MULTIPLICITY OF BOARDMAN STRATA AND DEFORMATIONS OF MAP GERMS

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Abstract. We define algebraically for each map germ $f: K^n, 0 \to K^p, 0$ and for each Boardman symbol $\mathbf{i} = (i_1, \dots, i_k)$ a number $c_{\mathbf{i}}(f)$ which is $\mathcal{A}$-invariant. If f is finitely determined, this number is the generalization of the Milnor number of f when $p = 1$, the number of cusps of f when $n = p = 2$, or the number of cross caps when $n = 2, p = 3$. We study some properties of this number and prove that, in some particular cases, this number can be interpreted geometrically as the number of $\Sigma^{\mathbf{i}}$ points that appear in a generic deformation of f. In the last part, we compute this number in the case that the map germ is a projection and give some applications to catastrophe map germs.

1. Introduction. The Milnor number of an analytic function germ $f: \mathbb{C}^n, 0 \to \mathbb{C}$ which has isolated singularity at zero is defined as

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathscr{E}_n}{J_f},$$

where $\mathscr{E}_n$ is the ring of germs $\mathbb{C}^n, 0 \to \mathbb{C}$ and J_f is the jacobian ideal generated by the partial derivatives $\partial f / \partial x_i$, for $i = 1, \dots, n$. It is well known that this number can be interpreted geometrically as the number of Morse points (or $\Sigma^{n,0}$ points if we use the Thom-Boardman singularities notation) that appear in a stable deformation of f.

Analogously, if we consider a finitely determined map germ $f: \mathbb{C}^2, 0 \to \mathbb{C}^2$, 0, we can define the number

$$c(f) = \dim_{\mathbb{C}} \frac{\mathscr{E}_2}{\langle J, p_x J_y - p_y J_x, q_x J_y - q_y J_x \rangle},$$

where $f = (p, q)$, J is the Jacobian determinant and subscripts indicate partial derivatives. Then, according to [**2**] or [**4**], we have that this number is the number of cusps (i.e., $\Sigma^{1,1,0}$ points) that appear in a stable deformation of f.

Finally, a similar result can be found in [**7**] for a finitely determined map germ $f: \mathbb{C}^2, 0 \to \mathbb{C}^3, 0$. The number $c(f)$ is defined as

$$c(f) = \dim_{\mathbb{C}} \frac{\mathscr{E}_2}{\langle J_1, J_2, J_3 \rangle},$$

where J_1, J_2, J_3 are the three 2-minors of the jacobian matrix of f. In this case, the number $c(f)$ is the number of cross caps (i.e., $\Sigma^{1,0}$ points) that appear in a stable deformation of f.

Here, we generalize the three constructions by defining a number $c_{\mathbf{i}}(f)$, for each Boardman symbol $\mathbf{i} = (i_1, \dots, i_k)$ and for each map germ $f: K^n, 0 \to K^p, 0$ (real analytic if $K = \mathbb{R}$ or analytic if $K = \mathbb{C}$). We prove that this number is $\mathcal{A}$-invariant and study some properties.

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The main part of this paper is dedicated to answer the following question: let $f:\mathbb{C}^n,0\to\mathbb{C}^p,0$ be a finitely determined map germ and $\mathbf{i}$ a Boardman symbol such that $\Sigma^{\mathbf{i}}$ has codimension n in the corresponding jet space $J^k(n,p)$, when is $c_{\mathbf{i}}(f)$ equal to the number of $\Sigma^{\mathbf{i}}$ points that appear in a generic deformation of f? Here, generic means generic in the sense of Thom-Boardman (that is, the jet extension of the map germ is transversal to all the Boardman submanifolds). We prefer the concept of generic deformation instead of stable deformation because a stable deformation does not always exist, unless you are in the "nice dimensions" of Mather [**5**]. Our partial answer to this question is that the result is true in three situations (see Theorem 4.4), namely: 1) $\mathbf{i}$ has length 1, 2) f is a singularity of type $\Sigma^{\mathbf{i}}$, or 3) f has rank $n-1$ and $\mathbf{i}=(1,\dots,1)$. We also show in Example 4.5 that the result is not true in general and give and example of a map germ $f:\mathbb{C}^3,0\to\mathbb{C}^3,0$ so that $c_{1,1,1}(f)$ is not equal to the number of $\Sigma^{1,1,1}$ points that appear in a generic deformation of f.

In the last section, we study this number in the case that the map germ is a projection $\pi:X,x\to K^n$, 0, where (X,x) is the set germ of zeros of a map germ $g:K^N$, $0\to K^p$, 0, which has 0 as a regular value. We compute the number $c_{\mathbf{i}}(\pi)$ in terms of the map germ g and prove that it depends only on the partial derivatives of g with respect to the coordinates which are not projected by π. In particular, this result has some applications to catastrophe map germs

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2. Definition of the invariant. Let $f:K^n,0\to K^p,0$ be a map germ (C^∞ or smooth if $K=\mathbb{R}$ or analytic if $K=\mathbb{C}$). We shall denote by $\mathscr{E}_n$ the ring of germs $K^n,0\to K$. We recall here a construction due to Morin (see [**9**]).

Definition 2.1. Let $f:K^n,0\to K^p$, be a map germ and $I\subset\mathscr{E}_n$ an ideal generated by elements $g_1,\dots,g_r\in I$. Then for each $m=1,\dots,n$ we define the *jacobian extension of rank m of* (f,I) as

$$\Delta_m(f,I)=I+I',$$

where I' is the ideal generated by the minors of order m of the jacobian matrix of $(f_1,\dots,f_p,g_1,\dots,g_r)$.

When $f=0$, we put $\Delta_m(0,I)=\Delta_m(I)$, and thus this construction coincides with the jacobian extension defined by other authors.

Lemma 2.2. *Suppose that $f:K^n,0\to K^p,0$ is a map germ, $I\subset\mathscr{E}_n$ an ideal, and $h:K^n$, $0\to K^n$, 0, $k:K^p$, $0\to K^p$, 0 are diffeomorphism germs. Then we have*

(i) *the ideal $\Delta_m(f,I)$ does not depend on the generators chosen for the ideal I;*
(ii) *$\Delta_m(f\circ h,h^*I)=h^*(\Delta_m(f,I))$, where $h^*:\mathscr{E}_n\to\mathscr{E}_n$ is the induced isomorphism of K-algebras;*
(iii) *$\Delta_m(k\circ f,I)=\Delta_m(f,I)$;*
(iv) *$\Delta_m(f,I)+I_f=\Delta_m(I+I_f)$, where $I_f=\langle f_1,\dots,f_p\rangle$.*

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Proof. We prove the second property, which is perhaps the least obvious. Suppose that the ideal I is generated by $g_1, \ldots, g_r$. Then h^*I is generated by $h^*g_1, \ldots, h^*g_r$. On the other hand, the chain rule gives that

$$\frac{\partial f_i \circ h}{\partial x_j} = \sum_{k=1}^{n} \left(\frac{\partial f_i}{\partial x_k} \circ h\right) \frac{\partial h_k}{\partial x_j} = \sum_{k=1}^{n} h^*\left(\frac{\partial f_i}{\partial x_k}\right) \frac{\partial h_k}{\partial x_j}.$$

This implies that every m-minor d of the jacobian matrix of $(f \circ h, h^*g)$ can be written as a linear combination $d = \sum a_i h^* d_i$, where $a_i \in \mathscr{E}_n$ and d_i are m-minors of the jacobian matrix of (f, g). Therefore $\Delta_m(f \circ h, h^*I) \subset h^*(\Delta_m(f, I))$.

The opposite inclusion follows by applying the same argument to the map germ $f' = f \circ h$, the ideal $J = h^*I$ and the diffeomorphism germ h. □

It will also be useful in the following definitions to take the convention that $\Delta_{n+1}(f, I) = I$ for any map germ f and ideal I.

DEFINITION 2.3. Let $f : K^n, 0 \to K^p, 0$ be a map germ and $\mathbf{i} = (i_1, \ldots, i_k)$ a Boardman symbol (i.e., $n \geq i_1 \geq \ldots \geq i_k \geq 0$). Then we define inductively the *iterated jacobian extension of f* by

$$J_{\mathbf{i}}(f) = \begin{cases} \Delta_{n-i_1+1}(f, \{0\}), & \text{if } k = 1, \\ \Delta_{n-i_k+1}(f, J_{i_1,\ldots,i_{k-1}}(f)), & \text{if } k > 1. \end{cases}$$

Moreover, we define the number $c_{\mathbf{i}}(f)$ by:

$$c_{\mathbf{i}}(f) = \dim_K \frac{\mathscr{E}_n}{J_{\mathbf{i}}(f)}.$$

EXAMPLES. When $p = 1$ and $\mathbf{i} = n$, we have that $J_n(f)$ is the ideal $\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle$. Thus, if f has isolated singularity at zero, $c_n(f)$ is just the Milnor number of f. In the complex case this number can be interpreted as the number of Morse points (i.e., Σ^n points) that appear in a stable deformation of f. In the real case, $c_n(f)$ would just be the maximum of this number.

When $p = n = 2$ and $\mathbf{i} = (1, 1)$, if the map germ is given by $f = (p, q)$, the ideal $J_{1,1}(f)$ is generated by $J, p_x J_y - p_y J_x, q_x J_y - q_y J_x$, where J is the Jacobian determinant and subscripts indicate partial derivatives. According to [**2, 3**] and again in the complex case, if f is finitely determined, $c_{1,1}(f)$ is the number of cusps (i.e., $\Sigma^{1,1}$ points) which appear in a stable deformation of f.

And finally, the same happens for the case $p = 2$, $n = 3$ and $\mathbf{i} = 1$ with the number of cross caps (see [**7**] for more details).

NOTATION. Suppose that we select a fixed set of coordinates $x_{i_1}, \ldots, x_{i_r}$ of K^n. We can construct the jacobian extension $\Delta_m(f, I)$ by looking only at the partial derivatives with respect to these coordinates. We shall denote this by putting $\Delta_m(f, I; x_{i_1}, \ldots, x_{i_r})$. Then we use $J_{\mathbf{i}}(f; x_{i_1}, \ldots, x_{i_r})$ for the corresponding iterated jacobian extension and $c_{\mathbf{i}}(f; x_{i_1}, \ldots, x_{i_r})$ for the number.

PROPOSITION 2.4. *The number $c_{\mathbf{i}}(f)$ is $\mathscr{A}$-invariant.*

Proof. Suppose that f, g are $\mathscr{A}$-equivalent. Then $g = k \circ f \circ h$ for some

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diffeomorphism germs h, k. By properties 2 and 3 of Lemma 2.2, we have that $J_{\mathbf{i}}(g) = h^*J_{\mathbf{i}}(f)$. Thus, $J_{\mathbf{i}}(f)$, $J_{\mathbf{i}}(g)$ are induced isomorphic and $c_{\mathbf{i}}(f) = c_{\mathbf{i}}(g)$. □

REMARK. Although the theory of Boardman symbols was introduced in the context of $\mathcal{K}$-equivalence, the number $c_{\mathbf{i}}(f)$ is not $\mathcal{K}$-invariant. For instance, consider the map germs $f(x, y) = (x, xy + y^3)$, $g(x, y) = (x, y^3)$, which are $\mathcal{K}$-equivalent. However, $c_{1,1}(f) = 1$ and $c_{1,1}(g) = \infty$.

Suppose now that $f: K^n, 0 \to K^p, 0$ is a map germ of rank r. We know that after a coordinate change in the source, f can be written as an unfolding of a map germ K^{n-r}, $0 \to K^{p-r}$, 0. That is, we can put $f(u, x) = (u, g(u, x))$, where u, x denote coordinates in K^r, K^{n-r} respectively, and $g: K^n$, $0 \to K^{p-r}$, 0 is a map germ. The next proposition says that in this case, the number $c_{\mathbf{i}}(f)$ is easier to compute.

PROPOSITION 2.5. *Suppose that* $f: K^n$, $0 \to K^p$, 0 *is a map germ given by* $f(u, x) = (u, g(u, x))$, *for* $u \in K^r, x \in K^{n-r}$ *and let* $\mathbf{i} = (i_1, \dots, i_k)$. *Then*

$$c_{\mathbf{i}}(f) = \begin{cases} 0, & \text{if } i_1 > n - r \\ c_{\mathbf{i}}(g; x), & \text{if } i_1 \le n - r. \end{cases}$$

Proof. The jacobian matrix of f has the form $\begin{pmatrix} I_r & A \\ 0 & B \end{pmatrix}$, where I_r is the identity matrix of order r, $A = (\partial g_i/\partial u_j)$ is the jacobian matrix of g with respect to the coordinates u_j and $B = (\partial g_i/\partial x_j)$ is the jacobian matrix with respect to the coordinates x_j.

In the case that $i_1 > n - r$, we have $n - i_1 + 1 \le r$. This gives that there is a minor of order $n - i_1 + 1$ which is equal to 1. Thus $J_{i_1}(f) = \mathcal{E}_n$ and $c_{\mathbf{i}}(f) = 0$.

Otherwise, $n - i_1 + 1 > r$ and every $(n - i_1 + 1)$-minor coincides with a minor of B of order $\ge n - r - i_1 + 1$. Reciprocally, every $(n - r - i_1 + 1)$-minor of B can be seen as a $(n - i_1 + 1)$-minor of the whole matrix. This proves that $J_{i_1}(f) = J_{i_1}(g; x)$.

Now we proceed by induction and applying a similar argument in each step, we get $J_{\mathbf{i}}(f) = J_{\mathbf{i}}(g; x)$, which concludes the proof. □

COROLLARY 2.6. *Suppose that* $f: K^n$, $0 \to K^p$, 0 *is a map germ of rank* $n - 1$ *given by* $f(u, x) = (u, g(u, x))$, *for* $u \in K^{n-1}$, $x \in K$ *and let* $\mathbf{i} = (i_1, \dots, i_k)$. *Then*

$$c_{\mathbf{i}}(f) = \begin{cases} 0, & \text{if } i_1 > 1 \\ \dim_K \dfrac{\mathcal{E}_n}{\left\langle \dfrac{\partial g_1}{\partial x}, \dots, \dfrac{\partial g_{p-n+1}}{\partial x}, \dots, \dfrac{\partial^s g_1}{\partial x^s}, \dots, \dfrac{\partial^s g_{p-n+1}}{\partial x^s} \right\rangle}, & \text{if } i_1 = 1, \end{cases}$$

being the number s *in the second case equal to the number of indices* i_l *which are not zero.*

3. Relation with the Thom-Boardman singularities. We recall the definition of the Thom-Boardman singularities, taking into account our notation. We say that a map germ $f: K^n$, $0 \to K^p$, 0 is a singularity of type $\Sigma^{\mathbf{i}}$, for a Boardman symbol $\mathbf{i} = (i_1, \dots, i_k)$, when

(i) the rank of f is $n - i_1$;

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(ii) for all $s=2,\dots,k$, the rank of (f,g) is $n-i_s$, being $g=(g_1,\dots,g_r)$ and $g_1,\dots,g_r$ generators of $J_{i_1,\dots,i_{s-1}}(f)$.

We denote by $\Sigma^{\mathbf{i}}(f)$ the set germ of points x, such that the germ of f at x is a singularity of type $\Sigma^{\mathbf{i}}$. Remember also that this set germ $\Sigma^{\mathbf{i}}(f)$ can be written as:

$$\Sigma^{\mathbf{i}}(f)=(j^kf)^{-1}(\Sigma^{\mathbf{i}}),$$

where $\Sigma^{\mathbf{i}}$ is the corresponding Boardman submanifold defined in the jet space $J^k(n,p)$.

To see the relationship between these sets and the ideals $J_{\mathbf{i}}(f)$ defined in the above section, we need to introduce some notation used by Morin in [**9**].

We define the *lexicographic* order, $\leq$, in the set of Boardman symbols, that saying that $\mathbf{i}\leq\mathbf{j}$ if writing $\mathbf{i}=(i_1,\dots,i_k)$ and $\mathbf{j}=(j_1,\dots,j_l)$, we have that either $\mathbf{i}=\mathbf{j}$ or $i_{r_0}<j_{r_0}$, where $r_0=\min\{r:i_r\neq j_r\}$.

The *length*, $|\mathbf{i}|$, of a Boardman symbol $\mathbf{i}=(i_1,\dots,i_k)$ is defined as the last r such that $i_r>0$.

Given a Boardman symbol $\mathbf{i}=(i_1,\dots,i_k)$, we define its *successor* as the symbol $\mathbf{i}'$ which is the following symbol for the lexicographic order among the symbols $\mathbf{j}$ such that $|\mathbf{j}|\leq|\mathbf{i}|$. That is, $\mathbf{i}'=(i_1,\dots,i_r,i_{r+1}+1)$, provided that $i_r>i_{r+1}=\dots=i_k>0$, or $\mathbf{i}'=(i_1+1)$, if $i_1=\dots=i_k>0$. Note that $\mathbf{i}'$ is not defined when $\mathbf{i}$ has the form $\mathbf{i}=(n,\dots,n)$.

If $\mathbf{i}=(i_1,\dots,i_k)$, we denote by $\mu(\mathbf{i})$ the number of Boardman symbols $\mathbf{j}=(j_1,\dots,j_k)$ such that $j_r\leq i_r$, for $r=1,\dots,k$ and $j_1>0$.

Finally, we define $\nu(\mathbf{i},n,p)$ as the number

$$(p-n+i_1)\mu(i_1,\dots,i_k)-(i_1-i_2)\mu(i_2,\dots,i_k)-\dots-(i_{k-1}-i_k)\mu(i_k),$$

provided that $\mathbf{i}=(i_1,\dots,i_k)$. It is shown in [**1**] that $\nu(\mathbf{i},n,p)$ is the codimension of the Boardman manifold $\Sigma^{\mathbf{i}}$ in the jet space $J^k(n,p)$. To simplify the notation, when the dimensions n,p are clear from the context, we shall use $\nu(\mathbf{i})$ instead of $\nu(\mathbf{i},n,p)$ (note that this number depends only on the difference $p-n$).

In next proposition we summarize some results of [**9**] that we are going to use.

PROPOSITION 3.1. *We have the following properties for Boardman symbols* $\mathbf{i},\mathbf{j}$ *and a map germ* $f:K^n,0\to K^p,0$:

(i) *If* $\mathbf{i}\leq\mathbf{j}$, *then* $J_{\mathbf{i}}(f)\subset J_{\mathbf{j}}(f)$;

(ii) $\Sigma^{\mathbf{i}}(f)=V(J_{\mathbf{i}}(f))\backslash V(J_{\mathbf{i}'}(f))$, *where* $V(I)$ *denotes the set germ of zeros in* $(K^n,0)$ *of an ideal* $I\subset\mathscr{E}_n$. (*We are using the convention that* $V(J_{\mathbf{i}'}(f))=\varnothing$ *when* $\mathbf{i}'$ *is not defined.*)

COROLLARY 3.2. *Let* $f;K^n,0\to K^p,0$ *be a map germ. For each Boardman symbol* $\mathbf{i}$ *we have that* $V(J_{\mathbf{i}}(f))=\Sigma^{\mathbf{i}}(f)\cup\Sigma^{\mathbf{i}'}(f)\cup\dots\cup\Sigma^{\mathbf{i}^{(l)}}(f)$, *where* $\mathbf{i}',\dots,\mathbf{i}^{(l)}$ *are the iterated successors of* $\mathbf{i}$. *Moreover*, $c_{\mathbf{i}}(f)\geq1$ *if and only if f is a singularity of type* $\Sigma^{\mathbf{i}}\cup\Sigma^{\mathbf{i}'}\cup\dots\cup\Sigma^{\mathbf{i}^{(l)}}$.

Proof. From the above proposition we deduce that

$$V(J_{\mathbf{i}}(f))=\Sigma^{\mathbf{i}}(f)\cup V(J_{\mathbf{i}'}(f)),$$

and then the required result follows by applying this recursively. The second part is an obvious consequence of the first one, since $c_{\mathbf{i}}(f)\geq1$ if and only if the ideal $J_{\mathbf{i}}(f)$ is proper, that is, $0\in V(J_{\mathbf{i}}(f))$. □

EXAMPLE 3.3. Note that we have $\overline{\Sigma^{\mathbf{i}}(f)}\subseteq V(J_{\mathbf{i}}(f))$, where $\overline{\Sigma^{\mathbf{i}}(f)}$ denotes the closure of

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$\Sigma^{\mathbf{i}}(f)$ in the Zariski topology. However, the equality is not true in general. For instance, consider the map germ $f:\mathbb{C}^5\to\mathbb{C}^5$ given by

$$f(u,v,w,x,y)=(u,v,w,xy,x^2+y^2+ux+vy).$$

It is a singularity of type $\Sigma^{2,0}$ and is $\mathcal{A}$-stable. For $\mathbf{i}=(1,1,1,1,1)$ we have that $\overline{\Sigma^{1,1,1,1,1}(f)}=\Sigma^{1,1,1,1,1}(f)=\varnothing$. But the above corollary gives that

$$V(J_{1,1,1,1,1}(f))=\Sigma^{1,1,1,1,1}(f)\cup\Sigma^2(f)\ldots\Sigma^5(f)\neq\varnothing.$$

On the other hand, the above corollary can be improved in some particular cases.

COROLLARY 3.4. *Let $f:K^n,0\to K^p,0$ be a map germ and* $\mathbf{i}$ *a Boardman symbol.*

(i) *Suppose that f is a singularity of type $\Sigma^{\mathbf{i}}$, then $V(J_{\mathbf{i}}(f))=\Sigma^{\mathbf{i}}(f)$. Moreover, if $c_{\mathbf{i}}(f)=1$ we have that f is a singularity of type $\Sigma^{\mathbf{i},0}$.*

(ii) *Suppose that f has rank $n-1$, then $V(J_{1,\ldots,1}(f))=\Sigma^{1,\ldots,1}(f)$.*

Proof. The fact that the rank is an upper semicontinuous function implies that if f is a singularity of type $\Sigma^{\mathbf{i}}$, then $\Sigma^{\mathbf{i}'}(f)=\ldots=\Sigma^{\mathbf{i}^{(r)}}(f)=\varnothing$, which gives the first part of (i).

For the second one, suppose that $c_{\mathbf{i}}(f)=1$. Then we have that $J_{\mathbf{i}}(f)=\mathcal{M}_n$, being $\mathcal{M}_n$ the maximal ideal of the local ring $\mathcal{E}_n$. This implies that $g=(g_1,\ldots,g_r)$ has rank n, where $J_{\mathbf{i}}(f)$ is generated by $g_1,\ldots,g_r$. Therefore, (f,g) has also rank n and f is a singularity of type $\Sigma^{\mathbf{i},0}$.

Finally, the same argument that the rank is an upper semicontinuous function gives that when f has rank $n-1$, then $V(J_{1,\ldots,1}(f))=\Sigma^{1,\ldots,1}(f)$. □

EXAMPLE. The converse of the second part of 1 in the above corollary is not true, even in the case that f is $\mathcal{A}$-stable. For instance, consider the map germ $f(x,y)=(x,y^2)$, which is of type $\Sigma^{1,0}$; however, $c_1=\infty$.

PROPOSITION 3.5. *Let $f:K^n,0\to K^p,0$ be a map germ of type $\Sigma^{\mathbf{i}}$ which is generic in the sense of Thom-Boardman, with $\nu(\mathbf{i})=n$ (and therefore of type $\Sigma^{\mathbf{i},0}$). Then $c_{\mathbf{i}}(f)=1$.*

Proof. Since f is generic and $\nu(\mathbf{i})=n$, f must be a singularity of type $\Sigma^{\mathbf{i},0}$. Then it follows from the definition of the Boardman symbol that we can select $g_1,\ldots,g_n\in J_{\mathbf{i}}(f)$ with rank n in 0. But this implies that $J_{\mathbf{i}}(f)=\langle g_1,\ldots,g_n\rangle=\mathcal{M}_n$, and hence $c_{\mathbf{i}}(f)=1$. □

4. Geometrical interpretation. In this section we restrict ourselves to the case $K=\mathbb{C}$. We want to determine when the number $c_{\mathbf{i}}(f)$ can be interpreted geometrically as the number of $\Sigma^{\mathbf{i}}$ points that appear in a generic deformation of f. To do this, we first study when the number $c_{\mathbf{i}}(f)$ is finite.

One would expect that when f is finitely determined and the codimension of $\Sigma^{\mathbf{i}}$ is large enough (for instance, $\nu(\mathbf{i})\geq n$), then $c_{\mathbf{i}}(f)<\infty$. However, this is not true. For instance, consider the map germ $f:\mathbb{C}^5\to\mathbb{C}^5$ of Example 3.3. It is a singularity of type $\Sigma^{2,0}$ and is $\mathcal{A}$-stable. On the other hand, the Boardman symbol $\mathbf{i}=(1,1,1,1,1)$ satisfies that $\nu(\mathbf{i})=5$. But a minor computation using Proposition 2.5 gives that $J_{\mathbf{i}}(f)\subset\langle u,v,x,y\rangle$ and thus $c_{\mathbf{i}}(f)=\infty$.

LEMMA 4.1. *Let* $\mathbf{i}$ *be a Boardman symbol such that $\nu(\mathbf{i}),\nu(\mathbf{i}'),\ldots,\nu(\mathbf{i}^{(r)})\geq n$, where $\mathbf{i}',\ldots,\mathbf{i}^{(r)}$ are the iterated successors of* $\mathbf{i}$. *If $f:\mathbb{C}^n,0\to\mathbb{C}^p,0$ is a finitely determined map germ, then $c_{\mathbf{i}}(f)<\infty$.*

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Proof. We have that $c_{\mathbf{i}}(f) = \dim_{\mathbb{C}}(\mathscr{E}_n/J_{\mathbf{i}}(f)) < \infty$ if and only if the Krull dimension of the ring $\mathscr{E}_n/J_{\mathbf{i}}(f)$ is zero. But this dimension coincides with $\dim V(J_{\mathbf{i}}(f))$ and by Corollary 3.2 this set can be written as

$$V(J_{\mathbf{i}}(f)) = \Sigma^{\mathbf{i}}(f) \cup \Sigma^{\mathbf{i}'}(f) \cup \ldots \cup \Sigma^{\mathbf{i}^{(r)}}(f).$$

On the other hand, we can use the Mather-Gaffney finite determinacy criterion, which says that there is a representative $f: U \to \mathbb{C}^p$ so that f is stable on $U \backslash \{0\}$ (see [**12**]). Then $j^k f$ is transversal to all the Boardman submanifolds on $U \backslash \{0\}$ and thus $V(J_{\mathbf{i}}(f)) \cap (U \backslash \{0\})$ is a finite union of submanifolds of codimension $\geq n$. By shrinking the neighbourhood U if necessary, we will have that $V(J_{\mathbf{i}}(f)) \cap (U \backslash \{0\}) = \varnothing$, which means that $V(J_{\mathbf{i}}(f)) \subset \{0\}$ and $\dim V(J_{\mathbf{i}}(f)) = 0$, as required. □

Again this result can be improved in some particular cases. The following lemma can be proved by using the same argument than in Lemma 4.1.

LEMMA 4.2. *Let $f: \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a finitely determined map germ and* $\mathbf{i}$ *a Boardman symbol such that* $\nu(\mathbf{i}) \geq$ n. Then $c_{\mathbf{i}}(f) < \infty$ *provided that either*

(i) *f is a singularity of type $\Sigma^{\mathbf{i}}$; or*
(ii) *f has rank $n-1$ and $\mathbf{i} = (1, \ldots, 1)$.*

Before stating the main theorem of this section, we give the following lemma. It is based on a standard argument and shows that the Cohen-Macaulay property in necessary in order to compute the number of f from the number $c_{\mathbf{i}}(f)$.

LEMMA 4.3. *Let $f: \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a finitely determined map germ and* $\mathbf{i}$ *a Boardman symbol such that $\nu(\mathbf{i}) = n$ and $\nu(\mathbf{i}'), \ldots, \nu(\mathbf{i}^{(r)}) > n$. Let $F(u, x) = (u, f_u(x))$ be a 1-parameter unfolding of f with the property that f_u is generic for $u \neq 0$. Then, the number of $\Sigma^{\mathbf{i}}$ points of f_u, for $u \neq 0$ is equal to $c_{\mathbf{i}}(f)$ if and only if the local ring $\mathscr{E}_{n+1}/J_{\mathbf{i}}(F)$ is Cohen-Macaulay.*

Proof. If $c_{\mathbf{i}}(f) = 0$, then $V(J_{\mathbf{i}}(f)) = \varnothing$ and since $V(J_{\mathbf{i}}(f_u)) = \Sigma^{\mathbf{i}}(f_u)$ for $u \neq 0$, f_u will not have any $\Sigma^{\mathbf{i}}$ point. Therefore, we can suppose that $c_{\mathbf{i}}(f) > 0$ and $V(J_{\mathbf{i}}(f)) = \{0\}$ by the above lemma.

In this case, the set germ $X = V(J_{\mathbf{i}}(F))$ is 1-dimensional and the projection $\pi: X \to \mathbb{C}$ given by $\pi(u, x) = u$ satisfies that $\pi^{-1}(0) = \{0\}$. Moreover, for $u \neq 0$, the cardinal of $\pi^{-1}(u)$ is equal to the number of $\Sigma^{\mathbf{i}}$ points that appear in f_u. But this number is equal, by the formula of Samuel (see for instance [**10**]), to the multiplicity $e(\langle \bar{u} \rangle, R)$, where $R = \mathscr{E}_{n+1}/J_{\mathbf{i}}(F)$ and $\bar{u}$ denotes the class of u in R.

On the other hand, since $\langle \bar{u} \rangle$ is a parameter ideal of R, we apply Theorem 17.11 of [**6**] and get that R is Cohen-Macaulay if and only if $e(\langle \bar{u} \rangle, R) = \dim_{\mathbb{C}} R/\langle \bar{u} \rangle$. Finally, note that

$$\dim_{\mathbb{C}} \frac{R}{\langle \bar{u} \rangle} = \dim_{\mathbb{C}} \frac{\mathscr{E}_{n+1}/J_{\mathbf{i}}(F)}{\langle \bar{u} \rangle} = \dim_{\mathbb{C}} \frac{\mathscr{E}_n}{J_{\mathbf{i}}(f)}.$$

□

THEOREM 4.4. *Let $f: \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a finitely determined map germ and* $\mathbf{i}$ *a Boardman symbol such that $\nu(\mathbf{i}) = n$. Then $c_{\mathbf{i}}(f)$ is the number of $\Sigma^{\mathbf{i}}$ points that appear in a generic deformation of f, provided that either*

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(i) *the length of* $\mathbf{i}$ *is* 1;
(ii) f *is a singularity of type* $\Sigma^{\mathbf{i}}$; *or*
(iii) f *has rank* $n-1$ *and* $\mathbf{i}=(1,\dots,1)$.

Proof. Let $F:\mathbb{C}^{n+1},0\to\mathbb{C}^{p+1},0$ be a 1-parameter unfolding of f, given by $F(u,x)=(u,f_u(x))$, and with the property that f_u is generic for $u\neq 0$. By the above lemma, we have to show that in the three cases, the ring $R=\mathscr{E}_{n+1}/J_{\mathbf{i}}(F)$ is Cohen-Macaulay.

In the first case, $J_{\mathbf{i}}(F)$ is defined by the $(n-i_1+1)$-minors of a matrix of order $n\times p$, being $\mathbf{i}=i_1$. Since $\nu(\mathbf{i})=i_1(p-n-i_1)=n$, we have that $\dim R=1=(n+1)-i_1(p-n-i_1)$, which implies that R is a determinantal ring and therefore is Cohen-Macaulay.

In the second case, F is also a singularity of type $\Sigma^{\mathbf{i}}$ and thus $V(J_{\mathbf{i}}(F))=\Sigma^{\mathbf{i}}(F)$. This means that the local ring R can be obtained as the pull back of the local ring of the Boardman submanifold $\Sigma^{\mathbf{i}}\subset J^k(n,p)$ through the map $j^kF:\mathbb{C}^{n+1},0\to J^k(n,p)$. Now, $\Sigma^{\mathbf{i}}$ is Cohen-Macaulay because it is smooth at every point and since $\operatorname{codim}\Sigma^{\mathbf{i}}=n=\operatorname{codim}R$, R is also Cohen-Macaulay.

In the last case, we have that F has rank n. By Corollary 2.6 we know that after a coordinate change in the source, the ideal $J_{\mathbf{i}}(F)$ is generated by n functions $g_1,\dots,g_n$. But $R=\mathscr{E}_{n+1}/J_{\mathbf{i}}(F)$ has dimension one and thus it is a complete intersection. In particular, it is Cohen-Macaulay (see **[6]** for instance, for the definitions and properties used here). □

Note that the first case of the above theorem includes the Milnor number for $p=1$ and the number of cross caps for $n=2$ and $p=3$. More generally, we have that $c_1(f)$ is the number of Σ^1 points of a finitely determined map germ $f:\mathbb{C}^n,0\to\mathbb{C}^{2n-1},0$, $c_2(f)$ is the number of Σ^2 points of $f:\mathbb{C}^{2n},0\to\mathbb{C}^{3n-2},0$, etc.

On the other hand, if we consider the general case of a finitely determined map germ $f:\mathbb{C}^n,0\to\mathbb{C}^p,0$ and a Boardman symbol $\mathbf{i}$ with $\nu(\mathbf{i})=n$, we can try to apply the above argument to prove that $c_{\mathbf{i}}(f)$ is the number of $\Sigma^{\mathbf{i}}$ points. After Corollary 3.2 and Lemma 4.1 it is obvious that we must add the condition that $\nu(\mathbf{i}'),\dots,\nu(\mathbf{i}^{(l)})>n$ in order to ensure that $c_{\mathbf{i}}(f)$ is finite and that f_u has only $\Sigma^{\mathbf{i}}$ points as isolated singularities. However, even in this case the result is not true in general. In fact, the local ring $R=\mathscr{E}_{n+1}/J_{\mathbf{i}}(F)$ that appears in the above proof is not Cohen-Macaulay in general and this is due to the fact that these rings do not have a reduced structure (it is well known that every one dimensional reduced local ring is Cohen-Macaulay). The following example will illustrate this with more detail.

Example 4.5. When $n=p=3$ the only Boardman symbol that satisfies $\nu(\mathbf{i})=3$ is $\mathbf{i}=(1,1,1)$. Moreover, its iterated successors are $\mathbf{i}'=2$, with $\nu(\mathbf{i}')=4$ and $\mathbf{i}''=3$, with $\nu(\mathbf{i}'')=9$.

Let $f:\mathbb{C}^3,0\to\mathbb{C}^3,0$ be the map germ given by $f(x,y,z)=(x,yz,y^2+z^2+xz)$. We will show that for this map germ the number $c_{1,1,1}(f)$ is 4, but the 1-parameter generic deformation $f_u(x,y,z)=(x,yz,y^2+z^2+xz+uy)$ just has two $\Sigma^{1,1,1}$ points for $u\neq 0$.

The first step is to compute the ideal $J_{1,1,1}(F)$. It is generated by the maximal minors of the matrix

$$\begin{pmatrix} z & 2y+u & -4y-u & 8y+u & 16z+6x \\ y & 2z+x & 4z+x & 8z+x & 16y+6u \end{pmatrix},$$

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In the case $u=0$, it is easy to see that $J_{1,1,1}(f)=\mathcal{M}_3^2$, where $\mathcal{M}_3=\langle x,y,z\rangle$, the maximal ideal of $\mathcal{E}_3$. Therefore $c_{1,1,1}(f)=4$.

On the other hand, we have $V(J_{1,1,1}(F))=V(4z+x,4y+u,x^2-u^2)$, so that for $u\neq 0$, the $\Sigma^{1,1,1}$ points of f_u are

$$P_1=\left(u,-\frac{u}{4},-\frac{u}{4}\right),\qquad P_2=\left(-u,-\frac{u}{4},\frac{u}{4}\right).$$

Finally, we see that f_u is in fact generic for $u\neq 0$, by showing that P_1 and P_2 are $\Sigma^{1,1,1,0}$ points. We must prove that the rank of f_u and the generators of $J_{1,1,1}(f_u)$ is equal to 3 at both points. We consider the minor given by the first and the last columns, which is equal to $-6xy+6uz$. Then the jacobian determinant of $(x,xy,-6xy+6uz)$ gives

$$6xy+6uz,$$

which is equal to $-3u^2$ at P_1 and $3u^2$ at P_2. This shows that the only singularities that appear in f_u are $\Sigma^{1,0},\Sigma^{1,1,0}$ or $\Sigma^{1,1,1,0}$. Then we can use the canonical forms of Morin [**8**] and deduce that f_u is generic at every point.

5. Singularities of projections Let $g:K^N,0\to K^p,0$ be a submersive map germ, so that $g^{-1}(0)$ is a submanifold germ of codimension p of K^N. Suppose that $K^N=K^n\times K^q$ and let $\pi:K^N,0\to K^n,0$ be the projection given by $\pi(x,y)=x$. Our purpose is to determine the number $c_{\mathbf{i}}(\pi|_{g^{-1}(0)})$ in terms of the partial derivatives of g with respect to the coordinates y_j.

THEOREM 5.1. *Let $g:K^N,0\to K^p,0$ be a submersive map germ and let $\pi:K^N,0\to K^n,0$ be the projection as above. Suppose that $\pi|_{g^{-1}(0)}$ has rank r. Then*

$$c_{\mathbf{i}}(\pi|_{g^{-1}(0)})=\dim_K\frac{\mathcal{E}_N}{I_g+J_{\mathbf{j}}(g;y)},$$

where $I_g=\langle g_1,\dots,g_p\rangle$ and

$$\mathbf{j}=\begin{cases}\mathbf{i}, & \text{when } n-p<r;\\ \mathbf{i}-(n-p-r)(1,\dots,1), & \text{when } n-p\geq r.\end{cases}$$

We start by showing that the ideal $I_g+J_{\mathbf{j}}(g;y)$ that appears in the above theorem, does not depend on the map germ g, but only depends on the submanifold $g^{-1}(0)$.

LEMMA 5.2. *Suppose that f and $g:K^N,0\to K^p,0$ are two submersive map germs such that $f^{-1}(0)=g^{-1}(0)$. Then*

$$I_g+J_{\mathbf{j}}(g;y)=I_f+J_{\mathbf{j}}(f;y).$$

Proof. We parameterize the submaifold $g^{-1}(0)=f^{-1}(0)$ by an immersion $\varphi:K^{N-p},0\to K^N,0$, which induces an epimorphism $\varphi^*:\mathcal{E}_N\to\mathcal{E}_{N-p}$. Then, by using the local form of an immersion/submersion, it is not very difficult to show that $I_f=I_g=\ker\varphi^*$.

Now, suppose that $\mathbf{j}=(j_1,\dots,j_k)$. We prove by induction on k the required condition. For $k=1$ we have

$$I_g+J_{j_1}(g;y)=I_g+\Delta_{n-j_1+1}(g,\{0\};y)=\Delta_{n-j_1+1}(I_g;y),$$

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where the last equality comes from property 4 of Lemma 2.2. Since the same can be stated for f, the result is a consequence of $I_f = I_g$.

Finally, a similar argument can be used to prove that if the result is true for $k-1$, then it is also true for k, which concludes the proof of the lemma. □

Suppose now that the map germ $\pi|_{g^{-1}(0)}$ has rank r and let $s = N - p - r$. We must distinguish the two cases: $n - p < r$ or $n - p \geq r$.

1. *Case* $n - p < r$. In order to simplify the notation we rewrite the coordinates of K^N as (z, u, v, w), being $z \in K^r$, $u \in K^{n-r}$, $v \in K^s$ and $w \in K^{q-s}$. With this notation, we can parameterize the submanifold $g^{-1}(0)$ by an immersion $\varphi: K^{N-p}, 0 \to K^N, 0$ of the form $\varphi(z, v) = (z, \psi(z, v), v, \eta(z, v))$, for some map germs $\psi: K^{N-p}, 0 \to K^{n-r}, 0$ and $\eta: K^{N-p}, 0 \to K^{q-s}, 0$.

Then we can apply the above lemma and suppose that g is defined by

$$g_i(z, u, v, w) = \begin{cases} \psi_i(z, v) - u_i, & \text{for } i = 1, \dots, n-r; \\ \eta_{i-(n-r)}(z, v) - w_{i-(n-r)}, & \text{for } i = n-r+1, \dots, p. \end{cases}$$

On the other hand, $\pi|_{g^{-1}(0)}$ is $\mathscr{A}$-equivalent to the map germ $\pi \circ \varphi$ given by

$$\pi \circ \varphi(z, v) = (z, \psi(z, v)).$$

2, *Case* $n - p \geq r$. This case is simpler than the above. Now we have $q < s$ and thus we only need to consider (z, u, v), with $z \in K^r$, $u \in K^{n-r}$ and $v \in K^s$, as coordinates of K^N.

The parameterization of $g^{-1}(0)$ is now given by the immersion $\varphi(z, v) = (z, \psi(z, v), v)$, and thus we can suppose that g is defined by

$$g_i(z, u, v) = \psi_i(z, v) - u_i, \qquad \forall i = 1, \dots, n-r.$$

Finally, the projection $\pi \circ \varphi$ has the same expression than above:

$$\pi \circ \varphi(z, v) = (z, \psi(z, v)).$$

Proof of Theorem 5.1. By Proposition 2.5, we have that

$$c_{\mathbf{i}}(\pi \circ \varphi) = c_{\mathbf{i}}(\psi; v) = \dim_K \frac{\mathscr{E}_{N-p}}{J_{\mathbf{i}}(\psi; n)}.$$

But the immersion φ induces a ring epimorphism $\varphi^*: \mathscr{E}_N \to \mathscr{E}_{N-p}$, whose kernel is given by the ideal I_g. Then we have an isomorphism

$$\varphi^*: \frac{\mathscr{E}_N}{I_g} \to \mathscr{E}_{N-p}.$$

Now, suppose that $\mathbf{i} = (i_1, \dots, i_k)$. We prove by induction on k that $\varphi^*(J_{\mathbf{j}}(g; v, w)) = J_{\mathbf{i}}(\psi; v)$ and thus we have the required result.

In the case $n - p < r$, the jacobian matrix of g with respect to the coordinates v, w has the form $\begin{pmatrix} A & 0 \\ B & -I_{q-s} \end{pmatrix}$, where $A = (\partial \psi_i / \partial v_j)$ is the jacobian matrix of ψ with respect the coordinates v_j, $B = (\partial \eta_i / \partial v_j)$ is the jacobian matrix of η with respect the coordinates v_j and I_{q-s} is the identity matrix of order $q - s$. The ideal generated by the minors of order $s - i_1 + 1$ of A is the same than the ideal generated by the minors of order

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$q - i_1 + 1$ of the whole matrix. Thus the above assertion is clear for $k = 1$, taking $\mathbf{j} = \mathbf{i}$. A similar argument shows that if the assertion is true for $k - 1$, then it is also true for k.

In the other case, $n - p \geq r$, the jacobian matrix of g with respect to the coordinates v is just the top row of the above matrix. Then A and the whole matrix have the same minors of order $s - i_1 + 1$. Therefore, we must adjust the size by taking $j_1 = i_1 - (s - q) = i_1 - (n - p - r)$, so that the assertion is true again. □

We conclude the paper with some applications of Theorem 5.1. For instance, if we consider the particular case $q = p = 1$, we have a submersive function germ $g: K^{n+1}, 0 \to K, 0$. Given any projection $\pi: K^{n+1}, 0 \to K^n, 0$, the restriction $\pi|_{g^{-1}(0)}$ will have rank at least n. Then we know from Corollary 2.6 that the only Boardman symbols $\mathbf{i}$ that give non trivial numbers $c_{\mathbf{i}}(\pi|_{g^{-1}(0)})$, are those of the form $\mathbf{i} = (1, \ldots, 1)$.

Corollary 5.3. *Let $g: K^{n+1}, 0 \to K, 0$ be a submersive function germ and let $\pi: K^{n+1}, 0 \to K^n, 0$ be the projection given by $\pi(x_1, \ldots, x_n, t) = (x_1, \ldots, x_n)$. Then*

$$c_{1, \overset{k \text{ times}}{\ldots}, 1}(\pi|_{g^{-1}(0)}) = \dim_K \frac{\mathscr{E}_{n+1}}{\left\langle g, \dfrac{\partial g}{\partial t}, \ldots, \dfrac{\partial^k g}{\partial t^k} \right\rangle}.$$

Other application of Theorem 5.1 can be observed for catastrophe maps. Suppose that we have a potential function $F: K^r \times K^n \to K$ given by $F(u, x) = F_u(x)$. Then the catastrophe manifold is defined as the set

$$M_F = \left\{ (u, x) \in K^r \times K^n : \frac{\partial F}{\partial x_i}(u, x) = 0, \forall i = 1, \ldots, n \right\} = (\nabla F)^{-1}(0),$$

where $\nabla F(u, x) = \nabla F_u(x)$ denotes the gradient vector of the potential function F_u with respect to the variables x_i. Now, the catastrophe map $\chi_F: M_F \to K^r$ is just the restriction of the projection $\pi: K^r \times K^n \to K^r$ given by $\pi(u, x) = u$ (see [**11**]).

Corollary 5.4. *Let $F: K^r \times K^n \to K$ be a potential function germ such that the gradient vector $\nabla F: K^r \times K^n \to K^n$ is a submersion. Suppose that χ_F has rank l. Then at each point of M_F we have*

$$c_{\mathbf{i}}(\chi_F) = \dim_K \frac{\mathscr{E}_{r+n}}{I_{\nabla F} + J_{\mathbf{j}}(\nabla F; x)},$$

where $I_{\nabla F} = \left\langle \dfrac{\partial F}{\partial x_1}, \ldots, \dfrac{\partial F}{\partial x_n} \right\rangle$ *and*

$$\mathbf{j} = \begin{cases} \mathbf{i}, & \text{when } r - n < l; \\ \mathbf{i} - (r - n - l)(1, \ldots, 1), & \text{when } r - n \geq l. \end{cases}$$

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