersion is also minimal. First we observe that in this case since M can never lie in an open hemisphere ([1] p.15) therefore the argument in [4] p.198 does not work. Our observation here is that certain local pinching condition (introduced in [3] p.644) leads to a condition on the codimension. We prove the following:

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THEOREM *Let M^n be a compact minimally immersed submanifold in S^{n+p}. Suppose its sectional curvature function K is locally $\frac{1}{2}$-pinched i.e. there exist a function $A : M \to \mathbb{R}$ and a constant δ such that*

(i) $0 < A < 1$ *and* $\frac{1}{2} < \delta \leqslant 1$

and (ii) $\delta A(x) \leqslant K_x \leqslant A(x)$ *for all* $x \in M$.

Then $p \geqslant n$.

Proof. Let R denote the scalar curvature function on M. By assumption (ii), we have

$$(1) \qquad R(x) \leqslant n(n-1)A(x)$$

for each $x \in M$.

In particular we see that M cannot be totally geodesic and hence it follows from [5] p.89, Theorem 12 (together with Gauss equation and our assumption $K_x \geqslant \delta A(x)$) that there exists a point $y \in M$ such that

$$(2) \qquad pn(1 - 2\delta A(y)) \geqslant n(n-1) - R(y)$$

Using (1) and (2), we have

$$pn(1 - 2\delta A(y)) \geqslant n(n-1) - n(n-1)A(y)$$

which implies

$$(3) \qquad (n-1-2p\delta)A(y) \geqslant n-1-p$$

Now suppose on the contrary that $p \leqslant n-1$. Then (3) implies that $n-1-2p\delta \geqslant 0$. This together with $A(y) < 1$ and (3) implies that

$$n-1-2p\delta \geqslant n-1-p$$

and hence we obtain

$$\delta \leqslant \tfrac{1}{2}$$

a contradiction to our hypothesis that $\delta > \frac{1}{2}$. Therefore, we must have $p > n-1$, that is

$$p \geqslant n .$$

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