ON STRONG UPPER DENSITIES

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Let ψ be the outer measure generated by a gauge g and a sequential covering class $\mathcal{C}$ of closed sets of a metric space X, and let D be the resulting strong upper density function. That is, for each $A \subseteq X$ and each $x \in X$:

$$\psi(A) = \lim_{\varepsilon \to 0^+} \inf \sum_\alpha g(I_\alpha)$$

the infimum being taken over all countable subcollections $\{I_\alpha\}$ of $\mathcal{C}$ such that $A \subseteq \bigcup_\alpha I_\alpha$ and, for all I_α in the subcollection, $d(I_\alpha) < \varepsilon$ where $d(I_\alpha)$ is the diameter of I_α. The strong upper density of A at x is

$$D(A, x) = \lim_{\varepsilon \to 0^+} \sup \frac{\psi(A \cap I)}{g(I)} ,$$

the supremum being taken over all $I \in \mathcal{C}$ such that $d(I) < \varepsilon$ and $x \in I$. We assume that $g(I) = 0$ if and only if I is empty.

It has been shown that if $\psi(A)$ is finite, then $D(A, x) \geqq 1$ for ψ-almost-all $x \in A$. [1, Theorem 6; 2, Theorem 3.2].

It is the purpose of this note to show that the condition that $\psi(A)$ be finite can be omitted from the above statement.

Let A be a subset of X and let

$$B = A \cap \{x : D(A, x) < 1\} .$$

Let $E \subseteq B$. Then $D(E, x) < 1$ for all $x \in E$, so $\psi(E)$ is either 0 or is infinite, by the known result. For each $x \in B$, the ratio

$$\frac{\psi(A \cap I)}{g(I)}$$

https://doi.org/10.4153/CMB-1969-066-8 Published online by Cambridge University Press

is eventually less than 1, so $\psi(B \cap I)$ - since it is either 0 or infinite - is eventually 0. Thus, for each $x \in B$ there is a number $\varepsilon(x) > 0$ with the property that, if $x \in I \in \mathcal{C}$ and $d(I) < \varepsilon(x)$, then $\psi(B \cap I) = 0$. Let

$$C_n = B \cap \{x : \varepsilon(x) < 1/n\}$$

and let $\{I_\alpha\}$ be a countable subcollection of $\mathcal{C}$ which covers C_n, each I_α being of diameter less than $1/n$. Then

$$\psi(C_n) \leq \sum_\alpha \psi(C_n \cap I_\alpha) = 0 .$$

Since B is the union of the C_n, it follows that $\psi(B) = 0$, which concludes the proof.

REFERENCES

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https://doi.org/10.4153/CMB-1969-066-8 Published online by Cambridge University Press