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SPECIAL FUNCTION POTENTIALS FOR THE LAPLACIAN

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1. Introduction. The purpose of this paper is to study the operator $\Delta + q$. Here Δ is the Laplace-Beltrami operator on a compact Lie group G and q is a matrix coefficient of a representation of G. We are able to calculate the powers of $\Delta + q$ acting on the function $q^k u$. This is done in Section 2 and the reader is refered there for definitions of the special functions q and u.

The interest in the operator $\Delta + q$ comes originally from physics and in particular from the Schrödinger equation. This is described in [**4**]. Here we are restricting ourselves to mathematical questions and shall not consider any applications to physics.

In this paper we take the heat equation with potential as

$$(1.1)\qquad (\Delta + q)f - (1/2\pi i)\partial f/\partial t = 0$$

with $t \in \mathscr{H}$, the upper half plane, and initial data $f(x, 0) = q^k(x)u(x)$. This equation can be solved and the solution is given by:

THEOREM 1.1. *The solution of equation* (1.1) *is*

$$f(x, t) = \frac{1}{\pi i}\int_{-\infty}^{\infty} e^{2\pi i t s} \sum_{j=0}^{\infty} \left(\prod_{r=0}^{j} \frac{1}{s - c_{k+r+1}} \right) q^{k+j} u ds,$$

where c_{k+r+1} *is an eigenvalue for the Laplacian on* G.

This theorem is proved in Section 4 as Theorem 4.1. In Section 2 we explain which eigenvalue is c_{k+r+1}. If $\sim$ denotes the Fourier transform

$$(1.2)\qquad \tilde{g}(t) = \int_{-\infty}^{\infty} e^{2\pi i t s} g(s) ds$$

then we can express this solution in terms of the Fourier transform. The result is

$$(1.3)\qquad f(x, t) = \frac{1}{i\pi} \sum_{j=0}^{\infty} \widetilde{\left(\prod_{r=0}^{j} \frac{1}{s - c_{k+r+1}} \right)} q^{k+j} u.$$

Some convention to deal with the poles of the function being transformed has to be adopted. This is explained in Section 4. To complete Section 4 we write down the semigroup property for solutions of equation (1.1).

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When we do this we obtain a rather complicated relation involving power series in q.

Of course there are differential equations other than the heat equation. In Section 5 there is a description of the solutions of a class of equations. Let F be an analytic function of one variable and write $L = \Delta + q$. Then for a function g we understand $F(tL)g$ as

$$(1.4) \qquad F(tL)g = \sum a_n t^n L^n g,$$

where $\sum a_n x^n$ is the Taylor series expansion of F. Then, formally at least, $F(tL)g$ satisfies the equation

$$(1.5) \qquad \frac{\partial}{\partial t}(F(tL)g) = LF'(tL)g.$$

When $F(x) = \exp(2\pi i x)$ we obtain the solution of the heat equation described above. Our result is:

THEOREM 1.2. *With F, L and c_{k+r} as before*

$$F(tL)q^k u = \sum_{j=0}^{\infty} \left(\sum_{r=1}^{j+1} F(tc_{k+r}) \prod_{l=1}^{j+1}{}' \frac{1}{c_{k+r} - c_{k+l}} \right) q^{k+j} u,$$

where in the product we omit the term with $l = r$.

This is proved in Section 5 as Theorem 5.1.

We can apply the result of Theorem 1.2 to the case of the heat equation. That is we put $F(x) = e^{2\pi i x}$. After some manipulation we obtain:

THEOREM 1.3. *The solution of the heat equation* (1.1) *is*

$$f = \sum_{r=1}^{\infty} \left(\sum_{j=r-1}^{\infty} \left(\prod_{l=1}^{j+1}{}' \frac{1}{c_{k+r} - c_{k+l}} \right) q^{k+j} u \right) e^{2\pi i c_{k+r} t}.$$

This is proved as Theorem 5.2. One consequence of this is worth comment.

COROLLARY 1.4. *The functions*

$$\sum_{j=r-1}^{\infty} \left(\prod_{l=1}^{j+1}{}' \frac{1}{c_{k+r} - c_{k+l}} \right) q^{k+j} u$$

are eigenfunctions of $\Delta + q$ with eigenvalues c_{k+r}.

This is proved as Corollary 5.3.

We refer to this as a partial isospectral result. It says that the intersection of the spectra of Δ and $\Delta + q$ is non-empty. Notice that the eigenfunctions of these operators are different even for the same eigenvalue and that both spectra may contain eigenvalues not in this intersection. Thus, only part of the spectra are common; hence, the reference to a partial isospectral result.

The paper is concluded by applying this result to the case $F(x) = x^n$. Having calculated $L^n q^k u$ in Section 2, we can now equate this with the expression given in Theorem 1.2. In this way we obtain the following identities:

THEOREM 1.5. *For each pair of positive integers j and n we have the following identities in $j + 1$ variables:*

a. *if $j \leqq n$ then*

$$\sum x_1^{s_1} \ldots x_{j+1}^{s_{j+1}} = \sum_{r=1}^{j+1} x_r^n \prod_{l=1}^{j+1} \frac{1}{x_r - x_l},$$

where the first summation is over $s_1, \ldots, s_{j+1}$ such that $s_i > 0$ and

$$s_1 + \ldots + s_{j+1} = n - j$$

and in the product we omit the term with $l = r$.

b. *if $j \geqq n + 1$ then*

$$0 = \sum_{r=1}^{j+1} x_r^n \prod_{l=1}^{j+1} \frac{1}{x_r - x_l},$$

where we again omit the term with $l = r$ from the product.

In this summary of the contents of this paper no description has been given of the contents of Section 2 and Section 3. These contain the technical results which make things work.

Finally I should express thanks to all the mathematicians whose discussions have been most helpful. In particular to W. Lichtenstein for pointing out the possibility of calculating $(\Delta + q)^n q^k u$.

2. The Laplacian plus a potential. In this section we introduce the functions q and u on G and then calculate $(\Delta + q)^r q^k u$.

Let G be a compact, connected, simply connected Lie group and $\pi_\lambda : G \rightarrow \mathrm{Aut}\, V_\lambda$ be the representation of G with highest weight λ. In V_λ let v_λ be the highest weight vector and let $\langle\ ,\ \rangle$ be a G-invariant inner product on V_λ. Then if we pick any two vectors v and w in V_λ we can define q and u as follows:

$$(2.1) \qquad q(g) = \langle v_\lambda, \pi_\lambda(g)v\rangle$$

and

$$(2.2) \qquad u(g) = \langle v_\lambda, \pi_\lambda(g)w\rangle.$$

These are matrix coefficients of the representation π_λ. The significance of the role of the highest weight vector v_λ occurs for us when we consider the product $q^k u$. Clearly $q^k u$ is a matrix coefficient in $\bigotimes^{k+1} V$, the $(k + 1)$-fold tensor product of V_λ. However, because of the use of v_λ in the definition of q and u we can identify the irreducible subrepresentation of

$\bigotimes^{k+1} V_\lambda$ which has $q^k u$ as a matrix coefficient. This subrepresentation is $V_{(k+1)\lambda}$, the representation with highest weight $(k+1)\lambda$.

Now that we have defined q and u the following result is immediately obvious.

LEMMA 2.1. *The function $q^k u$ is an eigenfunction for the Casimir operator with eigenvalue $c((k+1)\lambda)$.*

Here $c(\mu) = \langle \mu + 2\rho, \mu \rangle$ for a weight μ, ρ is half the sum of the positive roots and $\langle\ ,\ \rangle$ is the negative of the Killing form, see [**2**] for more details of the notation.

In this paper we shall use the notation $L = \Delta + q$. Now we can calculate $L^r q^k u = (\Delta + q)^r q^k u$. The result is given by:

THEOREM 2.2. *The powers of the Laplacian plus potential q satisfy*

$$L^r q^k u = \sum_{j=0}^{r} \left(\sum c_{k+1}^{s_1} \ldots c_{k+j+1}^{s_{j+1}}\right) q^{k+j} u,$$

where the second summation is over all s_i such that $s_i \geqq 0$ and

$$s_1 + \ldots + s_{j+1} = r - j.$$

The numbers c_i are $c_i = c(i\lambda)$.

Proof. The proof proceeds by induction on r. First consider the case $r = 1$ then

$$(2.3) \qquad L q^k u = c_{k+1} q^k u + q^{k+1} u.$$

Thus the result is true for $r = 1$. Now let

$$a_{r,j} = \sum c_{k+1}^{s_1} \ldots c_{k+j+1}^{s_{j+1}}$$

so that the result of the theorem reads

$$(2.4) \qquad L^r q^k u = \sum_{j=0}^{r} a_{r,j} q^{k+j} u.$$

By induction we suppose that this is true. Then

$$(2.5) \qquad L^{r+1} q^k u = L\left(\sum a_{r,j} q^{k+j} u\right)$$

so that

$$(2.6) \qquad L^{r+1} q^k u = \sum_{j=1}^{r} (c_{k+j+1} a_{r,j} + a_{r,j-1}) q^{k+j} u + c_{k+1}^{r+1} q^k u + q^{k+r+1} u.$$

Since the coefficients $a_{r,j}$ satisfy

$$a_{r+1,j} = c_{k+j+1} a_{r,j} + a_{r,j-1}$$

this completes the proof.

Now we observe that these coefficients $a_{r,j}$ satisfy the following:

LEMMA 2.3. *The infinite series* $\sum_{r=j}^{\infty} a_{r,j}t^{r-j}$ *has the sum*

$$\sum_{r=j}^{\infty} a_{r,j}t^{r-j} = \prod_{r=1}^{j+1} \frac{1}{1 - c_{k+r}t}.$$

The proof of this is elementary and is left to the reader.

Finally in this section we calculate $(1 - tL)^{-1}q^k u$. We use the well known expansion

$$(2.7) \qquad (1 - x)^{-1} = 1 + x + x^2 + \dots .$$

Substituting $x = tL$ into (2.7) yields

$$(2.8) \qquad (1 - tL)^{-1}q^k u = \sum_{r=0}^{\infty} t^r L^r q^k u.$$

Now by Theorem 2.2 we have

$$(2.9) \qquad (1 - tL)^{-1}q^k u = \sum_{r=0}^{\infty} \sum_{j=0}^{r} a_{r,j}t^r q^{k+j}u.$$

Interchanging the order of summation in (2.9) gives

$$(2.10) \qquad (1 - tL)^{-1}q^k u = \sum_{j=0}^{\infty} \sum_{r=j}^{\infty} a_{r,j}t^{r-j}t^j q^{k+j}u,$$

which by Lemma 2.3 is

$$(2.11) \qquad (1 - tL)^{-1}q^k u = \sum_{j=0}^{\infty} \left(\prod_{r=1}^{j+1} \frac{1}{1 - tc_{k+r}} \right) t^j q^{k+j}u.$$

We collect this together as the following result.

THEOREM 2.4. *The operator* $(1 - tL)$ *has the inverse, when acting on* $q^k u$,

$$(1 - tL)^{-1}q^k u = \sum_{j=0}^{\infty} \left(\prod_{r=1}^{j+1} \frac{1}{1 - tc_{k+r}} \right) t^j q^{k+j}u.$$

3. Cauchy's formula for operators. The aim of this section is to take Cauchy's integral formula,

$$(3.1) \qquad f(z) = \frac{1}{2\pi i}\int_{\gamma} f(s)\, \frac{1}{s - z}\, ds,$$

and replace the complex number z by an operator.

We shall state the result

THEOREM 3.1. *Let* P *be a normal differential operator with eigenvalues* λ_j, $t \in \mathbf{C}$ *and* γ *a simple closed curve containing* $\{\lambda_j\}$. *If* G *is an analytic func-*

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tion then

$$G(tP)u = \frac{1}{2\pi i}\int_{\gamma} G(ts)\frac{1}{s-P}uds,$$

for a smooth function u.

Proof. First observe that if u is an eigenfunction of P with eigenvalue λ then the equation reads

$$(3.2) \qquad G(t\lambda)u = \frac{1}{2\pi i}\int_{\gamma} \frac{G(ts)}{s-\lambda}uds.$$

This is just Cauchy's integral formula. To complete the proof we write u as a linear combination of eigenfunctions. Since P is normal, that is $PP^* = P^*P$ for P^* the adjoint of P, the spectral theorem shows that we can find an eigenfunction expansion of u, see [**5**].

Unfortunately we wish to apply this result to the case when $P = L = \Delta + q$. In this case we have an increasing sequence of eigenvalues and so there is no closed contour γ enclosing all of the eigenvalues. In this section we shall content ourselves with explaining the modification of Theorem 3.1 in the case when $G(x) = e_0^{ikx}$.

THEOREM 3.2. *Let P be a normal differential operator with eigenvalues $\{\lambda_j\}$ and $t \in \mathbf{C}$ such that $\{t\lambda_j\}$ is contained in the upper half plane. Then*

$$e^{iktP}u = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{e^{iks}}{s-tP}uds.$$

Proof. This follows by exactly the same argument as Theorem 3.1 once we have established

$$(3.3) \qquad e^{iktz} = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{e^{iks}}{s-tz}ds.$$

This result is a standard result from the theory of Fourier transforms, see [**3**].

We now define a class of functions, $\mathscr{C}$, with the property that $f \in \mathscr{C}$ if and only if

$$(3.4) \qquad f(tz) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{f(s)}{s-tz}ds.$$

Equation (3.3) shows that $\mathscr{C}$ is non-empty since $f(x) = e_0^{ikx}$ is in $\mathscr{C}$. The following result is immediately obvious from the proof of Theorem 3.2.

COROLLARY 3.3. *Let $G \in \mathscr{C}$ then*

$$G(tP)u = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{G(s)}{s-tP}uds.$$

4. The heat equation. In this section we apply the results of the previous section to give an integral expression for the solution of the equation

$$Lf - (1/2\pi i)\partial f/\partial t = 0$$

subject to the initial data

$$f(x, 0) = q(x)^k u(x).$$

Formally the solution is given by

$$f(x, t) = e^{2\pi i t L} q(x)^k u(x). \tag{4.1}$$

By Theorem 3.2 this is given by

$$f(x, t) = \frac{1}{i\pi}\int_{-\infty}^{\infty} e^{2\pi i s}\frac{1}{s - tL} q^k u ds. \tag{4.2}$$

Now it is clear that to give the integral expression for f we must recall the expression for $(1 - tL)^{-1}q^k u$. Since we have

$$(s - L)^{-1}q^k u = s^{-1}(1 - s^{-1}L)^{-1}q^k u \tag{4.3}$$

then from Theorem 2.4 we find:

$$(s - L)^{-1}q^k u = \sum_{j=0}^{\infty}\left(\prod_{r=1}^{j+1}\frac{1}{s - c_{k+r}}\right) q^{k+j}u. \tag{4.4}$$

We state the next result as a theorem.

THEOREM 4.1. *The solution of the heat equation*

$$Lf - (1/2\pi i)\partial f/\partial t = 0$$

subject to the initial data $f(x, 0) = q(x)^k u(x)$ *is*

$$f(x, t) = \frac{1}{i\pi}\int_{-\infty}^{\infty} e^{2\pi i t s}\sum_{j=0}^{\infty}\left(\prod_{r=1}^{j+1}\frac{1}{s - c_{k+r}}\right) q^{k+j}u ds.$$

As a corollary to this we observe the following formula.

COROLLARY 4.2. *The solution to the above heat equation can be expressed in terms of Fourier transforms as*

$$f(x, t) = \frac{1}{i\pi}\sum_{j=0}^{\infty}\widetilde{\left(\prod_{r=1}^{j+1}\frac{1}{s - c_{k+r}}\right)} q^{k+j}u.$$

Here we make the following notes on the Fourier transform. It is defined by

$$\tilde{f}(t) = \int_{-\infty}^{\infty} e^{2\pi i t s} f(s) ds. \tag{4.5}$$

The functions $1/(s - c_{k+r})$ have poles on the real axis. In this case the Fourier transform is understood to be

$$(4.6)\qquad \tilde{f}(t) = e^{2\pi\alpha t}\tilde{g}_\alpha(t)$$

where $g_\alpha(s) = f(s - i\alpha)$ for α a positive real number. Notice that Equation (4.6) is independent of the choice of α.

One consequence of this is the following identity. This is a straight forward application of the semigroup property for solutions of the heat equation.

$$\frac{1}{i\pi}\int_{-\infty}^{\infty} e^{2\pi i(t+\tau)s}\sum_{j=0}^{\infty}\left(\prod_{r=1}^{j+1}\frac{1}{s - c_{k+r}}\right) q^{k+j}uds$$
$$= -\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i(tx+\tau y)}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\left(\prod_{r=1}^{j+l+1}\frac{1}{s - c_{k+r}}\right)$$
$$\times\frac{1}{s - c_{k+j+1}}q^{k+j+l}udxdy.$$

5. Other differential equations. In this section we give a formula for the solution of a class of differential equations.

Let F be an analytic function of one variable. Then we understand $F(tL)g$ as

$$(5.1)\qquad F(tL)g = \sum a_n t^n L^n g,$$

where $\sum a_n x^n$ is the Taylor series of F. Clearly if g is an eigenfunction of L then the series (5.1) converges from some t and

$$(5.2)\qquad F(tL)g = F(t\lambda)g$$

where $Lg = g\lambda$. If g is a linear combination of eigenfunctions we shall still have convergence in (5.1) but in general we do not have an expression as simple as (5.2). As one would expect the functions $F(tL)g$ satisfy differential equations. For $F(tL)g$ this equation is

$$(5.3)\qquad \frac{\partial}{\partial t}(F(tL)g) = LF'(tL)g.$$

So for example if $F(x) = \exp(2\pi ix)$ then we obtain a solution of the heat equation. This was seen in the previous section.

We shall now state our theorem.

THEOREM 5.1. *Let $L = \Delta + q$ and $c_j = c(j\lambda)$ be as before. If F is an analytic function of one variable then*

$$F(tL)q^k u = \sum_{j=0}^{\infty}\left(\sum_{r=1}^{j+1}F(tc_{k+r})\prod_{l=1}^{j+1}{}'\frac{1}{c_{k+r} - c_{k+l}}\right)q^{k+j}u.$$

The prime with the product sign denotes that we omit the term with $l = r$.

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Proof. First we order the eigenvalues of $L : \lambda_1, \lambda_2, \ldots$, and the corresponding eigenfunctions $u_1, u_2, \ldots$. Let γ_n be a simple closed curve containing $\lambda_1, \ldots, \lambda_n$. Then as in Theorem 3.1 we have

$$(5.4) \qquad F(tL)f = \frac{1}{2\pi i}\int_{\gamma_n} F(ts)\frac{1}{s-L}f ds,$$

for any function f which is a linear combination of $u_1, \ldots, u_n$. Now choose a sequence of functions g_n so that $q^k u - g_n$ is a linear combination of $u_1, \ldots, u_n$. Applying (5.4) to $f = q^k u - g_n$ gives

$$(5.5) \qquad F(tL)(q^k u - g_n) = \frac{1}{2\pi i}\int_{\gamma_n} \frac{F(ts)}{s-L}(q^k u - g_n) ds.$$

Hence by Theorem 2.4

$$(5.6) \qquad F(tL)(q^k u - g_n) = \frac{1}{2\pi i}\int_{\gamma_n} F(ts)\sum_{j=0}^{\infty}\left(\prod_{r=1}^{j+1}\frac{1}{s-c_{k+r}}\right) q^{k+j} u ds - \frac{1}{2\pi i}\int_{\gamma_n} F(ts)\frac{1}{s-L} g_n ds.$$

Now by the residue theorem

$$(5.7) \qquad F(tL)(q^k u - g_n) = \sum_{j=0}^{\infty}\sum_{r=1}^{j+1}{}' F(tc_{k+r})\left(\prod_{l=1}^{j+1}{}'\frac{1}{c_{k+r}-c_{k+l}}\right) q^{k+j} u - \frac{1}{2\pi i}\int_{\gamma_n}\frac{F(ts)}{s-L} g_n ds.$$

In Equation (5.7) we have a prime with both the second sum and product. The prime with the sum indicates that we only have those r such that c_{k+r} lies inside γ_n and the prime with the product indicates that we omit the term with $l = r$.

To complete the proof of the theorem we let n tend to infinity. By the spectral theorem $q^k u$ can be expanded in terms of $u_1, u_2, \ldots$. Hence g_n can be expanded in terms of $u_{n+1}, u_{n+2}, \ldots$. Thus the term

$$(-1/2\pi i)\int_{\gamma_n} F(ts)(s-L)^{-1} g_n ds$$

only involves $u_{n+1}, u_{n+2}, \ldots$. Since for any j we can find an n so $c_{k+1}, \ldots, c_{k+j+1}$ lie inside γ_n this means that as n tends to infinity the right hand side of the Equation (5.7) tends to the right hand side of the equation in the theorem, thus completing the proof.

We can apply this to the heat equation. To do this take $F(x) = e^{2\pi i x}$ in Theorem 5.1.

THEOREM 5.2. *The solution of the equation*

$$Lf - (1/2\pi i)\partial f/\partial t = 0$$

subject to the initial data $f(x, 0) = q^k u$ *is*

$$f = \sum_{r=1}^{\infty} \left(\sum_{j=r-1}^{\infty} \left(\prod_{l=1}^{j+1}{}' \frac{1}{c_{k+r} - c_{k+l}} \right) q^{k+j} u \right) e^{2\pi i c_{k+r} t}.$$

Proof. In Theorem 5.1 take $F = e^{2\pi i x}$ to give

$$(5.8) \quad f = \sum_{j=0}^{\infty} \left(\sum_{r=1}^{j+1} e^{2\pi i c_{k+r} t} \prod_{l=1}^{j+1}{}' \frac{1}{c_{k+r} - c_{k+l}} \right) q^{k+j} u.$$

Now interchange the order of summation. The expression in this theorem we shall refer to as the "Fourier" expansion of f.

To justify interchanging the order of summation we need to consider questions of convergence. Since $c_j = c(j\lambda)$ we have

$$(5.9) \quad c_j = \langle j\lambda, j\lambda + 2\rho \rangle = j^2 \langle \lambda, \lambda \rangle + 2j \langle \lambda, \rho \rangle$$

so $c_j = O(j^2)$. Thus there is the bound

$$(5.10) \quad \left| \prod_{l=1}^{j+1}{}' \frac{1}{c_{k+j} - c_{k+l}} \right| < \frac{c^j}{(r!)^2},$$

where $r =$ integer part $[j + 1/2]$ and c is a constant. Notice that this bound is not the best possible. On the other hand, since t lies in the upper half plane the real part of $2\pi i t$ is negative. Hence we have

$$(5.11) \quad |e^{2\pi i c_{k+r} t}| < e^{-(k+r)\alpha}$$

for some positive constant α. The bounds (5.10) and (5.11) assure convergence for all the manipulations performed.

One consequence of Theorem 5.2 is the following result.

COROLLARY 5.3. *The functions*

$$\sum_{j=r-1}^{\infty} \left(\prod_{l=1}^{j+1}{}' \frac{1}{c_{k+r} - c_{k+l}} \right) q^{k+j} u$$

are eigenfunctions of L with eigenvalues c_{k+r}.

Proof. This function is the coefficient of $e^{2\pi i c_{k+r} t}$ in the "Fourier" expansion of the solution to the equation $Lf = (1/2\pi i)\partial f/\partial t$.

This result is a partial isospectral result. That is part of the spectrum of Δ, namely c_{k+r}, also appears as part of the spectrum of L. Notice, however, that the eigenfunctions of Δ and L are different, even for the same eigenvalue, and that the whole spectrum of Δ may be different from that of L.

We can check more directly the result contained in Equation (5.8). To do this we expand f as a power series in $q^{k+j}u$. It is convenient to write this as

$$(5.12) \quad f(x, t) = \sum_{j=0}^{\infty} e^{2\pi i c_{k+j+1} t} g_j(t) q^{k+j}(x) u(x).$$

The functions $g_j(t)$ then satisfy the initial value problem:

(5.13) $g_0(0) = 1,\ g_j(0) = 0,\ j \neq 0,$

$$\frac{d}{dt} g_j(t) = 2\pi i e^{2\pi i(c_{k+j}-c_{k+j+1})t} g_{j-1}.$$

We found this system from the initial data $f(x, 0) = q^{k+j}u$ and the equation

$$Lf = (1/2\pi i)\partial f/\partial t.$$

It is easy to solve the system (5.13), for example by taking the Laplace transform, and after some manipulation we obtain (5.8).

6. Some polynomial identities. In this section we take the formula from the previous section and calculate it in a special case.

First let us recall the result of Theorem 5.1:

$$(6.1)\qquad F(tL)q^k u = \sum_{j=0}^{\infty} \left(\sum_{r=1}^{j+1} F(tc_{k+r}) \prod_{\substack{l=1\\ l\neq r}}^{j+1} \frac{1}{c_{k+r} - c_{k+l}} \right) q^{k+j}u.$$

In this formula we use $F(x) = x^n$. This gives us the following

$$(6.2)\qquad L^n q^k u = \sum_{j=0}^{\infty} \left(\sum_{r=1}^{j+1} c_{k+r}^n \prod_{l=1}^{j+1}{}' \frac{1}{c_{k+r} - c_{k+l}} \right) q^{k+j}u.$$

In Section 2 we calculated $L^n q^k u$ as

$$(6.3)\qquad L^n q^k u = \sum_{j=0}^{n} \left(\sum c_{k+1}^{s_1} \dots c_{k+j+1}^{s_{j+1}}\right) q^{j+k}u.$$

If we equate (6.3) and (6.2) we obtain an identity. Then equating coefficients of $q^{k+j}u$ we obtain the following.

a) For $j \leqq n$ we have

$$(6.4)\qquad \sum c_{k+1}^{s_1} \dots c_{k+j+1}^{s_{j+1}} = \sum_{r=1}^{j+1} c_{k+r}^n \prod_{l=1}^{j+1}{}' \frac{1}{c_{k+j} - c_{k+l}}$$

where the first summation is over all $s_1, \dots, s_{j+1}$ such that $s_1 + \dots + s_{j+1} = n - j$ and $s_i > 0$.

b) In the case $j \geqq n + 1$ we have

$$(6.5)\qquad 0 = \sum_{r=1}^{j+1} c_{k+r}^n \prod_{l=1}^{j+1}{}' \frac{1}{c_{k+j} - c_{k+l}}.$$

In fact we have established rather more that (6.4) and (6.5). We state the result as a theorem.

THEOREM 6.1. *For each pair of positive integers j and n there is the following identity in $j + 1$ variables $x_1, \dots, x_{j+1}$:*

a) *if* $j \leqq n$ *then*

$$\sum x_1^{s_1} \ldots x_{j+1}^{s_{j+1}} = \sum_{r=1}^{j+1} x_r^{\,n} \prod_{l=1}^{j+1}{}' \frac{1}{x_r - x_l},$$

where the first summation is over s_i *such that* $s_1 + \ldots + s_{j+1} = n - j$ *and* $s_i \geqq 0$.

b) *if* $j \geqq n + 1$ *then*

$$0 = \sum_{r=1}^{j+1} x_r^{\,n} \prod_{l=1}^{j+1}{}' \frac{1}{x_r - x_l}.$$

Proof. Both of these identities are polynomial identities involving symmetric homogeneous polynomials. By Equations (6.4) and (6.5) we know that they are true for $x_r = c_{k+r}$. Taking values of $k = 1, 2, \ldots$ we obtain infinitely many distinct values of $(x_1, \ldots, x_{j+1})$ for which these identities are true. This completes the proof since we now have infinitely many zeros of a polynomial identity.

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