A NEW CONSTRUCTION OF THE INJECTIVE HULL

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The definition of injectivity, and the proof that every module has an injective extension which is a subextension of every other injective extension, are due to R. Baer [B]. An independent proof using the notion of essential extension was given by Eckmann-Schopf [ES]. Both proofs require the preliminary construction of some injective overmodule. In [F] I showed how the latter proof could be freed from this requirement by exhibiting a set F in which every essential extension could be embedded. Subsequently J.M. Maranda pointed out that F has minimal cardinality. It follows that F is equipotent with the injective hull. Below I construct the injective hull by equipping F itself with a module structure.

All modules will be unitary over a fixed ring R.

If x is an element in an extension of a module M, then the mapping $f(\lambda) = \lambda x$ defined on $\{\lambda \varepsilon R: \lambda x \varepsilon M\}$ is a homomorphism from an ideal of R to M; conversely, every such homomorphism can be realized with an extension of M by a single element, which it determines up to isomorphism.

The homomorphism is <u>irreducible</u> [B] if it cannot be extended to a larger ideal of R. An extension is <u>essential</u> [ES] if every non-zero element has a non-zero multiple in M. The extension by x is essential if and only if f is irreducible. Indeed, if f can be extended to $\lambda \notin I$ then $\lambda x - f(\lambda)$ is annihilated by every multiple sending it into M; conversely, if this is true of $\lambda x - y$ then every linear relation between λ and the elements of I holds as well between y and their images in M, so one may extend with $f(\lambda) = y$.

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Fix M and let F be the set of those well-ordered sequences $\{f_\alpha\}$ of non-zero irreducible homomorphisms for which f_α annihilates I_β, the domain of f_β, for $\beta < \alpha$. I propose to equip F with the structure of an essential extension of M in such a way that $\lambda\{f_\alpha\} = \lambda\{f_\beta\}_{\beta<\alpha} + f_\alpha(\lambda)$ for $\lambda \in I_\alpha$. Suppose this already done for a subset F* of F containing with every $\{f_\alpha\}$ all of its $\{f_\beta\}_{\beta<\alpha}$ as well as the elements of M identified as the void sequence and those single-termed sequences whose unique ideal is all of R. Let $\{f^*_\alpha\} \notin F^*$ be either a single-termed sequence or one without last term all of whose $\{f^*_\beta\}_{\beta<\alpha} \in F^*$. For the former alternative the unique term furnishes a homomorphism into (M hence) F*; for the latter, $\lambda\{f^*_\beta\}_{\beta<\alpha}$, being on $\bigcup I^*_\alpha$ independent of α for large α, yields a homomorphism of this ideal into F*. In either case, extend it in any way to an irreducible homomorphism and construct the corresponding essential extension M* of F*. The identification of F* with a subset of F will be extended, one element at a time, to one of M* (and thus the module operations transferred to the new elements): At each stage the identified elements shall include with $\{f_\alpha\}$ all their $\{f_\beta\}_{\beta<\alpha}$; and the next element x shall be identified with the unused sequence $\{f_\alpha\}$ of minimal length such that f_α is the irreducible homomorphism for the extension of M by $x - \{f_\beta\}_{\beta<\alpha}$. Observe that $\{f^*_\alpha\}$ will be used (at the latest) by the time the generator of the extension is identified; also, a multi-termed sequence with last term, $\{f_\beta\}_{\beta\leq\alpha}$, must be used by the time $\{f_\beta\}_{\beta<\alpha} + \{f_\alpha\}$ is identified. Thus all of F is finally made into a module.

Replacing F* by F shows that the latter admits no essential extension M* : hence it is injective. Since it is an essential extension it is embeddable over M in every other injective extension. These properties determine F up to isomorphism: If an injective extension were embeddable over M in F it would be essential, hence the embedding of F into it would have to be onto.

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