TRACE FORMS ON LIE ALGEBRAS

RICHARD BLOCK

1. Introduction. If L is a Lie algebra with a representation Δ: $a \to a\Delta$ (a in L) (of finite degree), then by the trace form $f = f_\Delta$ of Δ is meant the symmetric bilinear form on L obtained by taking the trace of the matrix products:

$$f_\Delta(a, b) = \mathrm{tr}((a\Delta)(b\Delta)).$$

Then f is invariant, that is, f is symmetric and $f(ab, c) = f(a, bc)$ for all a, b, c in L. By the Δ-radical $L^\perp = L^\perp(\Delta)$ of L is meant the set of a in L such that $f(a, b) = 0$ for all b in L. Then $L^\perp$ is an ideal and f induces a bilinear form $\bar{f} = \bar{f}_\Delta$, called a *quotient trace form*, on $L/L^\perp$. Thus an algebra $\bar{L}$ has a quotient trace form if and only if there exists a Lie algebra L with a representation Δ such that $\bar{L} \cong L/L^\perp(\Delta)$. A quotient trace form is, in particular, a non-degenerate invariant form, but it follows from the results below that there are simple Lie algebras with a non-degenerate invariant form but no quotient trace form, as well as simple algebras with a quotient trace form but no non-degenerate trace form.

Zassenhaus (**14**; **15**) determines the structure of Lie algebras with a quotient trace form in terms of simple Lie algebras which again have a quotient trace form. It is the main purpose of the present paper to identify these simple algebras and to determine which of them have a (non-zero) trace form.

In particular suppose F is an algebraically closed field of characteristic $p > 3$. Then it will be shown that if L is any simple Lie algebra over F with a quotient trace form, then L has a Cartan decomposition for which the axioms of Mills and Seligman (**10**) hold. But the simple algebras satisfying the Mills-Seligman axioms were completely classified (with the use of the usual vector diagrams) in (**10**) and (**9**). They are the analogues over a field of characteristic $\neq 2, 3$ of the complex simple Lie algebras, and are called algebras of classical type (including those of the five exceptional types).*

When F has characteristic $p > 5$, the algebras of classical type are known to have a quotient trace form and thus the result stated above implies that the possession of a quotient trace form is both a necessary and a sufficient

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*After this was written, the writer was informed by Zassenhaus that he had already known that simple algebras over F with a quotient trace form are of classical type. The proof of this which he had in mind is based on his own version of the results Seligman gives in (**11**); this version of Zassenhaus' differs substantially from that of (**11**), and has never been published. In the present paper the result on simple algebras with quotient trace form is proved by extending (**11**; **3**) so that (**10**) becomes applicable.

condition for a simple algebra over F to be of classical type. A similar statement could be made for $p = 5$, except that in that case information is lacking about the algebra of type E_8.

The determination of the simple algebras with quotient trace form generalizes **(11)** and **(3)**, in which it was proved that any simple algebra over F with a non-degenerate trace form is of classical type (**(9)** being needed when $p = 5$ or 7). But it has not been known whether all simple algebras of classical type have a non-degenerate trace form. When $p > 5$ it has been known that they all do except possibly for the algebras of type PA—an algebra L of characteristic p being said to be of type PA if for some multiple n of p, L is isomorphic to the $(n^2 - 2)$-dimensional Lie algebras of all $n \times n$ matrices (with entries in the base field) of trace 0 modulo scalar matrices.

A complete classification of the simple algebras over F with a non-degenerate trace form will be obtained here, when $p > 5$, by showing that the algebras of type PA do not have a non-zero trace form. In this connection, it will be proved that if L is a simple Lie algebra of characteristic $p > 3$, then any absolutely irreducible representation of L with a non-degenerate trace form is a restricted representation.

It is shown in **(12)** that if L is a Lie algebra over F (as above) then any two Cartan subalgebras of L which satisfy the axioms of Mills and Seligman are conjugate by an invariant automorphism. The result on quotient trace forms stated above thus implies an affirmative answer to the question of the conjugacy of the Cartan subalgebras of the algebras of type PA over F.

Finally, the non-existence of trace forms on algebras of type PA allows one to prove that if L is a semi-simple Lie algebra of characteristic $p > 3$ with a non-degenerate trace form, then all derivations of L are inner.

2. Restrictedness of algebras with a quotient trace form. The facts stated in the following lemma are essentially given in **(15)**.

LEMMA 2.1. *Let K be a Lie algebra over a field F of characteristic* $\neq 2, 3$, *with a representation Γ such that $K/K^{\perp}(\Gamma)$ is perfect.* Then there is a Lie algebra L over F with representation Δ such that all of the following four properties hold:*

(2.1) $$L/L^{\perp}(\Delta) \cong K/K^{\perp}(\Gamma);$$

(2.2) Δ *is faithful;*

(2.3) $$L^{\perp}(\Delta) \subseteq zL,$$

(2.4) $$\Delta = \Delta_1 \oplus \ldots \oplus \Delta_r$$

(*the direct sum representation*) *where Δ_i is irreducible and* $L^{\perp}(\Delta_i) \neq L$, $i = 1, \ldots, r$. *The same conclusion holds with* (2.4) *replaced by*

(2.5) *L is perfect.*

*The following notation is used in this paper: If L is a Lie algebra, DL denotes the derived algebra LL of L, and zL denotes the centre of L. Also L is called perfect if $L = DL$. Brackets are used to denote multiplication only for actual commutator products in an associative algebra.

Proof. By Theorem 2 of **(15)** and its proof, there is an algebra L over F with representation Δ such that (2.1) and (2.4) hold, and such that

$$LL^{\perp}(\Delta) \subseteq \text{Ker } \Delta \subseteq L^{\perp}(\Delta)$$

where Ker Δ is the kernel of Δ. By dividing L by Ker Δ, we may clearly suppose that (2.1), (2.2), (2.3), and (2.4) hold. Now for some term $M = D^i L$ in the derived series of L, M is perfect. Since $L/L^{\perp}(\Delta)$ is perfect, $L = M + L^{\perp}(\Delta)$. Thus, writing Θ for the restriction of Δ to M, we have $M^{\perp}(\Theta) = M \cap L^{\perp}(\Delta) \subseteq zM$. It follows that $M/M^{\perp}(\Theta) \cong L/L^{\perp}(\Delta)$. Moreover Θ is faithful. Thus (2.1), (2.2), (2.3), and (2.5) hold, with M, Θ in place of L, Δ, and the lemma is proved.

In order to prove our results about the Cartan decompositions of certain Lie algebras, we shall need to know that the algebras are restricted.

THEOREM 2.1. *Let $\bar{L}$ be a perfect Lie algebra with a quotient trace form, over a field F of characteristic p. If $p = 2$ or 3 suppose also that $\bar{L}$ is of the form $\bar{L} = L/L^{\perp}(\Delta)$ where* (2.3) *holds. Then $\bar{L}$ is restricted.*

Proof. By Lemma 2.1 (or its proof, under the added hypothesis, when $p = 2$ or 3), we may assume that $\bar{L}$ is of the form $L/L^{\perp}(\Delta)$, where (2.2), (2.3), and (2.5) hold. We may then identify L with a Lie algebra of matrices, with Δ the identity mapping. Now if $B \in L$ and $C \in L^{\perp}$ then we wish to determine $\text{tr}(CB^p)$. In doing this we may regard the matrices of L as having entries in the algebraic closure of F. By transforming under similarity, we may also assume that the matrices of L are in simultaneous block triangular form, where the blocks on the diagonal are irreducible. Denote the ith diagonal block of an A in L by A_i. Then for all i, by (2.3), and Schur's lemma, C_i is a scalar, say $s_i I_i$, and since L is perfect, $\text{tr}\, B_i = 0$. Now

$$\text{tr}(CB^p) = \sum_i \text{tr}(C_i B_i^p) = \sum_i s_i (\text{tr}\, B_i)^p = 0.$$

Thus B^p annihilates $L^{\perp}$ under the trace bilinear form, and so induces a linear functional on $\bar{L}$. Since the quotient trace form on $\bar{L}$ is non-degenerate, there is a D in L such that $\text{tr}(AD) = \text{tr}(AB^p)$ for all A in L. But for any A in L, $[A, B^p] = A(\text{ad}B)^p$ (where ad$B$ denotes the transformation $A \to [A, B]$ of L). Thus $D - B^p$ is in the normalizer N of L in the total matrix algebra over F. But the annihilator of L (under the trace form) in N forms an ideal Q of N. Thus for all A in L,

$$[A, D - B^p] \in L \cap Q \subseteq L^{\perp}.$$

Hence the pth power of the inner derivation of $\bar{L}$ by $B + L^{\perp}$ is the inner derivation by $D + L^{\perp}$, so that $\bar{L}$ is restricted, and the theorem is proved.

By using Lemma 3.3 below (and the remark following it for $p = 3$), one may prove in addition that if a perfect Lie algebra L of characteristic $p \geqslant 3$ has a representation Δ such that (2.3) holds, then L itself is restricted.

3. The Cartan decomposition. Let L be a Lie algebra over a field F, and let H be a Cartan subalgebra of L, that is, a nilpotent subalgebra of L which is its own normalizer. Even if F is not algebraically closed, L still has a Cartan decomposition, that is, L is a vector space direct sum of non-zero subspaces $L_0 = H, L_1, \ldots, L_n$, such that for every h in H and $i = 0, \ldots, n$, L_i is invariant under ad h, and the minimal polynomial of the restriction of ad h to L_i is a power of an irreducible polynomial $p_{h,i}$; and such that if $i \neq j$ then there is some h in H for which $p_{h,i} \neq p_{h,j}$. It follows that if E is an extension of F then H_E is a Cartan subalgebra of the scalar extension L_E. If for every h in H, all of the characteristic roots of ad h are in F, then all $p_{h,i}$ are linear, that is, the spaces L_i are root spaces for the roots with respect to H, in the usual sense.

LEMMA 3.1. *Suppose that the Lie algebra L has a representation Δ such that* (2.3) *holds. Let $\bar{H}$ be a Cartan subalgebra of $\bar{L} = L/L^{\perp}(\Delta)$, and let H be the inverse image of $\bar{H}$ under the natural homomorphism $a \rightarrow \bar{a}$ of L onto $\bar{L}$. Then H is a Cartan subalgebra of L. If L has characteristic $\neq 2$ and if* (2.2) *and* (2.4) *also hold then H and $\bar{H}$ are abelian.*

Proof. Clearly H is nilpotent, by (2.3). If a in L normalizes H then $\bar{a}$ normalizes $\bar{H}$, so that $\bar{a} \in \bar{H}$ and $a \in H$. Thus H is a Cartan subalgebra. Then by Lemma 4 of **(15)**, when $p \neq 2$, the conditions (2.2), (2.3), and (2.4) imply that H is abelian, and, *a fortiori*, $\bar{H}$ is abelian.

LEMMA 3.2. *Let $\bar{L}$ be a perfect Lie algebra over a field F of characteristic $p > 3$, and suppose that $\bar{L}$ has a quotient trace form. Then for any $\bar{h}$ in a Cartan subalgebra $\bar{H}$ of $\bar{L}$, if $(\text{ad}\,\bar{h})^p = 0$ then $\bar{h} = 0$.*

Proof. If E is the algebraic closure of F, then clearly $\bar{L}_E$ also has a quotient trace form. Since $\bar{H}_E$ is a Cartan subalgebra of $\bar{L}_E$, it suffices to prove the lemma under the assumption that F is algebraically closed. Let L be a Lie algebra with a representation Δ such that $\bar{L} = L/L^{\perp}(\Delta)$. By Lemma 2.1 we may assume that (2.2), (2.3), and (2.4) are satisfied. Identify each a in L with its image $a\Delta$, and write $\pi: a \rightarrow \bar{a}$ for the natural homomorphism of L onto $\bar{L}$. Let $\bar{h}$ in $\bar{H}$ be such that $(\text{ad}\,\bar{h})^p = 0$, and suppose that $\bar{h} \neq 0$. The restriction of the quotient trace form $\bar{f}_\Delta$ to any Cartan subalgebra of $\bar{L}$ is a non-degenerate bilinear form. Hence there is some $\bar{k}$ in $\bar{H}$ such that $\bar{f}_\Delta(\bar{h}, \bar{k}) \neq 0$. Pick h, k such that $\pi(h) = \bar{h}$, $\pi(k) = \bar{k}$. Thus h and k are in H, $[h, k] = 0$ by Lemma 3.1, and $\text{tr}\, hk \neq 0$. Since $\bar{L}$ is perfect, $L = DL + L^{\perp}$, so that $k = k' + k''$, where $k' \in DL$ and $k'' \in L^{\perp}$. Thus $\text{tr}\, hk = \text{tr}\, hk'$. We shall now compute $\text{tr}\, hk'$.

By transforming under similarity, we may assume that the matrices in L are in simultaneous block triangular form, with irreducible blocks on the diagonal. Denote the ith diagonal block of any a in this set by a_i. Since $0 = [h, k] = [h, k']$, we may also assume, for all i, that h_i and k_i' are in simultaneous triangular form. Now for any a in L,

$$[a, h^{p^2}] = a\ (\text{ad}\ (h^p))^p = a(\text{ad}\ h)^{p^2}.$$

But $\bar{a}$ (ad $\bar{h})^p = 0$ in $\bar{L}$, so by (2.3),

$$[a, h^{p^2}] = 0.$$

Thus by Schur's lemma, for each i, $h_i^{p^2}$ is a scalar matrix. It follows that the diagonal entries of h_i are all equal, say to s_i. Thus $\text{tr}\, hk' = \sum_i \text{tr}\, h_i k_i' = \sum_i s_i \text{tr}\, k_i'$. Since $k' \in L^2$, $\text{tr}\, k_i' = 0$ for all i. Thus $\text{tr}\, hk = \text{tr}\, hk' = 0$, which contradicts the choice of k. Hence $\bar{h} = 0$, and the lemma is proved.

LEMMA 3.3. *Suppose that the hypotheses of Lemma* 3.2 *hold, and that* $\bar{H}$ *is a Cartan subalgebra of* $\bar{L}$ *such that for every* h *in* $\bar{H}$, *all the characteristic roots of* ad h *are in* F. *Then* $\bar{H}$ *acts diagonally, that is, each* ad h *acts as a scalar on each root space.*

Proof. Since $\bar{L}$ has a non-degenerate invariant form and is perfect, it is centreless. But then it is known that restrictedness and the conclusion of Lemma 3.2 imply that $\bar{H}$ acts diagonally (see **3**, Theorem 3, proof).

It may easily be seen that Lemmas 3.2 and 3.3 remain valid when $p = 3$ provided the hypothesis is added that $\bar{L}$ is of the form $L/L^{\perp}(\Delta)$ with (2.3) holding.

LEMMA 3.4. *Suppose the hypotheses of Lemma* 3.3 *hold, and denote the quotient trace form on* $\bar{L}$ *by* f. *Then every non-zero root* α *of* $\bar{H}$ *is non-isotropic with respect to* f, *that is, if* h_α *is the element of* $\bar{H}$ *such that* $f(h_\alpha, h) = \alpha(h)$ *for all* h *in* $\bar{H}$, *then* $\alpha(h_\alpha) \neq 0$.

Proof. We may assume that F is algebraically closed. But $\bar{L}$ is centreless and has a non-degenerate invariant form, and $\bar{H}$ acts diagonally, so by (**8**, p. 165), the roots are non-isotropic (see also **10**, p. 521, n. 2).

By using the remark following the proof of Theorem 2.1 and by modifying the proofs of Theorems 4.1 and 4.2 of (**11**), it is possible to obtain another proof of Lemma 3.4, a proof which is also valid when $p = 3$ (under the added hypothesis of the remark preceding Lemma 3.4). This was shown in (**3**, p. 379) in the case in which f is a non-degenerate trace form.*

The conclusions of Lemma 3.3 and 3.4 correspond to those of Theorems 3.4 and 4.2 of (**11**). In establishing results corresponding to axioms (i)–(v) of (**10**), Seligman makes no further use of the fact that the invariant form he works with arises from a representation (see the proofs of his Corollary 3.2 (**11**, p. 9) and Theorem 5.4 (**11**, pp. 15–18). Thus, under the hypotheses of Lemma 3.3, $\bar{L}$ and $\bar{H}$ satisfy all five axioms of (**10**) (even when $p = 5$ or 7), so we have proved the following result.

THEOREM 3.1. *Let* L *be a perfect Lie algebra with a quotient trace form, over a field* F *of characteristic* $p > 3$. *Suppose that* H *is a Cartan subalgebra of* L

*One point in (**3**, p. 379, ll. 11–16) needs clarification. One supposes that α is a non-zero root such that $\alpha(h_\alpha) = 0$; the functional λ is defined not on L, but on L_1, the three-dimensional subalgebra of L generated by e_α and $e_{-\alpha}$; and U' is a representation of L_1 not of L.

such that for every h in H, all the characteristic root of ad h *are in* F. *Then* L *and* H *satisfy the Mills–Seligman axioms, and hence* L *is a direct sum of simple Lie algebras of classical type.*

As noted above, this implies the following results, the first of which uses the fact that, if L_E is a scalar extension of L, then L_E has a quotient trace form if L does.

COROLLARY 3.1. *Let* L *be a perfect Lie algebra over a field* F *of characteristic* $p > 3$, *and let* E *be the algebraic closure of* F. *If* L *has a quotient trace form then the scalar extension* L_E *is a direct sum of simple Lie algebras of classical type. The converse holds when* $E = F$, *provided it is assumed when* $p = 5$ *that none of the direct summands is of type* E_8.

COROLLARY 3.2. *If* L *and* H *satisfy the hypotheses of Theorem* 3.1, *and if* H_1 *is another Cartan subalgebra of* L *satisfying the condition of Theorem* 3.1, *then* H *and* H_1 *are conjugate under an invariant automorphism.*

4. Remarks on non-classical Lie algebras. Suppose $p > 3$. The above results provide a method of showing that no members of certain classes of simple Lie algebras of characteristic p are isomorphic to algebras of classical type. Indeed, to show this non-isomorphism, one need only obtain a Cartan decomposition which does not have the properties of Cartan decompositions of algebras of classical type. This applies, for example, to the algebras S_n and $L(G, \delta, f)$ (see **2**), which have Cartan subalgebras which do not act diagonally. In particular this gives a quick proof of the fact (proved another way in **1**) that none of the $(p^{2m} - 2)$-dimensional simple algebras V_m are of type PA. Since S_3 and $L(G, \delta, f)$ are not of classical type, they provide examples of simple Lie algebras with a non-degenerate invariant form but no quotient trace form.

Similar results for the case when $p = 3$ may be obtained by using the remark following Lemma 3.3 above.

5. The restrictedness of representations with non-degenerate trace form. Let L be the simple three-dimensional Lie algebra over an algebraically closed field F of characteristic $p > 2$. Thus L has a basis e, f, h, with

$$ef = h, \; eh = e, \; fh = -f, \tag{5.1}$$

and L is restricted, with $e^p = f^p = 0$, $h^p = h$. The restricted irreducible representations of L are well known; we shall need some information about the other irreducible representations of L.

LEMMA 5.1. *Let* Δ *be an irreducible representation of* L *which is not restricted. Then* Δ *has degree* p, *the weights (with respect to* (h)*) are of the form* $\lambda + i\mu$, $i = 0, 1, \ldots, p-1$ *(where* $\mu(h) = 1$*), and each weight space is one-dimensional. If* $p > 3$, *the trace form of* Δ *vanishes identically, while if* $p = 3$, *the trace form of* Δ *is non-degenerate.*

Proof. Let M be the representation space of Δ and write E, F, H for $e\Delta$, $f\Delta$, $h\Delta$, respectively. For any a in L, $(a\Delta)^p - (a^p)\Delta$ is a scalar matrix, by Schur's lemma, and since L is perfect, this scalar matrix has trace 0. Thus there exist s_1, s_{-1}, s_0 in F such that $E^p = s_1 I$, $F^p = s_{-1}I$, $H^p - H = s_0 I$. Since Δ is not restricted, at least one of s_1, s_{-1}, s_0 is non-zero, and thus the degree of Δ is a multiple of p. If $xH = tx$, where $x \in M$, $t \in F$, then $xEH = (t+1)xE$ and $xFH = (t-1)xF$. Thus the characteristic vectors of H span an invariant subspace of M, so that H acts as a scalar on each weight space; and each weight space is invariant under FE. If there exists a non-zero weight vector which is annihilated by either E or F, pick one and call it y; if necessary, change notation (by replacing h by $-h$ and interchanging e and f) so that $yF = 0$. If, for every non-zero weight vector x, $xE \neq 0$ and $xF \neq 0$, take y to be a non-zero weight vector which is a characteristic vector for FE. Set $y_0 = y$, $y_i = yE^i$, $i = 1, 2, \ldots$. Then $y_p = s_1 y$. If $yF \neq 0$, with say $yFE = ty$, then by the definition of y, $s_1 \neq 0$, so that $(yF - (t/s_1)y_{p-1})E = 0$, and hence $yF = (t/s_1)y_{p-1}$, again by the definition of y. Thus, in any case, $y_0, y_1, \ldots, y_{p-1}$ span an invariant subspace of M (cf. Lemma 1 of **(10)**), and hence span all of M. Hence $y_0, \ldots, y_{p-1}$ are linearly independent and each spans a weight space. Thus if $yH = sy$, then

$$\operatorname{tr} H^2 = \sum_{i=0}^{p-1} (s+i)^2 = ps^2 + (p-1)ps + (p-1)p(2p-1)/6 = 0$$

when $p > 3$, and $\operatorname{tr} H^2 \neq 0$ when $p = 3$. The statements of the lemma about the trace form of Δ follow from this since L is simple, and the proof is complete.

A complete classification of the irreducible representations of L may easily be obtained from the above lemma, but we shall have no need for this.

THEOREM 5.1. *Let L be a perfect Lie algebra of characteristic $p > 3$ and let Δ be an absolutely irreducible representation of L with non-degenerate trace form. Then Δ is a restricted representation.*

Proof. By Theorem 2.1, L is restricted, so the conclusion is meaningful. By taking a scalar extension, we may suppose that the base field is algebraically closed. Take a Cartan subalgebra H of L. For each non-zero root α of H, pick elements e_α and f_α spanning the root spaces L_α and $L_{-\alpha}$, respectively, and such that e_α, f_α, and $h_\alpha = e_\alpha f_\alpha$ satisfy (5.1) (with α inserted)—these elements exist by Theorem 3.1) and the properties proved in **(10)**. The elements h_α obtained in this way span H. For the restricted pth power operation in L we have, for any α, $e_\alpha{}^p = f_\alpha{}^p = 0$ and (since $\beta(h_\alpha) \in F_p$ for any root β) $h_\alpha{}^p = h_\alpha$. Thus by Schur's lemma, there are scalars $s_{1\alpha}, s_{-1,\alpha}, s_{0\alpha}$ such that

$$(e_\alpha\Delta)^p = s_{1\alpha}I,\ (f_\alpha\Delta)^p = s_{-1,\alpha}I,\ (h_\alpha\Delta)^p - h_\alpha\Delta = s_{0\alpha}I. \tag{5.2}$$

To prove that Δ is restricted it suffices to show that $s_{i\alpha} = 0$ for all α and $i = -1, 0, 1$. Now suppose that for some non-zero root α, $s_{i\alpha} \neq 0$ for some $i = -1, 0, 1$. Consider the representation Δ_α of the subalgebra of L spanned by e_α, f_α, and h_α, obtained by restricting Δ to this subalgebra. Then for each irreducible constituent $\Delta_{\alpha k}$ of Δ_α, (5.2) is satisfied (with $\Delta_{\alpha k}$ in place of Δ). Thus for each k, $\Delta_{\alpha k}$ is not restricted. Hence by Lemma 5.1,

$$0 = \sum_k \operatorname{tr}(e_\alpha \Delta_{\alpha k})(f_\alpha \Delta_{\alpha k}) = \operatorname{tr}(e_\alpha \Delta)(f_\alpha \Delta).$$

This contradicts the non-degeneracy of the trace form of Δ, and the theorem is proved.

By using the remark following Theorem 2.1, one may generalize Theorem 5.1. to prove the following: If instead of assuming that the trace form of Δ is non-degenerate, one merely assumes that $L^\perp(\Delta) \subseteq zL$, then it is possible to define a restricted pth power operation on L in such a way that Δ is restricted.

6. The algebras of type PA**.** Let F be a field of characteristic $p > 3$ and let n be a positive integral multiple of p. Write $M = M_n(F)$ for the Lie algebra of all $n \times n$ matrices over F, and write E_{ij} for the element of M with 1 in the (i,j) place and 0 elsewhere. Now DM consists of all matrices of trace 0, and

$$0 \subset zM = (I) \subset DM \subset M.$$

Write $\bar{M} = M/(I)$, and for any A in M, write $\bar{A}$ for the element $A + (I)$ of $\bar{M}$. Then $\bar{M}$ is a restricted Lie algebra, $D\bar{M} = (DM)/(I)$ is a simple restricted Lie algebra of dimension $n^2 - 2$, and the algebras of type PA are, up to isomorphism, just the algebras $D\bar{M}$ obtained in this way.

Write

$$H = \{\bar{A} | A \in M, A \text{ diagonal}\},$$

and let α_{ij} denote the functional on H defined as follows: if

$$A = \sum_i s_i E_{ii}$$

then

$$\alpha_{ij}(\bar{A}) = s_j - s_i \qquad (i, j = 1, \ldots, n).$$

Then H is a Cartan subalgebra of $\bar{M}$, the non-zero roots of $\bar{M}$ with respect to H are the functionals α_{ij} $(i \neq j)$, and the root space of the non-zero root α_{ij} is spanned by $\bar{E}_{ij}$. Also $H' = H \cap D\bar{M}$ is a Cartan subalgebra of $D\bar{M}$. Denote the restriction of α_{ij} to H' by α_{ij}'. Then the non-zero roots of $D\bar{M}$ with respect to H' are the functionals α_{ij}' $(i \neq j)$. Take the basis $h_1, \ldots, h_{n-2}$ of H', where

$$h_1 = \bar{E}_{11} - \bar{E}_{nn}, \quad H_i = \bar{E}_{ii} - \bar{E}_{i-1,i-1}, \qquad (i = 2, \ldots, n-2),$$

and take the linear ordering of the roots α_{ij}' induced lexicographically by $-h_1, \ldots, -h_{n-2}$ (see **5**, p. 96; in $D\bar{M}$, root strings have length at most two, so that the root integers have value 0 or ± 1; hence, even if $p = 5$ or 7, the ordering is defined and Lemma 4 of (**5**) holds for $D\bar{M}$). Then $\alpha_{ij}' > 0$ if and only if $j > i$ $(i, j = 1, \ldots, n)$, and the roots

$$\beta_1 = \alpha_{12}', \quad \beta_2 = \alpha_{23}', \ldots, \quad \beta_{n-1} = \alpha_{n-1,n}'$$

form a maximal simple system in the sense of **(5, 6)**. Now take the corresponding linear ordering of the roots of $\bar{M}$ with respect to H, that is, set $\alpha_{ij} > \alpha_{kl}$ if and only if $\alpha_{ij}' > \alpha_{kl}'$.

Let Δ be a restricted representation of $\bar{M}$ (respectively, $D\bar{M}$) with representation space R. A non-zero element m of R is called a *maximal vector* if Δ has a weight λ such that $mh = \lambda(h)m$ for all h in H (respectively, H') and $me = 0$ for all elements e of the root spaces of positive roots. Any weight λ for which there exists such an m is called a *maximal weight* of Δ. If Δ is irreducible then it has at most *one* maximal weight; indeed, the proof of uniqueness given in **(6**, pp. 310–12**)** for simple algebras of classical type also is valid for $\bar{M}$.

THEOREM 6.1. *Let F be an algebraically closed field of characteristic $p > 3$. If Δ is an irreducible restricted representation of $\bar{M}$ ($= M_n(F)/(I)$, where $p|n$), then the restriction of Δ to $D\bar{M}$ is irreducible. Conversely, any irreducible restricted representation of $D\bar{M}$ is the restriction to $D\bar{M}$ of an irreducible restricted representation of $\bar{M}$.*

Proof. Let Δ be an irreducible restricted representation of $\bar{M}$, and denote its restriction to $D\bar{M}$ by Γ. Let R be the representation space of Δ and let S be an irreducible Γ-subspace of R. Thus S is an irreducible restricted $D\bar{M}$ module, and so contains a maximal vector m **(6**, p. 311**)**. Then m is a maximal vector of Γ; we shall show next that m is actually a maximal vector of Δ. Write $a = \bar{E}_{11}$, and let T be the subspace of R spanned by $m, ma, \ldots, m(a\Delta)^{p-1}$. Since $a^p = a$ and Δ is restricted,

$$(a\Delta)^p = a\Delta,$$

so that T is invariant under $a\Delta$. Let λ be the maximal weight of Γ for m, and let Q be the set of elements of T which are either 0 or maximal vectors of Γ with maximal weight λ. Clearly Q is a subspace of T and contains m. If $q \in Q$ and $h \in H'$ then

$$(qa)h = (qh)a = \lambda(h)qa,$$

while if e is in the root space of a positive root then $ae \in Fe$ and so

$$(qa)e = (qe)a + q(ae) = 0,$$

that is, $qa \in Q$. It follows that $Q = T$, and hence any characteristic vector of $a\Delta$ in T is a maximal vector of Δ. Let $\pi = \pi(\xi)$ be the minimal polynomial of the restriction of $a\Delta$ to T. Then π divides $\xi^p - \xi$ and hence is a product of distinct linear factors. There cannot be more than one such factor, since otherwise $a\Delta$ would have two characteristic vectors in T with distinct characteristic roots, and so Δ would have two maximal vectors with distinct maximal weights. Hence π is linear, so that $ma \in Fm$. Then by the Jacobi identity and induction, if $b_1, \ldots, b_k$ are in $D\bar{M}$, then

$$((\ldots(mb_1)\ldots)b_k)a \in S.$$

But S is spanned by m and elements of the form $((\ldots(mb_1)\ldots)b_k)$, so that S is invariant under $a\Delta$, $S = R$, and the first half of the theorem is proved.

Now let λ be a linear functional on H such that

$$\lambda(h) \in F_p, \qquad h = h_1, \ldots, h_{n-2}, \bar{E}_{11}. \tag{6.1}$$

We assert that then there exists an irreducible restricted representation of $\bar{M}$ having λ as a maximal weight. Indeed, the proof of the existence of an irreducible representation of given maximal weight in (**4**, p. 17.3; see also **6**) goes through for $\bar{M}$; it is only necessary to consider U as the (finite dimensional) universal associative algebra of the *restricted* Lie algebra $\bar{M}$, and to note that the representation Θ in (**4**) is restricted because of (6.1). The irreducible representation of $\bar{M}$ of maximal weight λ obtained by the proof in (**4**) is then restricted and of finite degree by the properties of U. The second half of our theorem* now follows from the first half and from the classification by maximal weights of the irreducible restricted representations of simple algebras of classical type, given in (**6**), applied to $D\bar{M}$.

We remark that the above proof may easily be extended to give a classification by maximal weights of all irreducible restricted representations of $\bar{M}$ (when F is algebraically closed)—there are p^{n-1} of them.

THEOREM 6.2. *Let F be a field of characteristic $p > 3$, and let E be the algebraic closure of F. If L is a Lie algebra over F such that the scalar extension L_E is of type PA, then L has no non-zero trace form.*

Proof. We may assume that F is algebraically closed and that L is the algebra $D\bar{M}$ considered in Theorem 6.1. Suppose that Γ is a representation of $D\bar{M}$ with non-zero trace form. Then some irreducible constituent Γ_i of Γ has non-zero trace form. By Theorems 5.1 and 6.1, Γ_i has an extension to a representation Δ of $\bar{M}$. Since the only proper ideal of $\bar{M}$ is $D\bar{M}$, the trace form of Δ is non-degenerate. Hence $(D\bar{M})^{\perp}(\Delta)$ is one-dimensional. But $(D\bar{M})^{\perp}(\Delta) = z\bar{M} = 0$, a contradiction, and the theorem is proved.

COROLLARY 6.1. *Let F be an algebraically closed field of characteristic $p > 5$. A semi-simple Lie algebra L over F has a non-degenerate trace form if and only if L is a direct sum of simple algebras each of which is of classical type other than PA.*

Proof. The result follows from Theorem 6.2, as noted in the introduction.

7. The derivations of algebras with a trace form.

LEMMA 7.1. *Let L be a simple Lie algebra which is of classical type other than PA. Then all derivations of L are inner.*

*The second half of the theorem may also be proved by using Theorem II.10 of (**13**) along with the first half of the theorem.

Proof. Take a Cartan subalgebra H of L such that the axioms of **(10)** hold, and let σ be a derivation of L. By subtracting an inner derivation from σ, we may suppose **(13**, pp. 57–8**)** that $H\sigma = 0$ and that $L_\alpha\sigma \subseteq L_\alpha$ for each root α. It follows from Lemmas 19 and 24 of **(10)** (see **(9)** for $p = 5$ or 7) that there is a set of roots which is a fundamental system in the sense of **(7**, p. 500**)**. The lemma now follows by an application of the argument of **(7**, p. 501**)** to this set of roots. (Actually the lemma need only have been proved for the type E_8 algebra of characteristic 5, since by **(11; 12)** the other algebras either have non-degenerate Killing form or are of type B, C, or D and thus are treated in **(7)**).

THEOREM 7.1. *Let L be a simple Lie algebra of classical type over an algebraically closed field F of characteristic $p > 3$, let Δ be an irreducible restricted representation of L by matrices of degree n, and let σ be a derivation of L. Then there is an $n \times n$ matrix S, with entries in F, such that*

$$(a\sigma)\Delta = [a\Delta, S]$$

for all a in L.

Proof. This follows from the lemma unless L is of type PA. But by **(13**, pp. 59–60**)**, every derivation of $D\bar{M}$ is induced by right multiplication by an element of $\bar{M}$. Thus the theorem holds for algebras of type PA by the second half of Theorem 6.1.

THEOREM 7.2. *Let L be a semi-simple Lie algebra over a field F of characteristic $p > 3$, and suppose that L has a non-degenerate trace form, Then all derivations of L are inner.*

Proof. It is enough to prove this for the case when L is simple. By Corollary 6.1 and Lemma 7.1, the theorem holds when F is algebraically closed. Now let E be the algebraic closure of F, and let n be the dimension of L. Then L_E has a non-degenerate trace form, so that all of its derivations are inner. Now if L had $n + 1$ linearly independent derivations, these derivations could be extended to $n + 1$ derivations of L_E which would be linearly independent over E, a contradiction. Thus the algebra of derivations of L is n-dimensional, and the theorem is proved.

It is interesting to note that (when $p > 5$) the algebras of classical type which do not have a non-degenerate trace form are precisely those which do have non-zero outer derivations.

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