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# A GENERALIZATION OF GASPER'S KERNEL FOR HAHN POLYNOMIALS: APPLICATION TO POLLACZEK POLYNOMIALS

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**1. Introduction.** In this paper we consider a generalization of the discrete Poisson kernel for the Hahn polynomials obtained recently by Gasper [**6**]. The Hahn polynomials of degree $n$ are defined by

$$Q_n^{(N)}(x) = Q_n(x;\alpha,\beta,N) = {}_3F_2\left[\begin{matrix}-n, n+\alpha+\beta+1, -x\\ \alpha+1, \quad -N\end{matrix}\right], \tag{1.1}$$

and are known to be orthogonal on the set of non-negative integers $x = 0, 1, \ldots, N$ provided $\mathrm{Re}\,\alpha, \beta > -1$ or $\mathrm{Re}\,\alpha, \beta < -N$ [**7**; **8**].

Gasper [**6**] first established the following formula for the product of two Hahn polynomials

$$\begin{aligned} Q_n^{(M)}(x)Q_n^{(N)}(y) &= \frac{(-1)^n(\beta+1)_n}{(\alpha+1)_n}\sum_{r=0}^{n}\sum_{s=0}^{n-r} \\ &\times \frac{(-n)_{r+s}(n+\alpha+\beta+1)_{r+s}(-x)_r(-y)_r(x-M)_s(y-N)_s}{(-M)_{r+s}(-N)_{r+s}(\alpha+1)_r(\beta+1)_s r!s!}\end{aligned} \tag{1.2}$$

and then used it to derive the Poisson kernel

$$\begin{aligned} S_z(x,y;\alpha,\beta,M,N) &\equiv \frac{(M-z)\Gamma(M+z)}{\Gamma(M+1)(M+\alpha+\beta+2)_z} \\ \times \sum_{r=0}^{z}\sum_{s=0}^{z-r} &\frac{(-z)_{r+s}\left(\frac{\alpha+\beta+2}{2}\right)_{r+s}\left(\frac{\alpha+\beta+3}{2}\right)_{r+s}(-x)_r(-y)_r(x-M)_s(y-N)_s}{\left(\frac{1-M-z}{2}\right)_{r+s}\left(\frac{2-M-z}{2}\right)_{r+s}(-N)_{r+s}(\alpha+1)_r(\beta+1)_s r!s!} \\ &= \sum_{n=0}^{z}\frac{(-z)_n(-1)^n}{n!}\cdot\frac{(\alpha+1)_n(\alpha+\beta+1)_n}{(\beta+1)_n(M+\alpha+\beta+2)_n}\cdot\frac{2n+\alpha+\beta+1}{\alpha+\beta+1} \\ &\qquad\cdot Q_n^{(M)}(x)Q_n^{(N)}(y),\end{aligned} \tag{1.3}$$

where $x, y, z, M, N$ are non-negative integers such that $0 \leqq x \leqq M$, $0 \leqq y \leqq N$, $0 \leqq z \leqq \min(M, N)$ and, $\alpha > -1$, $\beta > -1$. Under these conditions $S_z$ is positive, and, in the limit $M, N \to \infty$ with $x, y, z$ replaced by $M(1 - \cos 2\phi)/2$, $N(1 - \cos 2\psi)/2$, and $tM$ respectively, reduces to the Poisson kernel for

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Jacobi polynomials [**2**, p. 102]

$$\sum_{n=0}^{\infty} \frac{n!(\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n}(2n+\alpha+\beta+1)t^n P_n^{(\alpha,\beta)}(\cos 2\phi)P_n^{(\alpha,\beta)}(\cos 2\psi)$$
$$(1.4) \qquad = \frac{(\alpha+\beta+1)(1-t)}{(1+t)^{\alpha+\beta+2}} \times F_4[\tfrac{1}{2}(\alpha+\beta+2), \tfrac{1}{2}(\alpha+\beta+3); \alpha+1, \beta+1; a^2/k^2, b^2/k^2],$$

where

$$a = \sin\phi \sin\psi, \quad b = \cos\phi\cos\psi, \quad k = \tfrac{1}{2}(t^{1/2}+t^{-1/2}),$$

and

$$F_4[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha)_{r+s}(\beta)_{r+s}}{(\gamma)_m(\gamma')_n m!n!} x^m y^n$$

is an Appell function [**2**].

However, in deriving (1.3) Gasper does not make use of the orthogonality of the Hahn polynomials on the discrete set and, as he pointed out in his paper, relation (1.3) is valid even for arbitrary complex values of $x$, $y$, $M$, $N$, $\alpha$, $\beta$ satisfying some obvious regularity conditions, but with $z$ still a non-negative integer. It is this aspect of the kernel $S_z$ that allows one to generalize (1.3) so as to encompass other polynomial systems whose supports are not necessarily discrete sets. We state the principal result of this paper in the following theorem.

THEOREM. *Let $x$, $y$, $M$, $N$, $\alpha$, $\beta$ be arbitrary complex numbers such that when $x$, $y$, $M$, $N$ are non-negative integers we require* $0 \leqq x \leqq M$, $0 \leqq y \leqq N$ *and that $\alpha$, $\beta$ cannot have integral values between* $\max(-M, -N)$ *and* $-1$, *both inclusive. Let $z$ be a non-negative integer and $\{a_k\}_{k=0}^{\infty}$ be an arbitrary complex sequence. Then the kernel $K_z(x, y)$ defined by*

$$K_z(x, y) = K_z(x, y; \alpha, \beta, M, N)$$
$$(1.5) \qquad = \sum_{r=0}^{z}\sum_{s=0}^{z-r} \frac{(-z)_{r+s}a_{r+s}(-x)_r(-y)_r(x-M)_s(y-N)_s}{(-M)_{r+s}(-N)_{r+s}(\alpha+1)_r(\beta+1)_s r!s!}$$

*has a decomposition in terms of the Hahn polynomials:*

$$(1.6) \qquad K_z(x, y) = \sum_{n=0}^{z} \frac{(\alpha+1)_n(\alpha+\beta+1)_n}{(\beta+1)_n(\alpha+\beta+1)_{2n}n!} \lambda_n Q_n^{(M)}(x)Q_n^{(N)}(y),$$

*where*

$$(1.7) \qquad \lambda_n = \lambda_n(z; \alpha, \beta, M, N) = \sum_{k=0}^{z-n} \frac{(-z)_{n+k}a_{n+k}}{k!(\alpha+\beta+2+2n)_k}.$$

The important applications of this theorem include the cases when the sequence $\{a_k\}$ is such that $K_z(x, y)$ is demonstrably non-negative, as is the case with Gasper's kernel $S_z(x, y)$, or when the right hand side of (1.5) assumes simple forms, for example, an Appell function or a product of hypergeometric functions. The "eigenvalues" $\lambda_n$ need not be hypergeometric, but when they are, (1.6) can be shown to include many known bilinear sums for the Hahn, Meixner, Krawtchouk and Charlier polynomials on the discrete sets, and, for the Jacobi and Laguerre polynomials in the continuous limits. The use of an arbitrary sequence in a bilinear expansion of the type (1.6) is not a new idea; it has been used before by Verma [**14**] and Fields and Ismail [**5**], among others. The kernel considered in our recent work [**10**] is just the continuous analogue of (1.6). In the case of hypergeometric $\lambda_n$ interesting special situations occur when it is a Saalschutzian or well-poised ${}_3F_2$ or ${}_4F_3$ with argument 1. These special cases we shall discuss in § 4. But the one case which we should like to think as our main application of the theorem just stated concerns the not-too-familiar Pollaczek polynomials, [**4**, p. 221; **13**, p. 395] on the infinite interval, defined by

$$(1.8)\qquad P_n^{\lambda}(x;\varphi) = \frac{(2\lambda)_n}{n!} e^{in\varphi}\, {}_2F_1(-n, \lambda + ix; 2\lambda; 1 - e^{-2i\varphi}),$$

$$-\infty < x < \infty,\ \lambda > 0 \text{ and } 0 < \varphi < \pi.$$

These polynomials are orthogonal with respect to the weight function

$$(1.9)\qquad w^{(\lambda)}(x;\varphi) = \frac{(2\sin\varphi)^{2\lambda-1}}{\pi} e^{-(\pi-2\varphi)x}|\Gamma(\lambda + ix)|^2.$$

The orthogonality relation is given by

$$(1.10)\qquad \int_{-\infty}^{\infty} dx\, w^{\lambda}(x;\varphi)P_n^{\lambda}(x;\varphi)P_m^{\lambda}(x;\varphi) = \frac{\Gamma(2\lambda + n)}{n!} \csc\varphi\,\delta_{mn}.$$

In the final section of this paper we shall deduce a Poisson kernel for the Pollaczek polynomials, expressed in terms of an $F_3$ Appell function in contrast to an $F_4$ function for the Jacobi polynomials as in (1.4), where

$$(1.11)\qquad \mathrm{F}_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s}{(\gamma)_{r+s}r!s!} x^r y^s.$$

**2. Proof of the main theorem.** Using (1.2) on the right hand side of (1.6) and changing the dummy variable $n - r$ to $l$ we obtain

$$K_z(x, y) = \sum_{r=0}^{z} \sum_{l=0}^{z-r} \sum_{s=0}^{l} \sum_{k=0}^{z-r-l} \frac{(-1)^{r+l}(\alpha + \beta + 1)_{r+l}(\alpha + \beta + 2)_{2r+2l}}{(r + l)!(\alpha + \beta + 1)_{2r+2l}}$$
$$\cdot \frac{(-z)_{r+l+k}a_{r+l+k}(-r - l)_{r+s}(r + l + \alpha + \beta + 1)_{r+s} \times (-x)_r(-y)_r(x - M)_s(y - N)_s}{k!(\alpha + \beta + 2)_{2r+2l+k}(-M)_{r+s}(-N)_{r+s}(\alpha + 1)_r(\beta + 1)_s r!s!}.$$

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Now, set $k + l = m$. Then

$$\sum_{l=0}^{z-r} \sum_{k=0}^{z-r-l} \sum_{s=0}^{l} \to \sum_{m=0}^{z-r} \sum_{l=0}^{m} \sum_{s=0}^{l}$$

and

$$(2.1)\quad K_z(x, y) = \sum_{r=0}^{z} \sum_{m=0}^{z-r} \sum_{s=0}^{m} \frac{(-z)_{r+m} a_{r+m} (-x)_r (-y)_r (x - M)_s (y - N)_s}{(-M)_{r+s} (-N)_{r+s} (\alpha + 1)_r (\beta + 1)_s r! s!} T_{r,m,s}$$

where

$$T_{r,m,s} = \sum_{l=s}^{m} \frac{(-1)^{l-s} (\alpha + \beta + 1)_{r+l} (\alpha + \beta + 2)_{2r+2l} (r + l + \alpha + \beta + 1)_{r+s}}{(m - l)!(l - s)!(\alpha + \beta + 1)_{2r+2l} (\alpha + \beta + 2)_{2r+m+l}}$$

(2.2)

$$= \sum_{l=0}^{m-s} \frac{(-1)^{l} (\alpha + \beta + 1)_{r+s+l} (\alpha + \beta + 2)_{2r+2s+2l} (\alpha + \beta + 1 + r + s + l)_{r+s}}{l!(m - s - l)!(\alpha + \beta + 1)_{2r+2s+2l} (\alpha + \beta + 2)_{2r+m+s+l}}.$$

Using the easily verified identities

$$(a)_{m+n} = (a)_m (a + m)_n,$$

$$(a)_{2m} = \left(\frac{a}{2}\right)_m \left(\frac{a}{2} + \frac{1}{2}\right)_m 2^{2m},$$

$$(a + m + n)_n = \frac{(a)_{2n} (a + 2n)_m}{(a)_n (a + n)_m},$$

we obtain

$$T_{r,m,s} = \frac{(\alpha + \beta + 2)_{2r+2s}}{(m - s)!(\alpha + \beta + 2)_{2r+m+s}}$$

$$\times \sum_{l=0}^{m-s} \frac{(-m + s)_l (\alpha + \beta + 2r + 2s + 1)_l \left(\frac{\alpha + \beta + 3}{2} + r + s\right)_l}{l! \left(\frac{\alpha + \beta + 1}{2} + r + s\right)_l (\alpha + \beta + 2 + 2r + m + s)_l}$$

(2.3)

$$= \frac{(\alpha + \beta + 2)_{2r+2s}}{(m - s)!(\alpha + \beta + 2)_{2r+m+s}}$$

$$\times {}_3F_2 \left[ \begin{matrix} \alpha + \beta + 1 + 2r + 2s, \frac{\alpha + \beta + 3}{2} + r + s, \quad -m + s \\ \frac{\alpha + \beta + 1}{2} + r + s, \quad \alpha + \beta + 2 + 2r + m + s \end{matrix} \right].$$

The argument in the hypergeometric function ${}_3F_2$ above is 1. The same should be understood for all the ${}_3F_2$ or ${}_4F_3$ used in the sequel.

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For $m = s$, it is obvious that $T_{r,s,s} = 1$. For $m > s$, $T_{r,m,s}$ vanishes, since the ${}_3F_2$ on the right of (2.3) is of the type

$$ {}_3F_2\left[\begin{matrix} a, 1+\frac{1}{2}a, -n \\ \frac{1}{2}a, \quad w \end{matrix}\right] = \frac{(w-a-m-1)(w-a)_{n-1}}{(w)_n} $$

[**2**, p. 30], and, in our case

$$ w - a - n - 1 = (\alpha + \beta + 2 + 2r + m + s) - (\alpha + \beta + 1 + 2r + 2s) - (m - s + 1) = 0. $$

It follows that

$$ T_{r,m,s} = \delta_{m,s}. $$

Hence

$$ \begin{aligned} &K_z(x, y) \\ (1.5) \qquad &= \sum_{r=0}^{z} \sum_{m=0}^{z-r} \sum_{s=0}^{m} \frac{(-z)_{r+m}a_{r+m}(-x)_r(-y)_r(x-M)_s(y-N)_s}{(-M)_{r+s}(-N)_{r+s}(\alpha+1)_r(\beta+1)_s r!s!}\,\delta_{m,s} \\ &= \sum_{r=0}^{z} \sum_{s=0}^{z-r} \frac{(-z)_{r+s}a_{r+s}(-x)_r(-y)_r(x-M)_s(y-N)_s}{(-M)_{r+s}(-N)_{r+s}(\alpha+1)_r(\beta+1)_s r!s!}, \end{aligned} $$

which proves the theorem.

**3. Limiting results.** We shall now consider various limiting situations where (1.5) and (1.6) reduce to corresponding results for other classical polynomial systems, discrete and continuous.

*Krawtchouk polynomials.* These are defined in [**6**].

$$ (3.1) \qquad \begin{aligned} K_n(x; \rho, M) &= \lim_{\xi\to\infty} Q_n(x; \rho\xi, (1-\rho)\xi, M) \\ &= {}_2F_1(-n, -x; -M; \rho^{-1}), \quad 0 < \rho < 1. \end{aligned} $$

If we set

$$ (3.2) \qquad \alpha = \rho\xi, \quad \beta = (1-\rho)\xi $$

and

$$ (3.3) \qquad a_k = (\xi)_k\, b_k $$

in (1.5) and (1.6) and take the limit $\xi \to \infty$, we obtain

$$ \begin{aligned} &\sum_{r=0}^{z} \sum_{s=0}^{z-r} \frac{(-z)_{r+s}b_{r+s}(-x)_r(-y)_r(x-M)_s(y-N)_s}{(-M)_{r+s}(-N)_{r+s}r!s!}\,\rho^{-r}(1-\rho)^{-s} \\ (3.4) \qquad &= \sum_{n=0}^{z} \frac{(-z)_n}{n!}\left(\frac{\rho}{1-\rho}\right)^n \sum_{k=0}^{z-n} \frac{(-z+n)_k b_{n+k}}{k!} K_n(x; \rho, M) K_n(y; \rho, N). \end{aligned} $$

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Here $\{b_k\}$ is assumed to be another arbitrary sequence. If, instead of (3.3), we take

$$a_k = (\xi)_k\,(\xi)_k\,b_k$$

and proceed to the limit $\xi \to \infty$, we obtain equation (4.7) of Gasper [**5**].

*Meixner and Pollaczek polynomials.* Meixner polynomials $M_n(x;\beta, c)$ are also derivable from the Hahn polynomials in the following limit process [**6**]:

$$\text{(3.5)}\quad \begin{aligned} M_n(x;\beta,c) &= \lim_{M\to\infty} Q_n(x;\beta-1, M(c^{-1}-1), M) \\ &= \lim_{\xi\to\infty} Q_n(x; c\xi/(c-1), \xi/(1-c), -\beta) \\ &= {}_2F_1(-n, -x; \beta; 1-c^{-1}). \end{aligned}$$

They are also the same as the Krawtchouk polynomials (3.1) if we replace $M$ by $-\beta$ and $\rho$ by $c/(c-1)$. Meixner polynomials are orthogonal on the discrete set $x = 0, 1, \ldots$ with respect to the weight function $c^x(\beta)^x/x!$, $\beta > 0$, $0 < c < 1$. Translated in terms of these polynomials, Equation (3.4) then reads

$$\text{(3.6)}\quad \begin{aligned} &\sum_{r=0}^{z}\sum_{s=0}^{z-r} \frac{(-z)_{r+s}b_{r+s}(-x)_r(-y)_r(x+\beta)_s(y+\gamma)_s}{(\beta)_{r+s}(\gamma)_{r+s}r!s!}(1-c^{-1})^r(1-c)^s \\ &= \sum_{n=0}^{z}\frac{(-z)_n}{n!}(-c)^n\sum_{k=0}^{z-n}\frac{(-z+n)_k b_{n+k}}{k!}M_n(x;\beta,c)M_n(y;\gamma,c). \end{aligned}$$

Since this relation is valid for arbitrary complex values of $x$, $y$, $\beta$, $\gamma$ let us replace $x$, $y$, $\beta$, $\gamma$ by $-\lambda - ix$, $-\lambda' - iy$, $2\lambda$ and $2\lambda'$ respectively. Also let us set $c = e^{2i\phi}$. Then, using the hypergeometric representation (3.5) of $M_n(x;\beta,c)$ and the definition (1.8) of the Pollaczek polynomials, we get

$$\text{(3.7)}\quad \begin{aligned} &\sum_{r=0}^{z}\sum_{s=0}^{z-r}\frac{(-z)_{r+s}b_{r+s}(\lambda+ix)_r(\lambda'+iy)_r(\lambda-ix)_s(\lambda'-iy)_s}{(2\lambda)_{r+s}(2\lambda')_{r+s}r!s!} \\ &\times (1-e^{-2i\phi})^r\cdot(1-e^{2i\phi})^s \\ &= \sum_{n=0}^{z}\frac{(-z)_n n!}{(2\lambda)_n(2\lambda')_n}(-1)^n\sum_{k=0}^{z-n}\frac{(-z+n)_k b_{n+k}}{k!}P_n^{\lambda}(x;\phi)P_n^{\lambda'}(y;\phi). \end{aligned}$$

This will be the basis of discussion of the Poisson kernel for the Pollaczek polynomials in § 5.

Although we shall not be interested in the Charlier polynomials $C_n(x; a)$, we might mention in passing that the corresponding bilinear sums for these polynomials can be deduced from (3.4) or (3.6) by using the defining relation [**6**]

$$C_n(x; a) = \lim_{M\to\infty} K_n(x; a/M, M)$$

(3.8)
$$= \lim_{\beta\to\infty} M_n(x; \beta, a/\beta)$$
$$= {}_2F_0(-n, -x; -a^{-1}), \quad a > 0.$$

Charlier polynomials are orthogonal on the discrete set $x = 0, 1, \ldots$ with respect to the Poisson distribution $e^{-a}a^x/x!$.

*Continuous limit.* Replacing $x$, $y$ by $M(1 - x)/2$, $N(1 - y)/2$ respectively, and letting $M$, $N$, $z \to \infty$, $a_k \to 0$ with $(-z)_k\, a_k \to b_k$, we obtain equation (1.5) of [**10**].

**4. Some particular cases.** As we mentioned before, Equation (1.6) is a source of many bilinear sums for the Hahn and other related polynomials. We shall consider only a few examples in this section to illustrate how one can use the theory of generalized hypergeometric series (Bailey [**2**], Slater [**11**]) to derive simpler kernels from (1.6).

(i) $\lambda_n(z)$ *a Saalschutzian* ${}_3F_2$. Let

$$a_k = \frac{(\alpha_1)_k(\alpha_2)_k}{(\alpha_1 + \alpha_2 - \alpha - \beta - z - 1)_k}.$$

Then, from (1.8), we have

(4.1)
$$\lambda_n = \frac{(-z)_n(\alpha_1)_n(\alpha_2)_n}{(\alpha_1 + \alpha_2 - \alpha - \beta - z - 1)_n} \times {}_3F_2\left[\begin{matrix} -z + n, \alpha_1 + n, \alpha_2 + n \\ \alpha_1 + \alpha_2 - \alpha - \beta - z - 1 + n, \alpha + \beta + 2 + 2n \end{matrix}\right].$$

Since the sum of the denominator parameters in this ${}_3F_2$ exceeds that of the numerator parameters by 1 and $z - n$ is a nonnegative integer we can express $\lambda_n$ in terms of Pochhammer products by using Pfaff-Saalschutz theorem [**1**; **2**]. Thus

$$\lambda_n = \frac{(-z)_n(\alpha_1)_n(\alpha_2)_n}{(\alpha_1 + \alpha_2 - \alpha - \beta - z - 1)_n} \cdot \frac{(\alpha + \beta + 2 - \alpha_1 + n)_{z-n}(\alpha + \beta + 2 - \alpha_2 + n)_{z-n}}{(\alpha + \beta + 2 + 2n)_{z-n}(\alpha + \beta + 2 - \alpha_1 - \alpha_2)_{z-n}}$$
$$= \frac{(-1)^n(-z)_n(\alpha + \beta + 2 - \alpha_1)_z(\alpha + \beta + 2 - \alpha_2)_z}{(\alpha + \beta + 2)_z(\alpha + \beta + 2 - \alpha_1 - \alpha_2)_z} \cdot \frac{(\alpha + \beta + 2)_{2n}(\alpha_1)_n(\alpha_2)_n}{(\alpha + \beta + 2 - \alpha_1)_n(\alpha + \beta + 2 - \alpha_2)_n(\alpha + \beta + 2 + z)_n}.$$

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The corresponding bilinear sum becomes

$$\sum_{r=0}^{z}\sum_{s=0}^{z-r}\frac{(-z)_{r+s}(\alpha_1)_{r+s}(\alpha_2)_{r+s}(-x)_r(-y)_r(x-M)_s(y-N)_s}{(\alpha_1+\alpha_2-\alpha-\beta-z-1)_{r+s}(-M)_{r+s}(-N)_{r+s}(\alpha+1)_r(\beta+1)_s r!s!}$$

$$(4.2)\quad =\frac{(\alpha+\beta+2-\alpha_1)_z(\alpha+\beta+2-\alpha_2)_z}{(\alpha+\beta+2)_z(\alpha+\beta+2-\alpha_1-\alpha_2)_z}$$

$$\times\sum_{n=0}^{z}\frac{(-1)^n(-z)_n(\alpha_1)_n(\alpha_2)_n(\alpha+1)_n(\alpha+\beta+1)_n}{(\beta+1)_n(\alpha+\beta+2-\alpha_1)_n(\alpha+\beta+2-\alpha_2)_n(\alpha+\beta+2+z)_n n!}$$

$$\cdot\frac{2n+\alpha+\beta+1}{\alpha+\beta+1}\cdot Q_n^{(M)}(x)Q_n^{(N)}(y).$$

The kernel on the left is evidently positive if $\alpha+1>0$, $\beta+1>0$ and either (i) $\alpha_1>0$, $\alpha_2>0$ and $\alpha_1+\alpha_2<\alpha+\beta+2$ or (ii) $\alpha_1$, $\alpha_2$ both negative integers.

(ii) $\lambda_n$ *related to a well-poised* ${}_3F_2$. Let

$$a_k=\frac{(z+a)_k\left(\frac{\alpha+\beta+2}{2}\right)_k}{\left(\frac{a+1}{2}\right)_k}.$$

Then

$$\lambda_n=\frac{(-z)_n(z+a)_n\left(\frac{\alpha+\beta+2}{2}\right)_n}{\left(\frac{a+1}{2}\right)_n}$$

$$(4.3)\quad\times\,{}_3F_2\left[\begin{matrix}-z+n,z+a+n,\frac{\alpha+\beta+2}{2}+n\\ \frac{a+1}{2}+n,\alpha+\beta+2+2n\end{matrix}\right].$$

The ${}_3F_2$ series on the right is related to a well-poised ${}_3F_2$ which can be summed by Dixon's theorem [**2**, p. 13]. However, Watson's theorem [**11**] gives a value of this ${}_3F_2$ directly:

$${}_3F_2\left[\begin{matrix}-z+n,z+a+n,\frac{\alpha+\beta+2}{2}+n\\ \frac{a+1}{2}+n,\alpha+\beta+2+2n\end{matrix}\right]$$

$$(4.4)\quad=\begin{cases}\dfrac{\left(\frac{1}{2}\right)_{(z-n)/2}\left(\frac{a-\alpha-\beta-1}{2}\right)_{(z-n)/2}}{\left(\frac{a+1}{2}+n\right)_{(z-n)/2}\left(\frac{\alpha+\beta+3}{2}+n\right)_{(z-n)/2}}, & \text{if } z-n \text{ is even}\\ 0, & \text{if } z-n \text{ is odd.}\end{cases}$$

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This leads to the following bilinear sum

$$\sum_{r=0}^{z}\sum_{s=0}^{z-r}\frac{(-z)_{r+s}(z+a)_{r+s}\left(\frac{\alpha+\beta+2}{2}\right)_{r+s}(-x)_r(-y)_r(x-M)_s(y-N)_s}{(-M)_{r+s}(-N)_{r+s}\left(\frac{a+1}{2}\right)_{r+s}(\alpha+1)_r(\beta+1)_s r!s!}$$

(4.5)

$$=\sum_{\substack{n=0\\ z-n\text{ even}}}^{z}\frac{(-z)_n(z+a)_n\left(\frac{1}{2}\right)_{(z-n)/2}\left(\frac{a-\alpha-\beta-1}{2}\right)_{(z-n)/2}(\alpha+1)_n(\alpha+\beta+1)_n}{n!\left(\frac{a+1}{2}\right)_{(z+n)/2}\left(\frac{\alpha+\beta+3}{2}\right)_{(z+n)/2}(\beta+1)_n 2^{2n}}\cdot\frac{2n+\alpha+\beta+1}{\alpha+\beta+1}\cdot Q_n^{(M)}(x)Q_n^{(N)}(y).$$

This, of course, reduces to Gasper's formula (1.3) [**6**] when $a = \alpha + \beta + 1$.

(iii) $\lambda_n$ *a Saalschutzian* ${}_4F_3$. Let us now take

$$a_k = \frac{(\alpha_1)_k(\alpha_2)_k(-M)_k}{(\beta_1)_k(\alpha_1+\alpha_2-\beta_1-M-z-\alpha-\beta-1)_k}.$$

Then

(4.6)

$$\lambda_n = \frac{(-z)_n(-M)_n(\alpha_1)_n(\alpha_2)_n}{(\beta_1)_n(\alpha_1+\alpha_2-\beta_1-M-z-\alpha-\beta-1)_n}\cdot {}_4F_3\left[\begin{matrix}\alpha_1+n,\alpha_2+n,-M+n,-z+n\\ \beta_1+n,\alpha_1+\alpha_2-\beta_1-M-z-\alpha-\beta-1+n,\alpha+\beta+2+2n\end{matrix}\right].$$

Note that the parameters have been chosen such that the ${}_4F_3$ series on the right is Saalschutzian.

We may now apply Whipple's transformation formula for a Saalschutzian ${}_4F_3$ series [**2**, p. 56].

(4.7)

$$ {}_4F_3\left[\begin{matrix}x,y,z,-m\\ u,v,w\end{matrix}\right] = \frac{(v-z)_m(w-z)_m}{(v)_z(w)_z}\,{}_4F_3\left[\begin{matrix}u-x,y-u,z,-m\\ 1-v+z-m,1-w+z-m,u\end{matrix}\right],$$

where $u + v + w = x + y + z - m + 1$.

Choosing the parameters appropriately we then obtain

(4.8)

$$\lambda_n = \frac{(\beta_1+M)_z(\alpha_1+\alpha_2-\beta_1-z-\alpha-\beta-1)_z}{(\beta_1)_z(\alpha_1+\alpha_2-\beta_1-M-z-\alpha-\beta-1)_z}\cdot\frac{(-z)_n(-M)_n(\alpha_1)_n(\alpha_2)_n}{(1-\beta_1-M-z)_n(\alpha+\beta+2-\alpha_1-\alpha_2+\beta_1)_n}$$
$$\cdot {}_4F_3\left[\begin{matrix}\alpha+\beta+2-\alpha_1+n,\alpha+\beta+2-\alpha_2+n,-M+n,-z+n\\ 1-\beta_1-M-z+n,\alpha+\beta+2-\alpha_1-\alpha_2+\beta_1+n,\alpha+\beta+2+2n\end{matrix}\right].$$

Gasper's Equation (1.4) of [**6**] corresponds to the case $\alpha_1 = (\alpha + \beta + 3)/2$, $\alpha_2 = (\alpha + \beta + 2)/2$, $\beta_1 = (1 - M - z)/2$ while his Equation (3.7) follows by setting $\alpha_1 = (\alpha + \beta + 2)/2$, $\alpha_2 = (\alpha + \beta + 1)/2$, $\beta_1 = -(M + z)/2$.

**5. A Poisson kernel for Pollaczek polynomials.** Let us now go back to equation (3.7) and let $z \to \infty$ and $b_k \to 0$ such that $(-z)_k b_k \to (-1)^k (2\lambda')_k t^k$. Then, in the limit, we get

$$\begin{aligned}(1+t)^{2\lambda'} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} & \frac{(\lambda + ix)_r (\lambda' + iy)_r (\lambda - ix)_s (\lambda' - iy)_s}{(2\lambda)_{r+s} r! s!} \\ & \times \{t(e^{-2i\phi} - 1)\}^r \{t(e^{2i\phi} - 1)\}^s \\ &= \sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} \left(\frac{t}{1+t}\right)^n P_n^{\lambda}(x;\phi) P_n^{\lambda'}(y;\phi). \end{aligned} \tag{5.1}$$

But the double series on the left is simply an $F_3$ Appell function defined in (1.11). Thus we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} \left(\frac{t}{1+t}\right)^n P_n^{\lambda}(x;\phi) P_n^{\lambda'}(y;\phi) &= (1+t)^{2\lambda'} \\ \times F_3(\lambda + ix, \lambda - ix, \lambda' + iy, \lambda' - iy; 2\lambda; & \, t(e^{-2i\phi} - 1), t(e^{2i\phi} - 1)). \end{aligned} \tag{5.2}$$

Note that this could also be derived as a special case of Equation (3.5) of Srivastava [**12**] (I would like to thank the referee for bringing this reference to my attention.).

It is easy to see that in spite of the presence of complex variables both sides are real if $\lambda$, $\lambda'$ and $t$ are real. The parameters of the $F_3$ function are such that we can apply the following reduction formula for an $F_3$ to an $F_1$ function [**3**, p. 241].

$$\begin{aligned} & F_3(\alpha, \alpha', \beta, \beta'; \alpha + \alpha'; x, y) \\ & \qquad = (1-y)^{-\beta'} F_1(\alpha, \beta, \beta', \alpha + \alpha'; x, y/(y-1)) \end{aligned}$$

where

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = \sum_{m,n} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n.$$

Thus we have

$$\begin{aligned} & F_3(\lambda + ix, \lambda - ix, \lambda' + iy, \lambda' - iy; 2\lambda; t(e^{-2i\phi} - 1), t(e^{2i\phi} - 1)) \\ & = (1 + t - te^{2i\phi})^{-\lambda' + iy} \\ & \times F_1\left(\lambda + ix, \lambda' + iy, \lambda' - iy, 2\lambda; t(e^{-2i\phi} - 1), \frac{t(1 - e^{2i\phi})}{1 + t(1 - e^{2i\phi})}\right). \end{aligned} \tag{5.3}$$

Assuming only that $\lambda > 0$ we have an integral representation of the above $F_1$ function [**3**, p. 231]:

$$(5.4)\quad F_1(\lambda + ix, \lambda' + iy, \lambda' - iy, 2\lambda; \xi, \eta) = \frac{\Gamma(2\lambda)}{|\Gamma(\lambda + ix)|^2} \times \int_0^1 du\, u^{\lambda+ix-1}(1-u)^{\lambda-ix-1}(1-\xi u)^{-\lambda'-iy}(1-\eta u)^{-\lambda'+iy}.$$

Transforming the integration variable $u$ to $(1-u)/2$ we then obtain

$$\sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n}\left(\frac{t}{1+t}\right)^n P_n^{\lambda}(x;\phi)P_n^{\lambda'}(y;\phi) = \frac{\Gamma(2\lambda)(1+t)^{2\lambda'}}{2^{2\lambda-1}|\Gamma(\lambda+ix)|^2} \times \int_{-1}^{1} du(1-u)^{\lambda+ix-1}(1+u)^{\lambda-1-ix}\{1+it\sin\phi e^{-i\phi}(1-u)\}^{-\lambda'-iy} \cdot \{1-it\sin\phi e^{i\phi}(1+u)\}^{-\lambda'+iy}.$$

Replacing $t/(1+t)$ by $r$ with $|r| < 1$ we get the final form of the kernel

$$(5.5)\quad \sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} r^n P_n^{\lambda}(x;\phi)P_n^{\lambda'}(y;\phi) = \frac{\Gamma(2\lambda)}{2^{2\lambda-1}|\Gamma(\lambda+ix)|^2}\int_{-1}^{1} du\,(1-u)^{\lambda+ix-1}(1+u)^{\lambda-1-ix} \times \{1-r+ir\sin\phi e^{-i\phi}(1-u)\}^{-\lambda'-iy} \cdot \{1-r-ir\sin\phi e^{i\phi}(1+u)\}^{-\lambda'+iy}.$$

When we set $\lambda = \lambda'$ the $F_1$ function representing the right hand side of (5.5) reduces to an ordinary hypergeometric function [**3**, p. 238] which could also be obtained directly by using Equation (12) of [**3**, p. 85]. Thus we get

$$(5.6)\quad \sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} r^n P_n^{\lambda}(x;\phi)P_n^{\lambda}(y;\phi) = (1-r)^{-2\lambda} e^{(x+y)\psi}\left\{1+\frac{4r\sin^2\phi}{(1-r)^2}\right\}^{i(x+y)/2} \times {}_2F_1\left(\lambda-ix, \lambda+iy; 2\lambda; -\frac{4r\sin^2\phi}{(1-r)^2}\right)$$

where $\tan\psi = r\sin 2\phi/(1 - r\cos 2\phi)$. The argument of the ${}_2F_1$ function in (5.6) is positive and less than 1 if $-1 < r < 0$ but becomes negative and numerically exceeds 1 as $r$ goes through 0 to positive values $<1$. When this happens, one can use the transformation formula [**3**, p. 105]

$$(5.7)\quad {}_2F_1(a, b; c; \xi) = (1-\xi)^{-a}\,{}_2F_1(a, c-b; c; \xi/(\xi-1))$$

to convert the hypergeometric function into a convergent series. It therefore suffices to discuss the kernel on the right of (5.6) for $-1 < r < 0$. Apart from the two positive factors the sign of the kernel evidently depends on the sign of the function

(5.8) $$G(x, y; \xi) \equiv (1-\xi)^{i(x+y)/2}\, {}_2F_1(\lambda + ix, \lambda + iy; 2\lambda; \xi)$$

where $\xi \equiv -4r\sin^2\phi/(1-r)^2$. That $G(x, y; \xi)$ is real can be easily checked by using another transformation formula for ${}_2F_1$:

(5.9) $${}_2F_1(a, b; c; \xi) = (1-\xi)^{c-a-b}\, {}_2F_1(c-a, c-b; c; \xi).$$

However, we have not been able to prove that $G(x, y; \xi)$ is non-negative for $0 < \xi < 1$ and for $-\infty < x < \infty$, $-\infty < y < \infty$. In fact, our conjecture is that $G(x, y; \xi)$ oscillates like a trigonometric function, for, we can always write

(5.10) $$G(x, y; \xi) = |{}_2F_1(\lambda + ix, \lambda + iy; 2\lambda; \xi)| \cos(\theta_1 + \theta_2),$$

where $\theta_1 = \arg[(1-\xi)^{i(x+y)/2}]$ and $\theta_2 = \arg[{}_2F_1(\lambda + ix, \lambda + iy; 2\lambda; \xi)]$. It seems unlikely that for general values of $x$ and $y$ and for all $\xi$, $0 < \xi < 1$, $\theta_1 + \theta_2$ would be such as to make $\cos(\theta_1 + \theta_2)$ non-negative.

We shall now show that if $\lambda > \lambda' > 0$ then the limit of the kernel in (5.5) exists and is positive as $r \to 1-$. Let us set

$$\rho_1 e^{i\kappa_1} = 1 - r + ir\sin\phi e^{-i\phi}(1-u),$$
$$\rho_2 e^{i\kappa_2} = 1 - r - ir\sin\phi e^{i\phi}(1+u),$$

and choose the principal branch of the function $\{e^{i(\kappa_1-\kappa_2)}\}^{iy}$. Then the integral on the right of (5.5) reduces to

$$\int_{-1}^{1} du(1-u)^{\lambda+ix-1}(1+u)^{\lambda-ix-1}\rho_1^{-\lambda'-iy}\rho_2^{-\lambda'+iy}e^{(\kappa_1-\kappa_2)y}.$$

As $r \to 1-$, $\rho_1 \to |\sin\phi|(1-u)$, $\rho_2 \to |\sin\phi|(1+u)$, $\tan\kappa_1 \to \cot\phi$ and $\tan\kappa_2 \to -\cot\phi$. Hence $\kappa_1 - \kappa_2 \to \pi - 2\phi$ and we obtain

(5.11) $$\sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} P_n^{\lambda}(x;\phi)P_n^{\lambda'}(y;\phi) = \frac{\Gamma(2\lambda)\, e^{(\pi-2\phi)y}}{2^{2\lambda-1}|\sin\phi|^{2\lambda'}|\Gamma(\lambda+ix)|^2} \times \int_{-1}^{1} du(1-u)^{\lambda-\lambda'+i(x-y)-1}(1+u)^{\lambda-\lambda'-i(x-y)-1}$$
$$= \frac{\Gamma(2\lambda)e^{(\pi-2\phi)y}}{(2\sin\phi)^{2\lambda'}|\Gamma(\lambda+ix)|^2} \cdot \frac{|\Gamma(\lambda-\lambda'+i(x-y))|^2}{\Gamma(2\lambda-2\lambda')},$$

since $0 < \phi < \pi$. If $\lambda = \lambda'$ but $x \neq y$ then the right hand side vanishes because of the factor $\Gamma(2\lambda - 2\lambda')$ in the denominator. Also, if $x = y$ and $\lambda \to \lambda'$ the limit does not exist in the ordinary sense but it does in the sense of distributions. For $\mu > 0$ and $-\infty < t < \infty$, we have

$$\frac{|\Gamma(\mu+it)|^2}{\Gamma(2\mu)} = \frac{|\Gamma(\mu+1+it)|^2}{\Gamma(2\mu+1)} \cdot \frac{2\mu}{(\mu+it)(\mu-it)}$$
$$= \frac{|\Gamma(\mu+1+it)|^2}{\Gamma(2\mu+1)}\left\{\frac{1}{\mu+it} + \frac{1}{\mu-it}\right\}.$$

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It is well-known in the theory of distributions [**15**, p. 67] that

$$\lim_{\mu\to 0+} \frac{1}{\mu \pm it} = \pi\delta(t) \mp i \operatorname{Pv}\frac{1}{t}, \tag{5.12}$$

where $\delta(t)$ is Dirac's delta function and Pv indicates the Cauchy principal value. Using this distributional limit we immediately obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} P_n^{\lambda}(x;\phi)P_n^{\lambda}(y;\phi) &= \frac{\pi\Gamma(2\lambda)\, e^{(\pi-2\phi)x}}{2^{2\lambda-1}(\sin\phi)^{2\lambda}|\Gamma(\lambda+ix)|^2}\delta(x-y)\\ &= \frac{\Gamma(2\lambda)\csc\phi}{w^{\lambda}(x;\phi)}\delta(x-y).\end{aligned} \tag{5.13}$$

This is a formal expression of the completeness of the Pollaczek polynomials on $L_2(-\infty,\infty)$ [**9**, p. 266]. If there is a continuous function $g(x) \in L_2(-\infty,\infty)$ which is orthogonal to all $P_n^{\lambda}(x)$, $n = 0, 1, \ldots$ with respect to the weight function $w^{\lambda}(x;\phi)$ then multiplying (5.13) by $w^{\lambda}(x,\phi)g(x)$ and integrating formally, one finds that $g(x)$ must be identically zero.

In conclusion we would like to mention that there is no choice of $\{b_k\}$ in (3.7) that could lead to the Poisson kernel for the Pollaczek polynomials over the finite internal $[-1, 1]$ because the same $\phi$ occurs in both $P_n^{\lambda}(x;\phi)$ and $P_n^{\lambda'}(y;\phi)$. However, the Poisson kernel for this case can be easily derived from Equation (12) of [**3**, p. 85] by making appropriate choices of the parameters followed by a differentiation.

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