Canad. Math. Bull. Vol. **23** (2), 1980

A COMMUTATIVITY THEOREM FOR DIVISION RINGS

BY
ANTHONY RICHOUX

ABSTRACT. Let D be a division ring with center Z. Suppose for all $x \in D$, there exists a monic polynomial, $f_x(t)$, with integer coefficients such that $f_x(x) \in Z$. Then D is commutative.

Throughout this note, J is the ring of rational integers, Q is the field of rational numbers, and D is a division ring with center Z. A polynomial in $J[t]$ is said to be monic (co-monic) if its highest (lowest) non-zero coefficient is one.

In [1, Theorem 2] Herstein showed that if D satisfies

(1) for all $x \in D$, there exists a co-monic $f_x(t) \in J[t]$ such that $f_x(x) \in Z$,

then D is a field. In this note, we show that if D satisfies

(2) for all $x \in D$, there exists a monic $f_x(t) \in J[t]$ such that $f_x(x) \in Z$,

then D is a field. This answers in the affirmative a question posed by Chacron [2].

LEMMA 1. *Let $E \subseteq Z$ be a Euclidian domain containing infinitely many primes $\{p_i\}_{i=1}^{\infty}$. Let $x \in D$. Suppose for each integer $i \geq 1$, there exists $q_i(t) = \sum_{j=1}^{n_i} \alpha_{i,j} t^j \in E[t]$ such that*

(i) *for all $j \geq 2$, $p_i \mid \alpha_{i,j}$, but $p_i \nmid \alpha_{i,1}$ and*
(ii) *$q_i(x) \in Z$.*

Then if $\{\deg(q_i)\}_{i=1}^{\infty}$ is bounded, then $x \in Z$.

Proof. Since $\{\deg(q_i)\}_{i=1}^{\infty}$ is bounded, there is an integer $n \geq 1$ such that for infinitely many integers $i \geq 1$, $\deg(q_i) = n$. Let n be the least integer ≥ 1 with the property that there exists infinitely many primes $\{p_i\}_{i=1}^{\infty} \subseteq E$ and for each integer $i \geq 1$, there exists a polynomial $q_i(t) = \sum_{j=1}^{n} \alpha_{i,j} t^j \in E[t]$ such that $\deg(q_i) = n$ and $q_i(t)$ satisfies (i) and (ii). If $n = 1$, we are obviously done. Suppose $n > 1$. For each integer $i \geq 1$, let $f_i(t) = \alpha_{1,n} q_i(t) - \alpha_{i,n} q_1(t)$. Then $f_i(t) \in E[t]$, $f_i(x) \in Z$ and $\deg(f_i) < n$. Also if $f_i(t) = \sum_{j=1}^{n-1} \beta_{i,j} t^j$, then for $j \geq 2$, $p_i \mid \alpha_{1,n}\alpha_{i,j} - \alpha_{i,n}\alpha_{1,j} = \beta_{i,j}$. Now if $p_i \mid \beta_{i,1} = \alpha_{1,n}\alpha_{i,1} - \alpha_{i,n}\alpha_{1,1}$, then since $p_i \mid \alpha_{i,n}$ and $p_i \nmid \alpha_{i,1}$, $p_i \mid \alpha_{1,n}$. Since only finitely many primes can divide $\alpha_{1,n}$, we have infinitely many f_i's satisfying (i) and (ii), contradicting the minimality of n. Thus $n = 1$ and we are done.

Received by the editors September 25, 1978.

Note that a division ring of non-zero characteristic which satisfies (2) also satisfies (1) and is hence, by Herstein's Theorem, a field. We thus, throughout the remainder of this note, assume D is a division ring of characteristic zero which satisfies (2).

LEMMA 2. *If $x \in D$ is transcendental over Q, then $x \in Z$.*

Proof. Let $f(t) = t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_1 t + a$ where the α_i's are integers, $a \in Z$, and $f(x) = 0$. By hypothesis a is transcendental over Q. So $E = Q[a]$ is a Euclidean domain contained in Z and containing infinitely many primes $\{p_i\}_{i=1}^{\infty}$. For each integer $i \geq 1$, there exists a polynomial $q_i(t) = t^{n_i} + \beta_{i,n_i-1}t^{n_i-1} + \cdots + \beta_{i,1}t \in J[t]$ such that $q_i[p_i^{-1}(p_i x + 1)] \in Z$. Let

$$h_i(t) = p_i^{n_i} q_i[p_i^{-1}(p_i t + 1)] = \sum_{j=0}^{n_i} \gamma_{i,j} t^j.$$

Note that

$$h_i(t) = (p_i t + 1)^{n_i} + p_i \beta_{i,n_i-1}(p_i t + 1)^{n_i-1} + \cdots + p_i^{n_i-1}\beta_{i,1}(p_i t + 1)$$

and from this equation it is quite easy to see that for $j \geq 2$, $p_i^2 \mid \gamma_{i,j}$. Also,

$$\gamma_{i,1} = n_i p_i + (n_i - 1)\beta_{i,n_i-1} p_i^2 + \cdots + \beta_{i,1} p_i^{n_i}.$$

Hence, $p_i \mid \gamma_{i,1}$, but, since n_i is a unit in E, $p_i^2 \nmid \gamma_{i,1}$. Let

$$\hat{h}_i(t) = h_i(t) - \gamma_{i,0} = \sum_{j=1}^{n_i} \gamma_{i,j} t^j$$

and let $q_i(t) = p_i^{-1}\hat{h}_i(t)$. Then $q_i(t) \in E[t]$ satisfies both (i) and (ii) of lemma 1. Thus for each integer $i \geq 1$, there exists a polynomial

$$q_i(t) = \sum_{j=1}^{n_i} \gamma_{i,j} t^j \in E[t]$$

of *minimal degree* which satisfies both (i) and (ii) of Lemma 1. We will show $\{\deg(q_i)\}_{i=1}^{\infty}$ is bounded. Recall that

$$f(t) = t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_1 t + a \in E[t]$$

and $f(x) = 0$. Suppose $\deg(q_i) = n_i > n = \deg(f)$. Let $h_i(t) = q_i(t) - \beta_{i,n_i} t^{n_i - n} f(t)$. Then $h_i(t) \in E[t]$ and $h_i(t)$ satisfies both (i) and (ii). But $\deg(h_i) < n_i$ contradicting the minimality of $\deg(q_i)$. Thus $\{\deg(q_i)\}_{i=1}^{\infty}$ is bounded. By Lemma 1, we are done.

COROLLARY 1. *If D is not a field, then D is algebraic over Q.*

Proof. Pick $x \in D - Z$. Then for all $c \in Z$, $cx \in D - Z$ and so by Lemma 2, cx is algebraic over Q. For any $c \in Z$, since both x and cx are algebraic over Q, c is algebraic over Q. Thus Z is algebraic over Q. Since D is algebraic over Z and Z is algebraic over Q, D is algebraic over Q.

THEOREM. *D is a field.*

Proof. We may assume, by Corollary 1, that D is algebraic over Q. We will show D satisfies (1) and so, by Herstein's Theorem, is a field. Let $x \in D$. Let $q(t) = \alpha_n t^n + \cdots + \alpha_1 t + \alpha_0 \in J[t]$ such that $\alpha_0 \neq 0$ and $q(x) = 0$. Let $h(t) = -\alpha_n t^{n-1} - \cdots - \alpha_2 t - \alpha_1$. Then $h(t) \in J[t]$ and $\alpha_0 = xh(x)$. Pick

$$f(t) = t^m + \beta_{m-1} t^{m-1} + \cdots + \beta_1 t + \beta_0 \in J[t]$$

such that $f(\alpha_0^{-2} x) \in Z$. Let

$$\hat{f}(t) = \alpha_0^{2m} f(\alpha_0^{-2} t) = t^m + \alpha_0^2 \beta_{m-1} t^{m-1} + \cdots + \alpha_0^{2(m-1)} \beta_1 t.$$

Then $\hat{f}(x) \in Z$ and

$$\begin{aligned} \hat{f}(x) &= x^m + \alpha_0^2 \beta_{m-1} x^{m-1} + \cdots + \alpha_0^{2(m-1)} \beta_1 x \\ &= x^m + \beta_{m-1} x^{m+1} [h(x)]^2 + \cdots + \beta_1 x^{2m-1} [h(x)]^{2(m-1)}. \end{aligned}$$

Thus $\hat{f}(x) = p(x)$ for some co-monic $p(t) \in J[t]$.

ACKNOWLEDGEMENT. We take this opportunity to express our gratitude to Professors Jiang Luh and L. O. Chung for their valuable comments. We are also indebted to the North Carolina State University Mathematics Department which partially supported the research involved in this note.

REFERENCES

1. I. N. Herstein, *The Structure of a Certain Class of Rings*, Amer. J. Math. **75** (1953), 866–871.
2. M. Chacron, *On a Theorem of Herstein*, Canada J. Math. **21** (1969), 1348–1353.

DEPARTMENT OF MATHEMATICS
NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NORTH CAROLINA 27607