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MINIMALLY GENERATED MODULES

BY

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ABSTRACT. A non-zero module M having a minimal generator set contains a maximal submodule. If M is Artinian and all submodules of M have minimal generator sets then M is Noetherian; it follows that every left Artinian module of a left perfect ring is Noetherian. Every right Noetherian module of a left perfect ring is Artinian. It follows that a module over a left and right perfect ring (in particular, commutative) is Artinian if and only if it is Noetherian. We prove that a local ring is left perfect if and only if each left module has a minimal generator set.

In this paper rings have unity and modules are unitary.

It is well-known that any non-zero finitely generated module contains a maximum submodule.

THEOREM 1. *If a module $M \neq \{0\}$ has a minimal generator set then M contains a maximum submodule, and hence $M \neq \text{Rad}\, M$.*

Proof. Let $\{m_i\}$ be a minimal generator set for M. The submodule T generated by $\{m_i \mid i \neq j\}$ is proper. By Zorn's lemma there is a maximal proper submodule of M containing T.

THEOREM 2. *Let $N_0 = M$, $N_{i+1} = \text{Rad}\, N_i$. If M is Artinian and if each N_i (in particular, if each submodule of M) has a minimal generator set, then M is Noetherian.*

Proof. Since $N_i = N_{i+1} = \text{Rad}\, N_i$ for some i, $N_i = \{0\}$ by Theorem 1. Since $\text{Rad}(N_p/N_{p+1}) = \{0\}$ and N_p/N_{p+1} is Artinian, N_p/N_{p+1} has finite length, and since $N_i = \{0\}$, we have a composition series for M, so M is Noetherian.

THEOREM 3. *Every left module of a left perfect ring has a minimal generator set, so every left Artinian module is Noetherian.*

Proof. Let M be a left A-module and let $R = \text{Rad}\, A$. Since A/R is semi-simple, M/RM is a direct sum of simple modules and so it has a minimal generator set $\{m_j + RM\}$. Clearly $\{m_i\}$ is a minimal generator set for the module it spans, N. Since $M = N + RM$, $M/N = R(M/N)$, and since R is left T-nilpotent, $M/N = \{0\}$ (see 1, Lemma 2.6, p. 473), so $M = N$ and M has a

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minimal generator set. From Theorem 2 every left Artinian module is Noetherian.

LEMMA 1. *If M is a Noetherian module and all non-zero quotient modules of M have non-zero socles, then M is Artinian.*

Proof. Let $S_0 = \operatorname{Soc} M$ and $S_{i+1}/S_i = \operatorname{Soc}(M/S_i)$. Since $S_1 \subseteq S_2 \subseteq \ldots$, we have $S_k = S_{k+1}$ for some k. Since $\operatorname{Soc}(M/S_k) = S_{k+1}/S_k = \{0\}$ and non-zero quotient modules have non-zero socles, we have $M = S_k$. Now $S_1/S_0, S_2/S_1, \ldots, M/S_{k-1}$ are semi-simple, and since M is Noetherian, they are finitely generated so they have finite length. Thus we have a composition series for M, hence M is Artinian.

THEOREM 4. *Every right Noetherian module of a left perfect ring is Artinian.*

Proof. From [1, Theorem P] each right module of a left perfect ring has a non-zero socle.

COROLLARY 1. *A module of a left and right perfect ring is Artinian if and only if it is Noetherian.*

THEOREM 5. *A local ring is left perfect if and only if each left module has a minimal generator set.*

Proof. If A is left perfect each left module has a minimal generator set by Theorem 3. For the converse, let M be a flat left module with minimal generator set $\{m_i\}$. Let F be a free module with $\{x_i\}$ as a basis, and let K be the kernel of the homomorphism from F to M sending $\sum a_i x_i$ to $\sum a_i m_i$. If $k = \sum a_i x_i \in K$, then $\sum a_i m_i = 0$; and since $\{m_i\}$ is minimal, each a_i is a non-unit and so $a_i \in R$, the maximal ideal of A; Thus $K \subseteq RF$. Now $M \approx F/K$, and since M is flat, $RF \cap K = RK$, and since $K \subseteq RF$, $K = RK$. If $\{k_i\}$ is a minimal generator set for K, since $K = RK$, $k_i = \sum r_i k_j$ for some $r_j \in R$, so $k_i = (1 - r_i)^{-1} \sum_{j \neq i} r_j k_j$. Since $\{k_i\}$ is minimal we have $r_j = 0$ if $j \neq i$, so $k_i = 0$, thus $K = \{0\}$ and M is free. From [4, Theorem 1] R is left T-nilpotent, so A is a left perfect ring.

COROLLARY 2. *A commutative ring is perfect if and only if each left module has a minimal generator set.*

Proof. Theorem 3 and Theorem 5.

THEOREM 6. *Let A be any ring and M an A-module. If $K \subseteq \operatorname{Rad} M$ and K is a minimally generated and injective, then $K = \{0\}$.*

Proof. Let $M = N \oplus K$. Since $M/N \approx K$, M/N has a minimal generator set. But $M = N + \operatorname{Rad} M$, so $\operatorname{Rad}(M/N) = M/N$ and by Theorem 1, $M = N$, so $K = \{0\}$.

COROLLARY 3. *If M is a left module of a left perfect ring,* Rad M *does not contain a non-zero injective submodule.*

COROLLARY 4. *If M has no proper maximal submodules, then M does not contain a non-zero minimally generated, injective submodule.*

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