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Non-stable K_1-functors of Multiloop Groups

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Abstract. Let k be a field of characteristic 0. Let G be a reductive group over the ring of Laurent polynomials $R = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Assume that G contains a maximal R-torus, and that every semisimple normal subgroup of G contains a two-dimensional split torus $\mathbf{G}_m^2$. We show that the natural map of non-stable K_1-functors, also called Whitehead groups, $K_1^G(R) \to K_1^G\big(k((x_1))\cdots((x_n))\big)$ is injective, and an isomorphism if G is semisimple. As an application, we provide a way to compute the difference between the full automorphism group of a Lie torus (in the sense of Yoshii–Neher) and the subgroup generated by exponential automorphisms.

1 Introduction

Let R be a commutative ring with 1 and let G be a reductive group scheme over R in the sense of [SGA3]. We say that the group scheme G is isotropic, if it contains a proper parabolic subgroup P, or, equivalently, the automorphism group of G contains a split 1-dimensional torus $\mathbf{G}_{\mathrm{m}}$. Under this assumption one can consider the following "large" subgroup of $G(R)$ generated by unipotent elements, $E_P(R) = \langle U_P(R), U_{P^-}(R)\rangle$, where U_P and U_{P^-} are the unipotent radicals of P and any opposite parabolic subgroup P^-. If R is a field of characteristic 0 and G is the automorphism group of a $\mathbb{Z}$-graded finite-dimensional Lie algebra L over R, then $E_P(R)$ can be visualized as the subgroup generated by $\exp(\mathrm{ad}(x))$, where x runs over all elements of non-zero grading in L.

The set of (left) cosets

$$G(R)/E_P(R) = K_1^{G,P}(R)$$

is called the non-stable K_1-functor associated with G and P [HV, W, S78]. When $G(R)/E_P(R)$ is a group, it is also sometimes denoted by $W_P(R,G)$ and called the *Whitehead group* of G [A, ChGP3, G2]. Both names go back to Bass' founding paper [B], where the case $G = \mathrm{GL}_n$ was considered. We prefer the name "non-stable K_1-functor" over "Whitehead group", since it suggests the existence of other non-stable K-functors. Indeed, as a functor on the category of smooth algebras over a field, the

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non-stable K_1-functor coincides with the first of non-stable Karoubi–Villamayor K-functors in the sense of J. F. Jardine [J]; see [W, St13].

It is known that $K_1^{G,P}(R)$ is a group, and it is independent of the choice of P provided R is a semilocal ring or every semisimple normal subgroup of G contains $(\mathbf{G}_m)^2$ locally in Zariski topology on Spec R [SGA3, Su, PSt1] (see also § 2). In this case P is omitted from the notation; *i.e.*, we write K_1^G instead of $K_1^{G,P}$.

The group $K_1^G(k) = G(k)/G(k)^+$, when $R = k$ is a field, has been systematically studied since the 1960s in relation to the Kneser–Tits problem [T1]; see the excellent survey [G2]. In particular, if G is simple and simply connected, this group is known to be torsion, trivial in many cases (*e.g.*, if G is k-rational), and abelian except possibly for some groups of type E_8 [ChM, G2]. The situation for arbitrary commutative rings is much less clear. The next simplest case seems to be the one of polynomial rings over a field k. It is known that if G is "constant", *i.e.*, defined over k, then $K_1^G(k[x]) = K_1^G(k)$ [M], and if, moreover, every semisimple normal subgroup of G contains $(\mathbf{G}_m)^2$, then

$$K_1^G(k[x_1,\dots,x_n]) = K_1^G(k)$$

for any $n \geq 1$ [Su, A, St13]. If G is simply connected, then

$$K_1^G\big(k[x_1^{\pm1},\dots,x_n^{\pm1}]\big) = K_1^G(k)$$

for any $n \geq 1$ [Su, St13].

The non-constant case, where G is defined over the polynomial ring itself, is not as well understood. However, significant progress was recently made by V. Chernousov P. Gille, and A. Pianzola in the case of a Laurent polynomial ring [ChGP2, ChGP3]. Their work is motivated by applications to the theory of infinite-dimensional Lie algebras, namely, to classification and conjugacy problems for extended affine Lie algebras (EALAs), which are higher nullity generalizations of affine Kac–Moody algebras [AABGP]. Any EALA can be reconstructed from its centerless core, which is a Lie torus in the sense of [Y, N], while the Realization theorem [ABFP, Theorem 3.3.1] implies that all Lie tori, except for just one class called quantum tori, are Lie algebras of some isotropic adjoint simple group schemes over $k[x_1^{\pm1},\dots,x_n^{\pm1}]$ (see § 5 for details).

In [ChGP3], V. Chernousov, P. Gille, and A. Pianzola showed that $K_1^G(k[x^{\pm1}]) = 1$ for a simply connected group G defined over $k[x^{\pm1}]$, provided that G contains $(\mathbf{G}_m)^2$, and either G is quasi-split, or k is algebraically closed. In [ChGP2] they obtained a general theorem relating groups of points

$$G\big(k[x_1^{\pm1},\dots,x_n^{\pm1}]\big) \quad \text{and} \quad G\big(k((x_1^{\pm1}))\cdots((x_n^{\pm1}))\big).$$

We state it here in a slightly simplified form.

Theorem 1.1 ([ChGP2, Theorem 14.3]) *Let k be a field of characteristic* 0*, and set $R = k[x_1^{\pm1},\dots,x_n^{\pm1}]$ and $K = k((x_1))\cdots((x_n))$. Let G be a reductive group over R having a maximal R-torus T. Then there exists a subgroup J of $G(K)$ such that*

- *J has no non-trivial quotient groups of finite exponent and*
- *$G(K) = G(R)\cdot J\cdot G(K)^+$, where $G(K)^+$ stands for the normal subgroup of $G(K)$ generated by the K-points of all K-subgroups of G_K isomorphic to $\mathbf{G}_{a,K}$.*

Note that a reductive group G over $R = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ always has a maximal R-torus if $n = 1$ [ChGP1, Propositions 5.9 and 5.10] or if k is algebraically closed and G is the adjoint group associated to a Lie torus [GP2, p. 532]. In general, there are counterexamples [GP3, Remark 6.6].

In the setting of Theorem 1.1, assume that G is semisimple and isotropic. Then the theorem implies that for any minimal parabolic R-subgroup P of G the natural map

$$K_1^{G,P}\big(k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]\big) \longrightarrow K_1^{G,P}\big(k((x_1))\cdots((x_n))\big) \tag{1.1}$$

is surjective; see [ChGP2, Remark 14.4] and Corollary 2.13. This result is essential for the proof of the main theorems in [ChGP2], but, unfortunately, it does not allow the computation of $K_1^G\big(k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]\big)$. V. Chernousov, P. Gille, and A. Pianzola then ask the following natural question [ChGP2, p. 316]: is the map (1.1) also injective, *i.e.*, an isomorphism? We answer this question positively in the following generality.

Theorem 1.2 *Let k be a field of characteristic 0. Let G be a reductive group scheme over $R = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ having a maximal R-torus T, and such that every semisimple normal subgroup of G contains $(\mathbf{G}_{\mathrm{m},R})^2$. Then the natural map*

$$K_1^G\big(k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]\big) \longrightarrow K_1^G\big(k((x_1))\cdots((x_n))\big)$$

is injective. If G is semisimple, this map is an isomorphism.

Note that this theorem implies the above-mentioned results of [ChGP3], since for a quasi-split simply connected group G, one has $K_1^G(K) = 1$ for any field K (see [G2]).

Theorem 1.2 is proved in Subsection 4.3 by combining the results of [PSt1, St13] on the structure of isotropic groups over general commutative rings with a special "diagonal argument" trick inspired by some unpublished work of I. Panin elaborating on [OPa, Prop. 7.1]; see Lemma 4.1. The assumption that G contains $(\mathbf{G}_{\mathrm{m},R})^2$ and not just $\mathbf{G}_{\mathrm{m},R}$ goes back to [Su, PSt1], the reason being that $\mathrm{SL}_2(k[x,y])$ is not equal to its subgroup $E_2(k[x,y])$ generated by upper and lower unitriangular matrices, and our methods fail. The actual statement of Theorem 1.2 for $K_1^G = \mathrm{SL}_2/E_2$ is trivially true if $n = 1$, since $k[x^{\pm 1}]$ is Euclidean, and false if $n \geq 3$ by [BaMo]; the case $n = 2$ is not known at present; see *e.g.*, [Ab].

As an immediate corollary of Theorem 1.2, we obtain the following result on Lie tori. Recall that a Lie torus is a $\Delta \times \Lambda$-graded Lie algebra, where Δ is an irreducible finite root system joined with 0 and $\Lambda \cong \mathbb{Z}^n$, satisfying certain axioms similar to the standard generators and relations axiomatics of complex simple Lie algebras; see Definition 5.2.

Theorem 1.3 *Let k be an algebraically closed field of characteristic 0, Δ be a finite root system of rank ≥ 2, and $\Lambda = \mathbb{Z}^n$, $n \geq 1$. Let $\mathcal{L}$ be a centerless Lie Λ-torus of type Δ over k that is finitely generated over its centroid $R \cong k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $G = \mathrm{Aut}_R(\mathcal{L})^\circ$ be the connected component of the algebraic automorphism group of $\mathcal{L}$ as an R-Lie algebra, and set*

$$E_{exp}(\mathcal{L}) = \big\langle \exp(\mathrm{ad}_x),\ x \in \mathcal{L}_\alpha^\lambda,\ (\alpha,\lambda) \in \Delta \times \Lambda,\ \alpha \neq 0 \big\rangle.$$

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Then there is an isomorphism of groups

$$G(R)/E_{exp}(\mathcal{L}) \cong K_1^G\big(k((x_1))\cdots((x_n))\big).$$

Using the same methods as in Theorem 1.2, we also prove a similar statement on $\mathcal{R}$-equivalence class groups of Yu. Manin. This application was suggested by P. Gille. The proof is given in Subsection 4.2.

Theorem 1.4 *Let k be a field of characteristic 0. Let G be a reductive group scheme over $R = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ having a maximal R-torus T. Then the natural map of $\mathcal{R}$-equivalence class groups*

$$G\big(k(x_1,\dots,x_n)\big)/\mathcal{R} \longrightarrow G\big(k((x_1))\cdots((x_n))\big)/\mathcal{R}$$

is an isomorphism.

2 Preliminaries

2.1 Elementary Subgroups, Non-stable K_1-functors, and $\mathcal{R}$-equivalence

Let A be a commutative ring. Let G be an isotropic reductive group scheme over A, and let P be a parabolic subgroup of G in the sense of [SGA3]. Since the base Spec A is affine, the group P has a Levi subgroup L_P [SGA3, Exp. XXVI Cor. 2.3]. There is a unique parabolic subgroup P^- in G that is opposite to P with respect to L_P, that is $P^- \cap P = L_P$ (*cf.* [SGA3, Exp. XXVI Th. 4.3.2]). We denote by U_P and U_{P^-} the unipotent radicals of P and P^-, respectively.

Definition 2.1 The *elementary subgroup $E_P(A)$ corresponding to P* is the subgroup of $G(A)$ generated as an abstract group by $U_P(A)$ and $U_{P^-}(A)$.

Note that if L'_P is another Levi subgroup of P, then L'_P and L_P are conjugate by an element $u \in U_P(A)$ [SGA3, Exp. XXVI Cor. 1.8], hence $E_P(A)$ does not depend on the choice of a Levi subgroup or of an opposite subgroup P^-, respectively. We suppress the particular choice of L_P or P^- in this context.

Definition 2.2 A parabolic subgroup P in G is called *strictly proper* if it intersects properly every normal semisimple subgroup of G.

The following theorem combines several results of [PSt1] and [SGA3, Exp. XXVI, §5].

Theorem 2.3 ([St13, Theorem 2.1]) *Let G be a reductive group scheme over a commutative ring A, and let R be a commutative A-algebra.*

(i) *Assume that A is a semilocal ring. Then the subgroup $E_P(R)$ of $G(R)$ is the same for any minimal parabolic A-subgroup P of G. If, moreover, G contains a strictly proper parabolic A-subgroup, the subgroup $E_P(R)$ is the same for any strictly proper parabolic A-subgroup P.*

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(ii) *If A is not necessarily semilocal, but for every maximal ideal m in A, every normal semisimple subgroup of G_{A_m} contains $(\mathbf{G}_{\mathrm{m},A_m})^2$, then the subgroup $E_P(R)$ of $G(R) = G_R(R)$ is the same for any strictly proper parabolic R-subgroup P of G_R.*

In both these cases, $E_P(A)$ is normal in $G(A)$.

Definition 2.4 Under the assumptions of Theorem 2.3(i) or (ii), we call $E_P(R)$ the *elementary subgroup* of $G(R)$ and denote it by $E(R)$.

Definition 2.5 The functor $K_1^{G,P}(R) = G(R)/E_P(R)$ on the category of commutative A-algebras R is called the *non-stable K_1-functor*, or the Whitehead group associated with G and P. Under the assumptions of Theorem 2.3(i) or (ii), we write K_1^G instead of $K_1^{G,P}$.

Note that the normality of the elementary subgroup implies that K_1^G is a group-valued functor.

Non-stable K_1 functors are closely related to $\mathcal{R}$-equivalence class groups introduced by Yu. Manin in [Ma].

Definition 2.6 Let X be an algebraic variety over a field k. Denote by $k[t]_{(t),(t-1)}$ the semilocal ring of the affine line $\mathbb{A}_k^1$ over k at the points 0 and 1. Two points $x_0, x_1 \in X(k)$ are called *directly $\mathcal{R}$-equivalent* if there is $x(t) \in X(k[x]_{(x),(x-1)})$ such that $x(0) = x_0$ and $x(1) = x_1$. The *$\mathcal{R}$-equivalence relation* on $X(k)$ is the equivalence relation generated by direct $\mathcal{R}$-equivalence. The *$\mathcal{R}$-equivalence class group* $G(k)/\mathcal{R}$ of an algebraic k-group G is the quotient of $G(k)$ by the $\mathcal{R}$-equivalence class of the neutral element $1 \in G(k)$.

It is easy to see that the $\mathcal{R}$-equivalence class of the neutral element $1 \in G(k)$ is a normal subgroup of $G(k)$, so $G(k)/\mathcal{R}$ is indeed a group. Apart from that, if G has a proper parabolic subgroup P over k, then all elements of $E_P(k)$ are $\mathcal{R}$-equivalent to 1, so $K_1^{G,P}(k)$ surjects onto $G(k)/\mathcal{R}$. If G semisimple and simply connected and P is strictly proper, then $K_1^{G,P}(k) = K_1^G(k) \cong G(k)/\mathcal{R}$ by [G2, Théorème 7.2].

In this paper, we are mainly interested in values of K_1^G on Laurent polynomial rings over a field. We will use the following result.

Theorem ([St13, Corollary 6.2]) *Let G be a simply connected semisimple algebraic group over a field k, such that every semisimple normal subgroup of G contains $(\mathbf{G}_{\mathrm{m},k})^2$. For any $m, n \geq 0$, there are natural isomorphisms*

$$K_1^G(k) \cong K_1^G\left(k[Y_1,\dots,Y_m,X_1,X_1^{-1},\dots,X_n,X_n^{-1}]\right).$$

We will also use the following lemma, which was established in [Su, Corollary 5.7] for $G = \mathrm{GL}_n$, and in [A, Prop. 3.3] for most Chevalley groups; for isotropic groups it was proved in [St13, Lemma 6.1], although the statement was slightly weaker than the present one. The idea goes back to [Q].

Lemma 2.7 *Let A be a commutative ring, and let G be a reductive group scheme over A such that every semisimple normal subgroup of G is isotropic. Assume, moreover,*

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that for any maximal ideal $m \subseteq A$, every semisimple normal subgroup of G_{A_m} contains $(\mathbf{G}_{m,A_m})^2$. Then for any monic polynomial $f \in A[t]$, the natural homomorphism

$$K_1^G(A[t]) \longrightarrow K_1^G(A[t]_f)$$

is injective.

Proof The proof goes exactly as in [Su, Corollary 5.7] using [St13, Theorem 1.1] in place of [Su, Theorem 5.1], and [St13, Lemma 2.3] in place of [Su, Lemma 3.7]. ∎

2.2 Torus Actions on Reductive Groups

Let R be a commutative ring with 1, and let $S = (\mathbf{G}_{m,R})^N = \operatorname{Spec}(R[x_1^{\pm1}, \dots, x_N^{\pm1}])$ be a split N-dimensional torus over R. Recall that the character group $X^*(S) = \operatorname{Hom}_R(S, \mathbf{G}_{m,R})$ of S is canonically isomorphic to $\mathbb{Z}^N$. If S acts R-linearly on an R-module V, this module has a natural $\mathbb{Z}^N$-grading

$$V = \bigoplus_{\lambda \in X^*(S)} V_\lambda,$$

where

$$V_\lambda = \left\{ v \in V \mid s \cdot v = \lambda(s)v \text{ for any } s \in S(R) \right\}.$$

Conversely, any $\mathbb{Z}^N$-graded R-module V can be provided with an S-action by the same rule.

Let G be a reductive group scheme over R in the sense of [SGA3]. Assume that S acts on G by R-group automorphisms. The associated Lie algebra functor $\operatorname{Lie}(G)$ then acquires a $\mathbb{Z}^N$-grading compatible with the Lie algebra structure,

$$\operatorname{Lie}(G) = \bigoplus_{\lambda \in X^*(S)} \operatorname{Lie}(G)_\lambda.$$

We will use the following version of [SGA3, Exp. XXVI Prop. 6.1].

Lemma 2.8 *Let $L = \operatorname{Cent}_G(S)$ be the subscheme of G fixed by S. Let $\Psi \subseteq X^*(S)$ be an R-subsheaf of sets closed under addition of characters.*

(i) *If $0 \in \Psi$, then there exists a unique smooth connected closed subgroup U_Ψ of G containing L and satisfying*

$$\operatorname{Lie}(U_\Psi) = \bigoplus_{\lambda \in \Psi} \operatorname{Lie}(G)_\lambda. \tag{2.1}$$

Moreover, if $\Psi = \{0\}$, then $U_\Psi = L$; if $\Psi = -\Psi$, then U_Ψ is reductive; if $\Psi \cup (-\Psi) = X^*(S)$, then U_Ψ and $U_{-\Psi}$ are two opposite parabolic subgroups of G with the common Levi subgroup $U_{\Psi \cap (-\Psi)}$.

(ii) *If $0 \notin \Psi$, then there exists a unique smooth connected unipotent closed subgroup U_Ψ of G normalized by L and satisfying (2.1).*

Proof The statement immediately follows by faithfully flat descent from the standard facts about the subgroups of split reductive groups proved in [SGA3, Exp. XXII]; see the proof of [SGA3, Exp. XXVI Prop. 6.1]. ∎

Definition 2.9 The sheaf of sets

$$\Phi = \Phi(S, G) = \{\lambda \in X^*(S) \setminus \{0\} \mid \mathrm{Lie}(G)_\lambda \neq 0\}$$

is called the *system of relative roots of G with respect to S*.

Remark 2.10 Choosing a total ordering on the $\mathbb{Q}$-space $\mathbb{Q} \otimes_{\mathbb{Z}} X^*(S) \cong \mathbb{Q}^n$, one defines the subsets of positive and negative relative roots Φ^+ and Φ^-, so that Φ is a disjoint union of Φ^+, Φ^-, and $\{0\}$. By Lemma 2.8 the closed subgroups

$$U_{\Phi^+ \cup \{0\}} = P, \qquad U_{\Phi^- \cup \{0\}} = P^-$$

are two opposite parabolic subgroups of G with the common Levi subgroup $\mathrm{Cent}_G(S)$. Thus, if a reductive group G over R admits a non-trivial action of a split torus, then it has a proper parabolic subgroup. The converse is true Zariski-locally; see Lemma 3.6.

2.3 Loop Reductive Groups and Maximal Tori

Let k be a field of characteristic 0. We fix once and for all an algebraic closure $\overline{k}$ of k and a compatible set of primitive m-th roots of unity $\xi_m \in \overline{k}$, $m \geq 1$.

P. Gille and A. Pianzola [GP3, Ch. 2, 2.3] compute the étale (or algebraic) fundamental group of the k-scheme

$$X = \mathrm{Spec}\, k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

at the natural geometric point $e\colon \mathrm{Spec}\, \overline{k} \to X$ induced by the evaluation $x_1 = x_2 = \cdots = x_n = 1$. Namely, let k_λ, $\lambda \in \Lambda$ be the set of finite Galois extensions of k contained in $\overline{k}$. Let I be the subset of $\Lambda \times \mathbb{Z}_{>0}$ consisting of all pairs (λ, m) such that $\xi_m \in k_\lambda$. The set I is directed by the relation $(\lambda, m) \leq (\mu, k)$ if and only if $k_\lambda \subseteq k_\mu$ and $m|k$. Consider

$$X_{\lambda,m} = \mathrm{Spec}\, k_\lambda[x_1^{\pm \frac{1}{m}}, \ldots, x_n^{\pm \frac{1}{m}}]$$

as a scheme over X via the natural inclusion of rings. Then $X_{\lambda,m} \to X$ is a Galois cover with the Galois group

$$\Gamma_{\lambda,m} = (\mathbb{Z}/m\mathbb{Z})^n \rtimes \mathrm{Gal}(k_\lambda/k),$$

where $\mathrm{Gal}(k_\lambda/k)$ acts on $k_\lambda[x_1^{\pm \frac{1}{m}}, \ldots, x_n^{\pm \frac{1}{m}}]$ via its canonical action on k_λ and each $(\overline{k}_1, \ldots, \overline{k}_n) \in (\mathbb{Z}/m\mathbb{Z})^n$ sends $x_i^{1/m}$ to $\xi_m^{k_i} x_i^{1/m}$, $1 \leq i \leq n$. The semi-direct product structure on $\Gamma_{\lambda,m}$ is induced by the natural action of $\mathrm{Gal}(k_\lambda/k)$ on $\mu_m(k_\lambda) \cong \mathbb{Z}/m\mathbb{Z}$. We have

$$\pi_1(X, e) = \varprojlim_{(\lambda,m)\in I} \Gamma_{\lambda,m} = \widehat{\mathbb{Z}}(1)^n \rtimes \mathrm{Gal}(k), \tag{2.2}$$

where $\widehat{\mathbb{Z}}(1)$ denotes the profinite group $\varprojlim_m \mu_m(\overline{k})$ equipped with the natural action of the absolute Galois group $\mathrm{Gal}(k) = \mathrm{Gal}(\overline{k}/k)$.

For any reductive group scheme G over X, we denote by G_0 the split, or Chevalley–Demazure reductive group in the sense of [SGA3] of the same type as G. The group G is a twisted form of G_0, corresponding to a cocycle class ξ in the étale cohomology set $H^1_{\text{ét}}(X, \mathrm{Aut}(G_0))$.

Definition 2.11 [GP3, Definition 3.4] The group scheme G is called *loop reductive* if the cocycle ξ is in the image of the natural map

$$H^1\big(\pi_1(X,e),\mathrm{Aut}(G_0)(\overline{k})\big) \longrightarrow H^1_{\text{ét}}\big(X,\mathrm{Aut}(G_0)\big).$$

Here $H^1(\pi_1(X,e),\mathrm{Aut}(G_0)(\overline{k}))$ stands for the non-abelian cohomology set in the sense of Serre [Se]. The group $\pi_1(X,e)$ acts continuously on $\mathrm{Aut}(G_0)(\overline{k})$ via the natural homomorphism $\pi_1(X,e)\to\mathrm{Gal}(\overline{k}/k)$.

We will use the following result.

Theorem ([GP3, Corollary 6.3]) *A reductive group scheme over X is loop reductive if and only if G has a maximal torus over X.*

The definition of a maximal torus is as follows.

Definition 2.12 ([SGA3, Exp. XII Déf. 3.1]) Let G be a group scheme of finite type over a scheme S and let T be a S-torus which is an S-subgroup scheme of G. Then T is a *maximal torus of G over S*, if $T_{\overline{k(s)}}$ is a maximal torus of $G_{\overline{k(s)}}$ for all $s\in S$.

2.4 Surjectivity Theorem of Chernousov–Gille–Pianzola

In this section we discuss Theorem 1.1 and its implications.

Proof of Theorem 1.1. In the original statement of [ChGP2, Theorem 14.3], one considers a linear algebraic k-group H whose connected component of identity H° is reductive, and a cocycle $\eta\in H^1(\pi_1(R,e),H(\overline{k}))$. Let $\mathfrak{H}$ be the R-group scheme that is the η-twisted form of H_R. Then there is a minimal parabolic (not necessarily proper) R-subgroup scheme $\mathfrak{P}$ of $\mathfrak{H}^\circ$, a Levi subgroup $\mathfrak{L}$ of $\mathfrak{P}$ that is a loop reductive group scheme, and a normal subgroup J of $\mathfrak{L}(K)$ such that

$$\mathfrak{H}(K)=\big\langle \mathfrak{H}(R),J,\mathfrak{H}(K)^+\big\rangle \tag{2.3}$$

and J is isomorphic to a quotient of a group admitting a composition series whose quotients are pro-solvable groups in k-vector spaces.

Clearly, such a group J has no non-trivial quotients of finite exponent. We also claim that in the above setting,

$$\mathfrak{H}^\circ(K)=\mathfrak{H}^\circ(R)\cdot J\cdot\mathfrak{H}^\circ(K)^+, \tag{2.4}$$

where $J\cdot\mathfrak{H}^\circ(K)^+$ is normal in $\mathfrak{H}^\circ(K)$. Since $\mathfrak{H}^\circ$ is a loop reductive group, by [GP3, Corollary 7.4] the parabolic subgroup $\mathfrak{P}_K$ of $\mathfrak{H}^\circ_K$ is also minimal. Then, since K has characteristic 0, one has $\mathfrak{H}^\circ(K)^+ = E_{\mathfrak{P}}(K)$ by Theorem 2.3(i) and [BT2, Proposition 6.2]. By [BT2, Proposition 6.11] one has

$$\mathfrak{H}^\circ(K)=\mathfrak{L}(K)E_{\mathfrak{P}}(K).$$

This implies that $J\cdot\mathfrak{H}^\circ(K)^+$ is normal in $\mathfrak{H}^\circ(K)$. It remains to note that $\mathfrak{H}(K)^+ = \mathfrak{H}^\circ(K)^+$, since $\mathbf{G}_{\mathbf{a},K}$ is connected. Now (2.3) and the equality $\mathfrak{H}^\circ(R)=\mathfrak{H}(R)\cap\mathfrak{H}^\circ(K)$ imply (2.4).

We proceed to show how the above facts imply the claim of our theorem.

Case 1*: G is a torus.* The proof of the theorem of Chernousov, Gille, and Pianzola for the case where $\mathfrak{H} = \mathfrak{H}^\circ = G$ is an R-torus does not use the assumption that $\mathfrak{H}$ is given by a cocycle with values in $H(\overline{k})$ [ChGP2, Proof of Theorem 14.3, Case 1, p. 314]. Therefore, (2.4) implies that our theorem holds for G.

Case 2*: G is adjoint.* Assume that G is a loop semisimple group of adjoint type over R. Then $G = \mathrm{Aut}(G)^\circ$, where $\mathrm{Aut}(G)$ is the R-group scheme of automorphisms of G. Since G is loop reductive, the group $\mathrm{Aut}(G) = \mathfrak{H}$ satisfies the conditions of [ChGP2, Theorem 14.3]. Then (2.4) shows that the claim of our theorem holds for G. Note that in this case $J \cdot G(K)^+$ is normal in $G(K)$.

Case 3*: G is semisimple.* Now assume that G is an arbitrary loop semisimple group scheme over R. Then there is a short exact sequence of R-group schemes

$$1 \longrightarrow \mathrm{Cent}(G) \longrightarrow G \xrightarrow{p} G^{\mathrm{ad}} \longrightarrow 1, \tag{2.5}$$

where G^{ad} is an adjoint semisimple group and $\mathrm{Cent}(G)$ is a finite group scheme of multiplicative type. Since G^{ad} has a maximal R-torus if and only if G does, G^{ad} is a loop semisimple group. By the previous case

$$G^{\mathrm{ad}}(K) = G^{\mathrm{ad}}(R) \cdot J \cdot G^{\mathrm{ad}}(K)^+,$$

where J has no non-trivial quotients of finite exponent. By [BT2, Corollaire 6.3] we have $p(G(K)^+) = G^{\mathrm{ad}}(K)^+$. Since $H^1_{\text{ét}}(K, \mathrm{Cent}(G))$ is a group of finite exponent, considering the "long" exact sequence of étale cohomology associated with (2.5), we conclude that $J \subseteq p(G(K))$. Set $I = p^{-1}(J) \subseteq G(K)$. Then clearly,

$$G(K) = p^{-1}(G^{\mathrm{ad}}(R)) \cdot I \cdot G(K)^+.$$

Since $H^i_{\text{ét}}(R, \mathrm{Cent}(G)) = H^i_{\text{ét}}(K, \mathrm{Cent}(G))$ for all $i \geq 0$ by [GP2, Prop. 3.4(2)], the "long" exact sequence also implies that

$$p^{-1}(G^{\mathrm{ad}}(R)) = \mathrm{Cent}(G)(K) \cdot G(R) = \mathrm{Cent}(G)(R) \cdot G(R) = G(R).$$

Assume that I has a proper normal subgroup I' such that I/I' has finite exponent. Since J has no non-trivial quotients of finite exponent, we have $I'/\mathrm{Cent}(G)(K) \cap I' = J$, and hence $I = \mathrm{Cent}(G)(K) \cdot I'$. Since $\mathrm{Cent}(G)(K)$ is finite, we can find a minimal subgroup $I' \leq I$ such that I' is normal in I and I/I' has finite exponent. One readily sees that such I' has no non-trivial quotients of finite exponent. Since $\mathrm{Cent}(G)(K) = \mathrm{Cent}(G)(R)$, we have

$$G(K) = p^{-1}(G^{\mathrm{ad}}(R)) \cdot I \cdot G(K)^+ = G(R) \cdot \mathrm{Cent}(G)(K) \cdot I' \cdot G(K)^+ = G(R) \cdot I' \cdot G(K)^+,$$

which proves the claim of the theorem for G.

Case 4*: G is reductive.* Let G be an arbitrary loop reductive group scheme over R. Let $\mathrm{der}(G)$ be the derived subgroup scheme of G and let $\mathrm{rad}(G)$ be the radical torus of G in the sense of [SGA3]. By [SGA3, Exp. XXII, Prop. 6.2.4] there is a short exact sequence of R-group schemes

$$1 \longrightarrow C \longrightarrow \mathrm{rad}(G) \times \mathrm{der}(G) \xrightarrow{f} G \longrightarrow 1,$$

where C is a finite group scheme of multiplicative type that is central in $\mathrm{rad}(G) \times \mathrm{der}(G)$. Arguing exactly as in [ChGP2, Proof of Theorem 14.3, Case 2, pp. 314–315] (except that the reference to Theorem 11.1 *ibid.* should be replaced by Theorem 14.1 *ibid.*), one concludes that

$$G(K) = G(R) \cdot f\big(\operatorname{der}(G)(K) \times \operatorname{rad}(G)(K)\big).$$

Note that $\operatorname{der}(G)$ is a loop semisimple group scheme, since it has a maximal R-torus once G does. Then $\operatorname{rad}(G)$ and $\operatorname{der}(G)$ are subject to the previous cases of the theorem, and one readily deduces the claim for G. ∎

Corollary 2.13 ([ChGP2, Remark 14.4]) *Let k, R, K, G be as in Theorem 1.1. Assume in addition that G is semisimple. Then for any minimal parabolic R-subgroup P of G, the map*

$$K_1^{G,P}\big(k[x_1^{\pm 1},\dots,x_n^{\pm 1}]\big) \longrightarrow K_1^{G,P}\big(k((x_1))\cdots((x_n))\big)$$

is surjective.

Proof Since G is a loop reductive group, by [GP3, Corollary 7.4] , any minimal parabolic subgroup P of G remains a minimal parabolic subgroup in G_K. Then since K has characteristic 0, one has $G(K)^+ = E_P(K)$ by [BT2, Proposition 6.2]. It was observed in [ChGP2, Remark 14.4] that if G is simply connected, then the surjectivity of the map in question follows from Theorem 1.1, since the group $K_1^{G,P}(K)$ has finite exponent by [G2, Remarque 7.6]. We claim that $K_1^{G,P}(K)$ has finite exponent whenever G is semisimple. Indeed, there is a short exact sequence

$$1 \longrightarrow C \longrightarrow G^{\mathrm{sc}} \longrightarrow G \longrightarrow 1, \tag{2.6}$$

where C is a finite group scheme of multiplicative type, contained in the center of $(G)^{\mathrm{sc}}$. Let $P^{\mathrm{sc}} \subseteq (G)^{\mathrm{sc}}$ be the parabolic subgroup that is the preimage of P. The "long" exact sequence of étale cohomology corresponding to (2.6) readily shows that $K_1^{G,P}(K)$ has finite exponent once $K_1^{G^{\mathrm{sc}},P^{\mathrm{sc}}}(K)$ does, since $H^1_{\text{ét}}(K,C)$ is an abelian torsion group. ∎

We also obtain the following immediate corollary on $\mathcal{R}$-equivalence class groups. Note that the group G is not required to have a maximal torus over $k[x_1^{\pm},\dots,x_n^{\pm 1}]$.

Corollary 2.14 *Let k be a field of characteristic* 0 *and let G be a reductive group over $k[x_1^{\pm 1},\dots,x_n^{\pm 1}]$. Set $F = k(x_1,\dots,x_n)$.*

(i) *The natural map of $\mathcal{R}$-equivalence class groups*

$$G\big(k(x_1,\dots,x_n)\big)/\mathcal{R} \longrightarrow G\big(k((x_1))\cdots((x_n))\big)/\mathcal{R}$$

is surjective.

(ii) *If G is semisimple, then for every strictly proper parabolic subgroup P of G_F the natural map*

$$K_1^{G_F,P}\big(k(x_1,\dots,x_n)\big) \longrightarrow K_1^{G_F,P}\big(k((x_1))\cdots((x_n))\big)$$

is surjective.

Proof The proof is by induction on n starting with $n = 1$. Set $A = k[x^{\pm 1}]$, $F = k(x)$, and $K = k((x))$. By [ChGP1, Propositions 5.9 and 5.10] every semisimple group scheme G over A is loop reductive, *i.e.*, contains a maximal A-torus.

By the definition of $\mathcal{R}$-equivalence the subgroup $G(K)^+$ is contained in the $\mathcal{R}$-equivalence class of the neutral element. By [V, §17.1, Corollary 2] the group $G\big(K\big)/\mathcal{R}$ has finite exponent. Therefore, by Theorem 1.1 the natural map $G(A) \to G(K)/\mathcal{R}$ is surjective. Since this map factors through the map $G(F)/\mathcal{R} \to G(K)/\mathcal{R}$, the latter map is surjective.

Now consider the non-stable K_1-functors. By [GP3, Corollary 7.4] minimal parabolic subgroups of G, G_F and G_K are of the same type. Then, since G_F contains a strictly proper parabolic F-subgroup P, we conclude that any minimal parabolic A-subgroup Q of G is strictly proper. Moreover, Q_F and Q_K are minimal parabolic subgroups of G_F and G_K, respectively. By Theorem 2.3 we have $K_1^{G,Q}(F) = K_1^{G_F,P}(F)$ and

$$K_1^{G,Q}(K) = K_1^{G_K,P_K}(K) = K_1^{G_F,P}(K).$$

By Corollary 2.13 the natural map $K_1^{G,Q}(A) \to K_1^{G,Q}(K)$ is surjective. Since this map factors through the map $K_1^{G,Q}(F) \to K_1^{G,Q}(K)$, the latter map is also surjective. Therefore, the map

$$K_1^{G_F,P}(F) \longrightarrow K_1^{G_F,P}(K)$$

is surjective.

Assume that $n > 1$. Let $\mathcal{F}(-)$ denote any of the functors $K_1^{G_F,P}(-)$ and $G(-)/\mathcal{R}$ on the category of field extensions of $k(x_1,\dots,x_n)$. By the case $n = 1$ the map

$$\mathcal{F}\big(k((x_1))\cdots k((x_{n-1}))(x_n)\big) \longrightarrow \mathcal{F}\big(k((x_1))\cdots((x_n))\big)$$

is surjective. Since this map factors through the map

$$\mathcal{F}\big(k(x_n)((x_1))\cdots((x_{n-1}))\big) \longrightarrow \mathcal{F}\big(k((x_1))\cdots((x_n))\big),$$

the latter map is also surjective. By the induction hypothesis the map

$$\mathcal{F}\big(k(x_1,\dots,x_n)\big) \longrightarrow \mathcal{F}\big(k(x_n)((x_1))\cdots((x_{n-1}))\big)$$

is surjective, which completes the proof. ■

3 K_1^G of Laurent Polynomials and Power Series Over General Rings

3.1 Results Over General Rings

In this section we discuss various relations between $K_1^G(R[[t]])$, $K_1^G(R[t,t^{-1}])$, and $K_1^G(R((t)))$, where R is an arbitrary commutative ring and G is a reductive algebraic group defined over R. Our keystone result is the following theorem.

Theorem 3.1 *Let R be a commutative ring and let G be a reductive group scheme over R, such that every semisimple normal subgroup of G contains $(\mathbf{G}_{m,R})^2$. Then*

$$E\big(R((t))\big) = E\big(R[[t]]\big)E\big(R[t,t^{-1}]\big).$$

The proof of this theorem uses the notions of relative roots and relative root subschemes of reductive groups introduced by V. Petrov and the author in [PSt1]. Their definitions and a sketch of construction are given in Subsection 3.2, after which we give a proof of Theorem 3.1. As for now, we discuss several easy corollaries of this

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theorem. We begin with a reformulation of Theorem 3.1 in terms of non-stable K_1-functors.

Corollary 3.2 *Let R, G be as in Theorem 3.1. Then the sequence of pointed sets*

$$1 \longrightarrow K_1^G(R[t]) \xrightarrow{g\mapsto(g,g)} K_1^G(R[[t]]) \times K_1^G(R[t,t^{-1}]) \xrightarrow{(g_1,g_2)\mapsto g_1 g_2{}^{-1}} K_1^G\big(R((t))\big)$$

is exact.

Proof Follows immediately from Theorem 3.1 and Lemma 2.7. ∎

Corollary 3.3 *Let R, G be as in Theorem 3.1. Then the natural homomorphism*

$$K_1^G(R[[t]]) \longrightarrow K_1^G(R((t)))$$

is injective.

Proof Assume that the class of $g \in G(R[[t]])$ trivializes in $K_1^G\big(R((t))\big)$, that is, $g \in G(R[[t]]) \cap E\big(R((t))\big)$. Then by Theorem 3.1 we can assume that $g \in G(R[[t]]) \cap E(R[t,t^{-1}])$. Since

$$G(R[t,t^{-1}]) \cap G(R[[t]]) = G(R[t]), \tag{3.1}$$

we have $g \in G(R[t]) \cap E(R[t,t^{-1}])$. By Lemma 2.7, this implies that $g \in E(R[t])$. Hence, $g \in E(R[[t]])$. ∎

The following corollary is what we will use in the proof of Theorem 1.2.

Corollary 3.4 *Let R, G be as in Theorem 3.1. If $G(R[t]) = G(R)E(R[t])$, then the natural homomorphism*

$$K_1^G(R[t,t^{-1}]) \longrightarrow K_1^G\big(R((t))\big)$$

is injective.

Proof Assume that the class of $g \in G(R[t,t^{-1}])$ trivializes in $K_1^G\big(R((t))\big)$, that is, $g \in G(R[t,t^{-1}]) \cap E\big(R((t))\big)$. Then by Theorem 3.1 we can assume that $g \in G(R[t,t^{-1}]) \cap E\big(R[[t]]\big)$. By (3.1) we have $g \in G(R[t]) \cap E(R[[t]])$. Since $G(R[t]) = G(R)E(R[t])$, we can write $g = g_0 g_1$ with $g_0 \in G(R)$, $g_1 \in E(R[t])$. Since $g \in E(R[[t]])$, setting $t = 0$ we deduce $g_0 \in E(R)$. Hence, $g \in E(R[t]) \subseteq E(R[t,t^{-1}])$. ∎

Remark 3.5 The main result of [St13] shows that the equality

$$G\big(R[t]\big) = G(R)E\big(R[t]\big)$$

holds if R is a regular ring containing a perfect field k, and G is defined over k. Using other results of [St13], Lemma 2.7, and the techniques of [PaStV], we can prove the same equality whenever R is a regular ring containing an infinite field k, and G is defined over R. The latter result is still unpublished, so we decided not to use it in this paper. Instead, we give an independent and much simpler proof in the case where R is a ring of Laurent polynomials, and G satisfies the same assumptions as in Theorem 1.2; see Lemma 4.5.

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3.2 Relative Roots and Relative Root Subschemes

In order to prove Theorem 3.1, we need to use the notions of relative roots and relative root subschemes. These notions were initially introduced and studied in [PSt1] and further developed in [St09].

Let R be a commutative ring. Let G be a reductive group scheme over R. Let P be a parabolic subgroup scheme of G over R, and let L be a Levi subgroup of P. By [SGA3, Exp. XXII, Prop. 2.8] the root system Φ of $G_{\overline{k(s)}}$, $s \in \operatorname{Spec} R$, is constant locally in the Zariski topology on $\operatorname{Spec} R$. The type of the root system of $L_{\overline{k(s)}}$ is determined by a Dynkin subdiagram of the Dynkin diagram of Φ, which is also constant Zariski-locally on $\operatorname{Spec} R$ by [SGA3, Exp. XXVI, Lemme 1.14 and Prop. 1.15]. In particular, if $\operatorname{Spec} R$ is connected, all these data are constant on $\operatorname{Spec} R$.

Lemma 3.6 *Let G be a reductive group over a connected commutative ring R, let P be a parabolic subgroup of G, let L be a Levi subgroup of P, and let $\overline{L}$ be the image of L under the natural homomorphism $G \to G^{\mathrm{ad}} \subseteq \operatorname{Aut}(G)$. Let D be the Dynkin diagram of the root system Φ of $G_{\overline{k(s)}}$ for any $s \in \operatorname{Spec} A$. We identify D with a set of simple roots of Φ. Let $J \subseteq D$ be the set of simple roots such that $D \setminus J \subseteq D$ is the subdiagram corresponding to $L_{\overline{k(s)}}$. Then there are a unique maximal split subtorus $S \subseteq \operatorname{Cent}(\overline{L})$ and a subgroup $\Gamma \leq \operatorname{Aut}(D)$ such that J is invariant under Γ, and for any $s \in \operatorname{Spec} R$ and any split maximal torus $T \subseteq \overline{L}_{\overline{k(s)}}$ the kernel of the natural surjection*

$$X^*(T) \cong \mathbb{Z}\Phi \xrightarrow{\pi} X^*(S_{\overline{k(s)}}) \cong \mathbb{Z}\Phi(S,G) \tag{3.2}$$

is generated by all roots $\alpha \in D \setminus J$, and by all differences $\alpha - \sigma(\alpha)$, $\alpha \in J$, $\sigma \in \Gamma$.

Proof We can assume that $G = G^{\mathrm{ad}}$ from the start, and $L = \overline{L}$. The radical $\operatorname{rad}(L) = \operatorname{Cent}(L)^\circ$ of L is a torus. Since $\operatorname{Spec} R$ is connected, it contains a unique maximal split subtorus $S \subseteq \operatorname{Cent}(L)$ by [SGA3, Exp. XXVI, 6.5]. In order to show that the kernel of the map (3.2) is as required, we use the notion of the Dynkin scheme of G.

By construction [SGA3, Exp. XXIV, §3.7], the Dynkin scheme $\operatorname{Dyn}(G)$ over R is an étale twisted form of the constant Dynkin scheme D_R over R. It is thus a finite étale scheme over R endowed with a subscheme $E \subseteq \operatorname{Dyn}(G) \times_R \operatorname{Dyn}(G)$ not intersecting the diagonal (the scheme of edges of the Dynkin diagram) and a morphism $\operatorname{Dyn}(G) \to \{1,2,3\}_R$ (the lengths of simple roots). Clearly, there is a finite Galois ring extension R'/R such that $\operatorname{Dyn}(G)_{R'} \cong D_{R'}$ is split. Since $\operatorname{Spec} R$ is connected, the scheme $\operatorname{Dyn}(G)$ is uniquely determined by D together with a subgroup Γ of $\operatorname{Aut}(D)$ that represents the action of the Galois group $\operatorname{Gal}(R'/R)$ on D. The orbits of Γ in D are in one-to-one correspondence with minimal clopen R-subschemes of $\operatorname{Dyn}(G)$. The parabolic subgroup P of G is defined over R, hence by [SGA3, Exp. XXVI, §3] $\operatorname{Dyn}(G)$ contains a clopen R-subscheme $t(P)$, called the type of P, which is a twisted form of $J_R \subseteq D_R$. In particular, J is a Γ-invariant subset of D. The subscheme $\operatorname{Dyn}(G) \setminus t(P)$ is the twisted form of $(D \setminus J)_R$ isomorphic to the Dynkin scheme $\operatorname{Dyn}(L)$.

Recall that there exists a quasi-split reductive group G_{qs} over R of the same type as G (in particular, adjoint) such that G is an inner twisted form of G_{qs}, that is, G is given by a cocycle class in $H^1_{\text{ét}}(R, G_{\mathrm{qs}})$ [SGA3, Exp. XXIV, 3.12]. One also has $\operatorname{Dyn}(G) \cong \operatorname{Dyn}(G_{qs})$ [SGA3, Théorème 3.11]. Let P_{qs} be a parabolic subgroup in G_{qs} of the same

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type as P that is standard, *i.e.*, contains a Killing couple $T_{qs} \subseteq B_{qs}$, and let L_{qs} be the standard Levi subgroup of P_{qs} containing T_{qs}.

First, we study the torus S in the case where $G = G_{qs}$, $P = P_{qs}$, and $L = L_{qs}$. There is an explicit presentation of T_{qs} as a product of Weil restrictions of $\mathbf{G}_m$ [SGA3, Exp. XXIV Prop. 3.13]:

$$T_{qs} \cong R_{\mathrm{Dyn}(G_{qs})/R}\big(\mathbf{G}_{m,\mathrm{Dyn}(G_{qs})}\big) \cong \prod_{O} R_{O/R}(\mathbf{G}_{m,O}),$$

where O runs over all minimal clopen subschemes of $\mathrm{Dyn}(G)$. This presentation is obtained by descent from the standard decomposition of a split maximal torus into a direct product of 1-dimensional tori corresponding to the vertices of D. By [PSt2, Prop. 1(2)], we have

$$\mathrm{Cent}(L_{qs}) \cong \prod_{O \notin t(P)} R_{O/R}(\mathbf{G}_{m,O}),$$

where O runs over all minimal clopen subschemes of $\mathrm{Dyn}(G)$ not contained in $t(P)$. Then, clearly,

$$S = \prod_{O \notin t(P)} \mathbf{G}_{m,R} \subseteq \mathrm{Cent}(L_{qs}), \tag{3.3}$$

where each $\mathbf{G}_{m,R}$ is the canonical split subtorus of $R_{O/R}(\mathbf{G}_{m,O})$. For any π as in the statement of the lemma, by (3.3) all roots $\alpha \in D \setminus J$, and all differences $\alpha - \sigma(\alpha)$, $\alpha \in J$, $\sigma \in \Gamma$, belong to $\ker \pi$. Since the rank of $X^*(S)$ is equal to the number of orbits of Γ in $D \setminus J$, these elements generate $\ker \pi$.

Now we consider S in the general case where $G \neq G_{qs}$. Let $\eta \in Z^1_{\text{ét}}(R, G_{qs})$ be a cocycle corresponding to the twisted form G of G_{qs}. Let $\bigsqcup \mathrm{Spec}\, R_\tau \to \mathrm{Spec}\, R$ be an étale covering (which we can and do assume to be affine for simplicity) such that $G_{R_\tau} \cong (G_{qs})_{R_\tau}$ for each τ, and let $g_{\sigma\tau} \in G_{qs}(R_\tau \otimes_R R_\sigma)$ be the elements representing η on this covering. For each τ, the pair (L_{R_τ}, P_{R_τ}), considered inside $(G_{qs})_{R_\tau}$, is conjugate to the pair $((L_{qs})_{R_\tau}, (P_{qs})_{R_\tau})$ locally in the étale topology on $\mathrm{Spec}\, R_\tau$ by [SGA3, Exp.XXVI, 4.5.2]. Refining our étale covering, we can assume that these pairs are conjugate already over R_τ, *i.e.*, $L_{R_\tau} = f_\tau (L_{qs})_{R_\tau} f_\tau^{-1}$ and $P_{R_\tau} = f_\tau (P_{qs})_{R_\tau} f_\tau^{-1}$ for an element $f_\tau \in G_{qs}(R_\tau)$. Note that $g_{\sigma\tau}$ preserves the pair $(L_{R_\tau\otimes_R R_\sigma}, P_{R_\tau\otimes_R R_\sigma})$, since L and P are defined over R. Since the normalizer of $((L_{qs})_{R_\tau\otimes_R R_\sigma}, (P_{qs})_{R_\tau\otimes_R R_\sigma})$ in $(G_{qs})_{R_\tau\otimes_R R_\sigma}$ by [SGA3, Exp. XXVI Prop. 1.2, Prop. 1.6] equals $(L_{qs})_{R_\tau\otimes_R R_\sigma}$, we conclude that

$$f_\sigma^{-1} g_{\sigma\tau} f_\tau \in L_{qs}(R_\tau \otimes_R R_\sigma). \tag{3.4}$$

Let S_{qs} be the maximal split R-subtorus of $\mathrm{Cent}(L_{qs})$. Since S_{qs} is central in L_{qs}, by (3.4) we have that

$$g_{\sigma\tau} f_\tau|_{(S_{qs})_{R_\tau\otimes_R R_\sigma}} = f_\sigma|_{(S_{qs})_{R_\tau\otimes_R R_\sigma}}$$

as $R_\tau \otimes_R R_\sigma$-group scheme morphisms from $(S_{qs})_{R_\tau\otimes_R R_\sigma}$ to $\mathrm{Cent}(L)_{R_\tau\otimes_R R_\sigma}$ induced by conjugation. By faithfully flat descent for affine morphisms [FGIKNV, Part 1, Theorem 4.33], there is a closed embedding of R-group schemes $i\colon S_{qs} \to \mathrm{Cent}(L)$ such that $i_{R_\tau} = f_\tau|_{(S_{qs})_{R_\tau}}$ for each τ. Clearly, $i(S_{qs})$ is contained in the maximal split subtorus S of $\mathrm{Cent}(L)$.

Note that we can interchange the roles of the groups (G_{qs}, P_{qs}, L_{qs}) and (G, P, L) in the argument of the previous paragraph. Indeed, since G is given by a cocycle with values in G_{qs}, conversely, G_{qs} is given by a cocycle with values in G; *cf.* [Se, Ch. I, Proposition 35]. Then we conclude that there is a closed embedding of R-group schemes $j: S \to \mathrm{Cent}(L_{qs})$ as well. Therefore, $i(S_{qs}) = S$, since these tori have the same rank. It remains to note that, since $\mathrm{Dyn}(G_{qs}) = \mathrm{Dyn}(G)$ and for any $s \in \operatorname{Spec} R$ the homomorphism $R \to \overline{k(s)}$ factors through one of the rings R_τ, the torus $S \subseteq \mathrm{Cent}(L)$ satisfies the claim of the lemma on $\ker \pi$, since S_{qs} does. ∎

In [PSt1], we introduced a system of relative roots Φ_P with respect to a parabolic subgroup P of a reductive group G over a commutative ring R. This system Φ_P was defined independently over each member $\operatorname{Spec} A = \operatorname{Spec} A_i$ of a suitable finite disjoint Zariski covering

$$\operatorname{Spec} R = \coprod_{i=1}^{m} \operatorname{Spec} A_i,$$

such that over each $A = A_i$, $1 \le i \le m$, the root system Φ and the Dynkin diagram D of G is constant. Namely, we considered the formal projection

$$\pi_{J,\Gamma}: \mathbb{Z}\Phi \longrightarrow \mathbb{Z}\Phi / \langle D \smallsetminus J;\ \alpha - \sigma(\alpha),\ \alpha \in J,\ \sigma \in \Gamma \rangle$$

and set $\Phi_P = \Phi_{J,\Gamma} = \pi_{J,\Gamma}(\Phi) \smallsetminus \{0\}$. The last claim of Lemma 3.6 allows us to identify $\Phi_{J,\Gamma}$ and $\Phi(S,G)$ whenever $\operatorname{Spec} R$ is connected.

Definition 3.7 In the setting of Lemma 3.6 we call $\Phi(S,G)$ a *system of relative roots with respect to the parabolic subgroup P over R* and denote it by Φ_P.

Example 3.8 If A is a field or a local ring, and P is a minimal parabolic subgroup of G, then Φ_P is nothing but the relative root system of G with respect to a maximal split subtorus in the sense of [BT1] or [SGA3, Exp. XXVI §7].

In [PSt1], we have also defined irreducible components of systems of relative roots, the subsets of positive and negative relative roots, simple relative roots, and the height of a root. These definitions are immediate analogs of the ones for usual abstract root systems, so we do not reproduce them here.

Let R be a commutative ring with 1. For any finitely generated projective R-module V, we denote by $W(V)$ the natural affine scheme over R associated with V; see [SGA3, Exp. I, §4.6]. Any morphism of R-schemes $\mathrm{W}(V_1) \to \mathrm{W}(V_2)$ is determined by an element $f \in \mathrm{Sym}^*(V_1^\vee) \otimes_R V_2$, where Sym^* denotes the symmetric algebra, and $V_1^\vee$ denotes the dual module of V_1. If $f \in \mathrm{Sym}^d(V_1^\vee) \otimes_R V_2$, we say that the corresponding morphism is homogeneous of degree d. By abuse of notation, we also write $f: V_1 \to V_2$ and call it *a degree d homogeneous polynomial map from V_1 to V_2*. In this context, one has $f(\lambda v) = \lambda^d f(v)$ for any $v \in V_1$ and $\lambda \in R$.

Lemma 3.9 ([PSt1]) *In the setting of Lemma 3.6, for any $\alpha \in \Phi_P = \Phi(S,G)$, there exists a closed S-equivariant embedding of R-schemes*

$$X_\alpha: W\big(\mathrm{Lie}(G)_\alpha\big) \longrightarrow G,$$

satisfying the following condition.

(∗) *Let R'/R be any ring extension such that $G_{R'}$ is split with respect to a maximal split R'-torus $T \subseteq L_{R'}$. Let e_δ, $\delta \in \Phi$, be a Chevalley basis of* $\mathrm{Lie}(G_{R'})$*, adapted to T and*

P, and $x_\delta\colon \mathbf{G}_{\mathrm{a}} \to G_{R'}$, $\delta \in \Phi$, *be the associated system of 1-parameter root subgroups* (e.g., $x_\delta = \exp_\delta$ *of* [SGA3, Exp. XXII, Th. 1.1]). *Let*

$$\pi\colon \Phi = \Phi(T, G_{R'}) \longrightarrow \Phi_P \cup \{0\}$$

be the natural projection. Then for any $u = \sum_{\delta\in\pi^{-1}(\alpha)} a_\delta e_\delta \in \mathrm{Lie}(G_{R'})_\alpha$, *one has*

$$X_\alpha(u) = \Big(\prod_{\delta\in\pi^{-1}(\alpha)} x_\delta(a_\delta)\Big)\cdot\prod_{i\geq 2}\Big(\prod_{\theta\in\pi^{-1}(i\alpha)} x_\theta(p^i_\theta(u))\Big), \tag{3.5}$$

where every $p^i_\theta\colon \mathrm{Lie}(G_{R'})_\alpha \to R'$ *is a homogeneous polynomial map of degree* i, *and the products over* δ *and* θ *are taken in any fixed order.*

Proof Proceeding exactly as in [PSt1, Th. 2], we prove the existence of a closed embedding

$$X_\alpha\colon W(V_\alpha) \longrightarrow G$$

satisfying condition $(*)$, where V_α is a finitely generated projective R-module of rank $|\pi^{-1}(\alpha)|$, implicitly constructed by descent. However, once we identify the system of relative roots Φ_P in the sense of [PSt1] with $\Phi(S, G)$ as discussed above, it follows from the proof of [PSt1, Th. 2] that V_α is canonically isomorphic to $\mathrm{Lie}(G)_\alpha$. The S-equivariance of X_α follows immediately from condition $(*)$. ■

Definition 3.10 Closed embeddings X_α, $\alpha \in \Phi_P$, satisfying the statement of Lemma 3.9, are called *relative root subschemes of G with respect to the parabolic subgroup P.*

Remark 3.11 Relative root subschemes of G with respect to P, actually, depend on the choice of a Levi subgroup L in P, but their essential properties stay the same, so we usually omit L from the notation.

Example 3.12 Let A be a connected commutative ring that contains $\mathbb{Q}$ and let G be a semisimple reductive group of adjoint type over A containing a parabolic subgroup P with a Levi subgroup L. We identify G with its image under the natural homomorphism $G \to \mathrm{Aut}_A(\mathrm{Lie}(G))$. Then the relative root A-subschemes X_α, $\alpha \in \Phi_P$, of Lemma 3.9 can be constructed as follows. For any ring extension B/A and any $v \in V_\alpha \otimes_A B = \mathrm{Lie}(G_B)_\alpha$, set

$$X_\alpha(v) = \exp(\mathrm{ad}_v) = \sum_{i=0}^{\infty}\frac{1}{i!}(\mathrm{ad}_v)^i \in \mathrm{Aut}_B(\mathrm{Lie}(G_B)).$$

Here the "infinite" sum is necessarily finite, since we have $(\mathrm{ad}_v)^i = 0$ for any $i > |\Phi_P|$. It is clear that $X_\alpha\colon W(\mathrm{Lie}(G)_\alpha) \to \mathrm{Aut}_A(\mathrm{Lie}(G))$ is a morphism of A-schemes.

Apart from that, we need to show that each X_α is an S-equivariant closed embedding that satisfies condition $(*)$ of Lemma 3.9. Let A'/A be any ring extension such that $G_{A'}$ is split with respect to a maximal split A'-torus $T \subseteq L_{A'}$. We recall that $x_\delta(t)$, $\delta \in \Phi$, $t \in A'$, coincide with $\exp(t\mathrm{ad}_{e_\delta})$ in the adjoint representation of $G_{A'}$; see, *e.g.*, [Che]. Then the Baker–Campbell–Hausdorff formula implies that (3.5) holds, and that each morphism

$$X_\alpha\colon W(\mathrm{Lie}(G_{A'})_\alpha) \longrightarrow G_{A'}$$

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is $S_{A'}$-equivariant. Denote by X_α^H the morphism X_α considered as a morphism from $W(\mathrm{Lie}(G_{A'})_\alpha)$ to the unipotent closed A'-subgroup

$$H = \prod_{i\geq 1} \prod_{\delta\in\pi^{-1}(i\alpha)} x_\delta(\mathbf{G}_{\mathrm{a}}) \cong W\Big(\sum_{i\geq 1} \mathrm{Lie}(G_{A'})_{i\alpha}\Big)$$

of $G_{A'}$. Then (3.5) readily implies that X_α^H is universally closed and universally injective, and hence the same is true for X_α. Since the tangent map $\mathrm{Lie}(X_\alpha)$, corresponding to the inclusion of $\mathrm{Lie}(G_{A'})_\alpha$ into $\mathrm{Lie}(G_{A'})$, is also injective, we conclude that X_α is formally unramified. Summing up, this implies that X_α is a closed embedding of $W(\mathrm{Lie}(G_{A'})_\alpha)$ into $G_{A'}$.

Finally, note that, locally in the étale topology, the group G over A is split with respect to a torus T contained in $L \subseteq P$, see [SGA3, Exp. XXII, Cor. 2.3; Exp. XXVI, Lemme 1.14]. Then faithfully flat descent implies that in order to prove that X_α is an S-equivariant closed embedding over A, it is enough to prove the same over every A' as above. Since the latter is already established, we conclude that X_α satisfies all the conditions present in Lemma 3.9.

We will use the following properties of relative root subschemes.

Lemma 3.13 ([PSt1, Theorem 2, Lemma 6, Lemma 9]) *Let X_α, $\alpha \in \Phi_P$, be as in Lemma 3.9. Set $V_\alpha = \mathrm{Lie}(G)_\alpha$ for short.*

(i) *There exist degree i homogeneous polynomial maps $q_\alpha^i\colon V_\alpha \oplus V_\alpha \to V_{i\alpha}$, $i > 1$, such that for any R-algebra R' and for any $v, w \in V_\alpha \otimes_R R'$, one has*

$$X_\alpha(v)X_\alpha(w) = X_\alpha(v+w)\prod_{i>1} X_{i\alpha}\big(q_\alpha^i(v,w)\big). \tag{3.6}$$

(ii) *For any $g \in L(R)$, there exist degree i homogeneous polynomial maps $\phi_{g,\alpha}^i\colon V_\alpha \to V_{i\alpha}$, $i \geq 1$, such that for any R-algebra R' and for any $v \in V_\alpha \otimes_R R'$, one has*

$$gX_\alpha(v)g^{-1} = \prod_{i\geq 1} X_{i\alpha}\big(\phi_{g,\alpha}^i(v)\big).$$

If $g \in S(R)$, then $\phi_{g,\alpha}^1$ is multiplication by a scalar and all $\phi_{g,\alpha}^i$, $i > 1$, are trivial.

(iii) (generalized Chevalley commutator formula) *For any $\alpha, \beta \in \Phi_P$ such that $m\alpha \neq -k\beta$ for all $m, k \geq 1$, there exist polynomial maps*

$$N_{\alpha\beta ij}\colon V_\alpha \times V_\beta \longrightarrow V_{i\alpha+j\beta}, \quad i, j > 0,$$

homogeneous of degree i in the first variable and of degree j in the second variable, such that for any R-algebra R' and for any for any $u \in V_\alpha \otimes_R R'$, $v \in V_\beta \otimes_R R'$ one has

$$[X_\alpha(u), X_\beta(v)] = \prod_{i,j>0} X_{i\alpha+j\beta}\big(N_{\alpha\beta ij}(u,v)\big) \tag{3.7}$$

(iv) *For any subset $\Psi \subseteq X^*(S) \setminus \{0\}$ that is closed under addition, the morphism*

$$X_\Psi\colon W\Big(\bigoplus_{\alpha\in\Psi} V_\alpha\Big) \longrightarrow U_\Psi, \qquad (v_\alpha)_\alpha \mapsto \prod_\alpha X_\alpha(v_\alpha),$$

where the product is taken in any fixed order, is an isomorphism of schemes.

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3.3 Proof of Theorem 3.1

By assumption, every semisimple normal subgroup of G contains $(\mathbf{G}_{m,R})^2$. We claim that there is a split subtorus S_0 of G such that $S_0 \cap H$ contains $(\mathbf{G}_{m,R})^2$ for every semisimple normal subgroup H of G. Indeed, if G is semisimple and simply connected, this follows from the fact that G is a direct product of its minimal normal semisimple subgroups [SGA3, Exp. XXIV, §5]. In general, G is a quotient of the direct product $G^{sc} \times \mathrm{rad}(G)$ by a central finite subgroup, where $\mathrm{rad}(G)$ is the radical of G, and G^{sc} is the simply connected cover of the derived group scheme of G. One readily sees that if S_0^{sc} is the split subtorus of G^{sc} whose intersection with every semisimple normal subgroup of G^{sc} contains $(\mathbf{G}_{m,R})^2$, then the same is true for its image S_0 in G.

Let P be a parabolic subgroup of G with a Levi subgroup $L = \mathrm{Cent}_G(S_0)$ constructed as in Remark 2.10. Clearly, P is strictly proper. Let P^- be the opposite parabolic subgroup to P, satisfying $L = P \cap P^-$. For any R-algebra A, we have

$$E(A) = \langle U_P(A), U_{P^-}(A) \rangle .$$

Now we show that, in order to prove the equality

$$E\big(R((t))\big) = E\big(R[[t]]\big)\, E\big(R[t^{\pm 1}]\big), \tag{3.8}$$

we can assume that R is connected. Fix an element $g \in E\big(R((t))\big)$. The commmutative ring R is a direct limit of its Noetherian subrings, $R = \varinjlim R_\alpha$. Fix an element $g \in E\big(R((t))\big)$. Since G, its semisimple normal subgroup schemes, P, P_- and L are all finitely presented R-group schemes, there is an index α and a reductive group scheme G' over R_α such that all these group schemes are defined over R_α, P is strictly proper over R_α, and $g \in E\big(R_\alpha((t))\big) \le E\big(R((t))\big)$; *cf.* [SGA3, Exp. XIX, Remarque 2.9]. Clearly, in order to show that g belongs to the right-hand side of (3.8), it is enough to prove the equality (3.8) for the Noetherian ring R_α in place of R. Thus, we can assume from the start that R is Noetherian. Then $R = \prod_{i=1}^m A_i$, where A_i, $1 \le i \le m$, are connected rings. Set $R_1 = R((t))$, $R_2 = R[t^{\pm 1}]$, and $R_3 = R[[t]]$. Then

$$E(R_j) = \big\langle U_P(R_j), U_{P^-}(R_j) \big\rangle = \prod_{i=1}^m E(A_i \otimes_R R_j)$$

for all R_j, $j = 1, 2, 3$. Therefore, it is enough to show that (3.8) holds with R replaced by each of the connected rings A_i. Thus, we can assume from now on that R is connected.

Let $S \subseteq \mathrm{Cent}(\overline{L})$ be the split torus constructed in Lemma 3.6, $\Phi_P = \Phi(S, G)$, and X_α, $\alpha \in \Phi_P$, be the relative root subschemes over R that exist by Lemma 3.9. Since, clearly, S contains the image of S_0 in G^{ad}, every irreducible component of Φ_P in the sense of [PSt1] has rank ≥ 2.

By Lemma 3.13(iv) the group $E(R((t)))$ is generated by root elements $X_\alpha(v)$, $\alpha \in \Phi_P$, $v \in V_\alpha \otimes_R R((t))$. Write $v = \sum_{j=-k}^{\infty} v_j t^j$, $v_j \in V_\alpha$, $k \ge 0$. By equality (3.6) of Lemma 3.13 we have

$$X_\alpha(v) = X_\alpha\Big(\sum_{j=-k}^{0} v_j t^j\Big)\, X_\alpha\Big(\sum_{j=1}^{\infty} v_j t^j\Big) \prod_{i \ge 2} X_{i\alpha}(u_i)$$

for some $u_i \in V_{i\alpha} \otimes_R R((t))$. Applying induction on the height of α, we conclude that $X_\alpha(v)$ decomposes into a product of elements from $E(R[t^{-1}])$ and $E(R[[t]])$, that is,

$$E\big(R((t))\big) = \big\langle E(R[t^{-1}]), E(R[[t]])\big\rangle.$$

Similarly, one concludes that $E(R[t,t^{-1}])$ is generated by elements $X_\alpha(t^n u)$, $n \in \mathbb{Z}$, $u \in V_\alpha$, $\alpha \in \Phi_P$. Consequently, in order to prove (3.8), it is enough to show that for any $\beta \in \Phi_P$, $v \in V_\beta \otimes_R R[[t]]$ we have

$$E\big(R[t,t^{-1}]\big) X_\beta(v) \subseteq E\big(R[[t]]\big) E\big(R[t,t^{-1}]\big). \tag{3.9}$$

For any R-algebra R', any ideal $I \subseteq R'$, and any additively closed set $\Psi \subseteq X^*(S) \setminus \{0\}$, we set

$$U_\Psi(I) = \langle X_\alpha(u),\ \alpha \in \Psi,\ u \in V_\alpha \otimes_R I\rangle \subseteq U_\Psi(R')$$

and

$$E(I) = \langle X_\alpha(u),\ \alpha \in \Phi_P,\ u \in V_\alpha \otimes_R I\rangle \subseteq E(R').$$

We show that for any $\beta \in \Phi_P$, $v \in V_\beta \otimes_R R[[t]]$, one has

$$X_\beta(v) \in E\big(t^N R[[t]]\big) E(R[t]) \quad \text{for any } N \geq 0. \tag{3.10}$$

More precisely, set $(\beta) = \{i\beta \mid i \geq 1\}$; we show that

$$X_\beta(v) \in U_{(\beta)}\big(t^N R[[t]]\big) \cdot U_{(\beta)}\big(R[t]\big) \tag{3.11}$$

arguing by descending induction on the height of β. Since V_β is a finitely generated projective R-module, we can write $v = v_1 + t^N v_2$, where $v_1 \in V_\beta \otimes_R R[t]$ and $v_2 \in V_\beta \otimes_R R[[t]]$. Then (3.6) of Lemma 3.13 implies that

$$X_\beta(v) = X_\beta(t^N v_2) \cdot \prod_{i>1} X_{i\beta}\big(q_\beta^i(v,-v_1)\big) \cdot \big(X_\beta(-v_1)\big)^{-1}. \tag{3.12}$$

By the induction hypothesis, for any $i > 1$ one has

$$\begin{aligned} X_{i\beta}\big(q_\beta^i(v,-v_1)\big) &\in U_{(i\beta)}\big(t^N R[[t]]\big) \cdot U_{(i\beta)}(R[t]) \\ &\subseteq U_{(\beta)}\big(t^N R[[t]]\big) \cdot U_{(\beta)}\big(R[t]\big). \end{aligned} \tag{3.13}$$

Note that by (3.7) of Lemma 3.13 the group $U_{(\beta)}(R[t])$ normalizes the group $U_{(\beta)}(t^N R[[t]])$. Then, clearly, (3.12) and (3.13) together imply (3.11). This finishes the proof of (3.10).

Next, we show that for any $n \in \mathbb{Z}$, $u \in V_\alpha$, $\alpha \in \Phi_P$, and $M \geq 0$ there is $N \geq 0$ such that

$$X_\alpha(t^n u) E\big(t^N R[[t]]\big) X_\alpha(t^n u)^{-1} \subseteq E\big(t^M R[[t]]\big). \tag{3.14}$$

Clearly, this statement and (3.10) together imply (3.9).

By [PSt1, Lemma 11] we know that if $N \geq 3$, then $E(t^N R[[t]])$ is contained in the subgroup of $E(R[[t]])$ generated by $X_\gamma(V_\gamma \otimes_R t^{\lfloor \frac{N}{3} \rfloor} R[[t]])$ for all $\gamma \in \Phi_P \setminus \mathbb{Z}\alpha$. On the other hand, for any such γ by the Chevalley commutator formula (3.7) of Lemma 3.13, we have

$$\Big[X_\alpha(t^n u), X_\gamma\big(V_\gamma \otimes_R t^{\lfloor \frac{N}{3} \rfloor} R[[t]]\big)\Big] \subseteq E\Big(t^{\lfloor \frac{N}{3} \rfloor - |\Phi_P| \cdot |n|} R[[t]]\Big).$$

This implies the claim (3.14). ∎

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4 Proof of the Main Results

4.1 Diagonal Argument for Loop Reductive Groups

Our main results are based on the following observation.

Lemma 4.1 ("diagonal argument") *Let k be a field of characteristic 0. Let G be a loop reductive group over $R = k[x_1^{\pm1},\dots,x_n^{\pm1}]$. For any integer $d > 0$, denote by $f_{z,d}$ (respectively, $f_{w,d}$) the composition of k-homomorphisms*

$$R \longrightarrow k[z_1^{\pm1},\dots,z_n^{\pm1},w_1^{\pm1},\dots,w_n^{\pm1}] \longrightarrow k[z_1^{\pm1},\dots,z_n^{\pm1},(z_1w_1^{-1})^{\pm\frac{1}{d}},\dots,(z_nw_n^{-1})^{\pm\frac{1}{d}}]$$

sending x_i to z_i (respectively, to w_i) for any $1 \le i \le n$. Then there is $d > 0$ such that

$$f_{z,d}^*(G) \cong f_{w,d}^*(G)$$

as group schemes over $k[z_1^{\pm1},\dots,z_n^{\pm1},(z_1w_1^{-1})^{\pm\frac{1}{d}},\dots,(z_nw_n^{-1})^{\pm\frac{1}{d}}]$.

Proof Let G_0 be a split reductive group over k such that G is a twisted form of G_0. Let $A_0 = \operatorname{Aut}(G_0)$ be the group scheme of automorphisms of G_0. Denote by $\overline{k}$ the algebraic closure of k, and by Γ the Galois group $\operatorname{Gal}(\overline{k}/k)$. We also introduce the following auxiliary notation. We write X_x for the k-scheme $\operatorname{Spec} k[x_1^{\pm1},\dots,x_n^{\pm1}]$, X_z for $\operatorname{Spec} k[z_1^{\pm1},\dots,z_n^{\pm1}]$, etc.

According to Definition 2.11, G is given by a cocycle η in $H^1\big(\pi_0(X_x,e),A_0(\overline{k})\big)$. Considering the description (2.2), we can assume that

$$\eta \in H^1\Big(\operatorname{Gal}\big(\overline{k}[x_1^{\pm\frac{1}{d}},\dots,x_n^{\pm\frac{1}{d}}]/k[x_1^{\pm1},\dots,x_n^{\pm1}]\big),A_0(\overline{k})\Big)$$

for some integer $d > 0$, and we know that

$$\operatorname{Gal}\big(\overline{k}[x_1^{\pm\frac{1}{d}},\dots,x_n^{\pm\frac{1}{d}}]/k[x_1^{\pm1},\dots,x_n^{\pm1}]\big) = M_x \rtimes \Gamma,$$

where $M_x \cong (\mathbb{Z}/d\,\mathbb{Z})^n$ acts on $\overline{k}[x_1^{\pm\frac{1}{d}},\dots,x_n^{\pm\frac{1}{d}}]$ by sending $x_i^{\frac{1}{d}}$ to $\xi_d^{k_i}x_i^{\frac{1}{d}}$, for any $(k_1,\dots,k_n) \in M_x$. We will denote by M_z and M_w respectively the group $(\mathbb{Z}/d\,\mathbb{Z})^n$ operating in the same way on $\overline{k}[z_1^{\pm\frac{1}{d}},\dots,z_n^{\pm\frac{1}{d}}]$ and $\overline{k}[w_1^{\pm\frac{1}{d}},\dots,w_n^{\pm\frac{1}{d}}]$.

Denote by i_z (respectively, i_w) the k-homomorphism

$$k[x_1^{\pm1},\dots,x_n^{\pm1}] \longrightarrow k[z_1^{\pm1},\dots,z_n^{\pm1},w_1^{\pm1},\dots,w_n^{\pm1}]$$

sending x_i to z_i (respectively, to w_i) for any $1 \le i \le n$. Consider the images $i_z^*(\eta)$ and $i_w^*(\eta)$ of η in $H^1\big(\pi_0(X_z \times_k X_w,e),A_0(\overline{k})\big)$ as elements of

$$H^1\big((M_z \times M_w) \rtimes \Gamma, A_0(\overline{k})\big).$$

Denote by Δ the diagonal subgroup of $M_z \times M_w$. The subgroup $\Delta \rtimes \Gamma$ of $(M_z \times M_w) \rtimes \Gamma$ is closed, and it is straightforward to check that

$$i_z^*(\eta)|_{\Delta\rtimes\Gamma} = i_w^*(\eta)|_{\Delta\rtimes\Gamma}.$$

Since

$$\big(\overline{k}[z_1^{\pm\frac{1}{d}},\dots,z_n^{\pm\frac{1}{d}},w_1^{\pm\frac{1}{d}},\dots,w_n^{\pm\frac{1}{d}}]\big)^{\Delta\rtimes\Gamma} = k[z_1^{\pm1},\dots,z_n^{\pm1},(z_1w_1^{-1})^{\pm\frac{1}{d}},\dots,(z_nw_n^{-1})^{\pm\frac{1}{d}}],$$

we conclude that $f_{z,d}^*(G) \cong f_{w,d}^*(G)$, as required. ∎

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We introduce additional notation that will be used every time when we apply Lemma 4.1 in proofs of other statements.

Notation 4.2 *In the setting of the claim of Lemma 4.1, set*

$$t_i = (z_i w_i^{-1})^{1/d}, \quad 1 \le i \le n,$$

where z_i, w_i, and d are as in that lemma. Note that this is equivalent to

$$z_i = w_i t_i^d, \quad 1 \le i \le n.$$

We denote by G_z the group scheme over $k[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$, which is the pull-back of G under the k-isomorphism

$$k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \xrightarrow{x_i \mapsto z_i} k[z_1^{\pm 1}, \dots, z_n^{\pm 1}].$$

The group scheme G_w over $k[w_1^{\pm 1}, \dots, w_n^{\pm 1}]$ is defined analogously. Note that G_z and G_w are isomorphic after pull-back to

$$k[z_1^{\pm 1}, \dots, z_n^{\pm 1}, t_1^{\pm 1}, \dots, t_n^{\pm 1}] = k[w_1^{\pm 1}, \dots, w_n^{\pm 1}, t_1^{\pm 1}, \dots, t_n^{\pm 1}].$$

4.2 Proof of Theorem 1.4 on $\mathcal{R}$-equivalence Class Groups

The surjectivity of the natural map

$$G\big(k(x_1, \dots, x_n)\big)/\mathcal{R} \longrightarrow G\big(k((x_1))\cdots((x_n))\big)/\mathcal{R}$$

follows from Corollary 2.14. To prove the injectivity, recall that since G has a maximal torus over $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, it is loop reductive by [GP3, Corollary 6.3]. Thus, we can apply Lemma 4.1 to G. We use Notation 4.2.

Consider the following commutative diagram, where the horizontal maps j_1 and j_2 are the natural ones:

$$\begin{array}{ccc}
G\big(k(x_1,\dots,x_n)\big)/\mathcal{R} & \xrightarrow{\quad j_1 \quad} & G\big(k((x_1))\cdots((x_n))\big)/\mathcal{R} \\
{\scriptstyle f_1 : x_i \mapsto z_i}\big\downarrow \cong & & {\scriptstyle f_2 : x_i \mapsto z_i}\big\downarrow \cong \\
G_z\big(k(z_1,\dots,z_n,t_1,\dots,t_n)\big)/\mathcal{R} & & G_z\big(k((z_1))\cdots((z_n))\big)/\mathcal{R} \\
{\scriptstyle g_1 : z_i \mapsto w_i t_i^d}\big\downarrow \cong & & {\scriptstyle g_2 : z_i \mapsto w_i t_i^d}\big\downarrow \\
G_w\big(k(w_1,\dots,w_n,t_1,\dots,t_n)\big)/\mathcal{R} & \underset{\cong}{\xrightarrow{j_2}} & G_w\big(k(w_1,\dots,w_n)((t_1))\cdots((t_n))\big)/\mathcal{R}
\end{array}$$

The map f_1 in this diagram is an isomorphism, since G_z is defined over $k(z_1, \dots, z_n)$, and by [V, §16.2, Proposition 2], for any reductive group H over an infinite field l one has $H(l)/\mathcal{R} \cong H\big(l(t)\big)/\mathcal{R}$. The map f_2 is an isomorphism by definition. The map g_1 is an isomorphism by Lemma 4.1. The map j_2 is an isomorphism, since G_w is defined over $k(w_1, \dots, w_n)$, and by [G1, Corollaire 0.3] for any reductive group H over a field l of characteristic $\ne 2$, one has $H(l)/\mathcal{R} \cong H\big(l((t))\big)/\mathcal{R}$.

Since

$$g_2 \circ f_2 \circ j_1 = j_2 \circ g_1 \circ f_1$$

is an isomorphism, we conclude that the map j_1 is injective. ∎

4.3 Proof of Theorem 1.2

In order to prove Theorem 1.2, we still need to prove some technical lemmas.

Lemma 4.3 *Let k be an arbitrary field, let A be a commutative k-algebra, and let G be a reductive group defined over $A[z_1^{\pm1},\dots,z_n^{\pm1}]$ such that every semisimple normal subgroup of G contains $(\mathbf{G}_{\mathrm{m},k})^2$. For any set of integers $d_i > 0$, $1 \le i \le n$, the map*

$$K_1^G\big(A[z_1^{\pm1},\dots,z_n^{\pm1},t_1,\dots,t_n]\big) \xrightarrow{z_i\mapsto w_i t_i^{d_i}} K_1^G\big(A\otimes_k k(w_1,\dots,w_n)[t_1^{\pm1},\dots,t_n^{\pm1}]\big)$$

is injective.

Proof We prove the claim by induction on $n \ge 0$. The case $n = 0$ is trivial. To prove the induction step for $n \ge 1$, it is enough to show that

$$\phi\colon K_1^G\big(A\otimes_k k[z_1^{\pm1},\dots,z_n^{\pm1},t_1,\dots,t_n]\big) \xrightarrow{z_1\mapsto w_1 t_1^{d_1}} K_1^G\big(A\otimes_k k(w_1)[t_1^{\pm1}][z_2^{\pm1},\dots,z_n^{\pm1},t_2,\dots,t_n]\big)$$

is injective. Indeed, after that we can apply the induction assumption with k substituted by $k(w_1)$ and A substituted by $A\otimes_k k(w_1)[t_1^{\pm1}]$. Set

$$B = A[z_2^{\pm1},\dots,z_n^{\pm1},t_2,\dots,t_n]$$

and omit for simplicity the subscript 1. Then we need to show that the map

$$\phi\colon K_1^G\big(B[z^{\pm1},t]\big) \xrightarrow{z\mapsto wt^d} K_1^G\big(B\otimes_k k(w)[t^{\pm1}]\big)$$

is injective. Here, G is defined over $B[z^{\pm1}]$. We have

$$B\otimes_k k(w)[t^{\pm1}] = \varinjlim_g B\otimes_k k[w^{\pm1}]_g[t^{\pm1}] = \varinjlim_g B\otimes_k k[w^{\pm1},t^{\pm1}]_g,$$

where $g = g(w)$ runs over all monic polynomials in $k[w]$ with $g(0) \ne 0$. Since $\phi(z) = wt^d$, we have $g(w) = g(\phi(z)t^{-d}) = t^{-Nd}f(t)$ for a suitable integer N, where $f(t)$ is a polynomial in t with coefficients in $k[\phi(z)^{\pm1}]$ such that its leading coefficient is invertible. Then by Lemma 2.7 the natural map

$$K_1^G\big(B[z^{\pm1},t]\big) \xrightarrow{z\mapsto wt^d} K_1^G\big(B\otimes_k k[w^{\pm1},t^{\pm1}]_g\big) = K_1^G\big(B\otimes_k k[\phi(z)^{\pm1},t]_{tf}\big)$$

is injective. Since K_1^G commutes with filtered direct limits, we conclude that ϕ is injective. ∎

Lemma 4.4 *Let k be an arbitrary field, let A be a commutative k-algebra, and let G be a reductive group scheme over A such that every semisimple normal subgroup of G contains $(\mathbf{G}_{\mathrm{m},A})^2$. For any $n \ge 0$, the natural map*

$$K_1^G\big(A[t_1^{\pm1},\dots,t_n^{\pm1}]\big) \longrightarrow K_1^G\big(A\otimes_k k(t_1,\dots,t_n)\big)$$

is injective.

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Proof We prove the claim by induction on n; the case $n = 0$ is trivial. Set $l = k(t_1,\dots,t_{n-1})$. By the inductive hypothesis, the map

$$K_1^G\big(A[t_1^{\pm1},\dots,t_n^{\pm1}]\big)\longrightarrow K_1^G\big(A[t_n^{\pm1}]\otimes_k l\big)=K_1^G\big(A\otimes_k l[t_n^{\pm1}]\big)$$

is injective, so it remains to prove the injectivity of the map

$$K_1^G\big(A\otimes_k l[t_n^{\pm1}]\big)\longrightarrow K_1^G\big(A\otimes_k l(t_n)\big).$$

We have $l(t_n)=\varinjlim_g l[t_n]_{t_ng}$, where $g\in l[t_n]$ runs over all monic polynomials coprime to t_n. Since K_1^G commutes with filtered direct limits, it remains to show that every map

$$K_1^G\big(A\otimes_k l[t_n^{\pm1}]\big)\longrightarrow K_1^G\big(A\otimes_k l[t_n]_{t_ng}\big) \tag{4.1}$$

is injective. Assume that

$$x\in G\big(A\otimes_k l[t_n^{\pm1}]\big)\cap E\big(A\otimes_k l[t_n]_{t_ng}\big).$$

By [St13, Lemma 2.3] there exist $x_1\in E(A\otimes_k l[t_n]_{t_n})$ and $x_2\in E(A\otimes_k l[t_n]_g)$ such that $x=x_1x_2$. We have $x,x_1\in G(A\otimes_k l[t_n^{\pm1}])$, therefore, $x_2\in G(A\otimes_k l[t_n^{\pm1}])$. Since

$$G\big(A\otimes_k l[t_n^{\pm1}]\big)\cap G\big(A\otimes_k l[t_n]_g\big)=G\big(A\otimes_k l[t_n]\big),$$

we have $x_2\in G(A\otimes_k l[t_n])\cap E(A\otimes_k l[t_n]_g)$. By Lemma 2.7 this implies that $x_2\in E(A\otimes_k l[t_n])$. Summing up, we have $x=x_1x_2\in E(A\otimes_k l[t_n^{\pm1}])$. Therefore, the map (4.1) is injective. ∎

Lemma 4.5 *Let k be a field of characteristic 0 and let G be a reductive group over $X=\operatorname{Spec} k[x_1^{\pm1},\dots,x_n^{\pm1}]$, having a maximal X-torus and such that every semisimple normal subgroup of G contains $(\mathbf{G}_{m,X})^2$. Then*

(i) *the natural map*

$$K_1^G\big(k[x_1^{\pm1},\dots,x_n^{\pm1}]\big)\longrightarrow K_1^G\big(k(x_1,\dots,x_n)\big)$$

is injective;

(ii) *one has $K_1^G\big(k[x_1^{\pm1},\dots,x_n^{\pm1}]\big)=K_1^G\big(k[x_1^{\pm1},\dots,x_n^{\pm1},y_1,\dots,y_m]\big)$ for any $m\geq0$.*

Proof First we show that for any $m\geq0$, the natural map

$$K_1^G\big(k[x_1^{\pm1},\dots,x_n^{\pm1},y_1,\dots,y_m]\big)\longrightarrow K_1^G\big(k(x_1,\dots,x_n)[y_1,\dots,y_m]\big)$$

is injective. This includes (i). In short, we write $\mathbf{y}$ instead of $y_1,\dots,y_m$.

As in Theorem 1.4, we note that G is loop reductive over $k[x_1^{\pm1},\dots,x_n^{\pm1}]$ by [GP3, Corollary 6.3]. We apply Lemma 4.1 to G, and we use Notation 4.2. Consider the following commutative diagram. In this diagram, the horizontal maps j_1 and j_2 are the natural ones, and all maps always take variables t_i to t_i, $1\leq i\leq n$, and $\mathbf{y}$ to $\mathbf{y}$. The isomorphisms g_1 and g_2 exist by Lemma 4.1:

$$\begin{array}{ccc}
K_1^G\big(k[x_1^{\pm 1},\dots,x_n^{\pm 1},\mathbf{y}]\big) & \xrightarrow{\;j_1\;} & K_1^G\big(k(x_1,\dots,x_n)[\mathbf{y}]\big) \\
\Big\downarrow{\scriptstyle f_1: x_i\mapsto z_i} & & \Big\downarrow{\scriptstyle f_2: x_i\mapsto z_i} \\
K_1^{G_z}\big(k[z_1^{\pm 1},\dots,z_n^{\pm 1},t_1,\dots,t_n,\mathbf{y}]\big) & & K_1^{G_z}\big(k(z_1,\dots,z_n,t_1,\dots,t_n)[\mathbf{y}]\big) \\
\Big\downarrow{\scriptstyle h: z_i\mapsto w_i t_i^d} & & \\
K_1^{G_z}\big(k(w_1,\dots,w_n)[t_1^{\pm 1},\dots,t_n^{\pm 1},\mathbf{y}]\big) & & \Big\downarrow{\scriptstyle g_2: z_i\mapsto w_i t_i^d}\,\cong \\
\Big\downarrow{\scriptstyle g_1}\,\cong & & \\
K_1^{G_w}\big(k(w_1,\dots,w_n)[t_1^{\pm 1},\dots,t_n^{\pm 1},\mathbf{y}]\big) & \xrightarrow{\;j_2\;} & K_1^{G_w}\big(k(w_1,\dots,w_n,t_1,\dots,t_n)[\mathbf{y}]\big)
\end{array}$$

In order to prove that j_1 is injective, it is enough to show that all maps j_2, g_1, h, f_1 are injective. The map j_2 is injective by Lemma 4.4. As explained above, g_1 is an isomorphism. The map h is injective by Lemma 4.3. Finally, the map f_1 is injective, since it has a retraction that sends z_i to x_i and t_i to 0. Therefore, the map j_1 is injective.

Now we prove (ii). Consider the commutative diagram

$$\begin{array}{ccc}
K_1^G\big(k[x_1^{\pm 1},\dots,x_n^{\pm 1}][\mathbf{y}]\big) & \xrightarrow{\;y_i\mapsto 0\;} & K_1^G\big(k[x_1^{\pm 1},\dots,x_n^{\pm 1}]\big) \\
\Big\downarrow & & \Big\downarrow \\
K_1^G\big(k(x_1,\dots,x_n)[\mathbf{y}]\big) & \xrightarrow{\;y_i\mapsto 0\;} & K_1^G\big(k(x_1,\dots,x_n)\big).
\end{array}$$

The bottom arrow is an isomorphism by [St13, Theorem 1.2]. The vertical arrows are injective by the previous paragraph. Therefore, the top arrow

$$K_1^G\Big(k[x_1^{\pm 1},\dots,x_n^{\pm 1}][\mathbf{y}]\Big) \xrightarrow{y_i\mapsto 0} K_1^G\Big(k[x_1^{\pm 1},\dots,x_n^{\pm 1}]\Big)$$

is also injective. Since it has a section, it is an isomorphism. ∎

Proof of Theorem 1.2. We prove the injectivity claim by induction on n starting with the trivial case $n = 0$. To prove the induction step, it is enough to show that the map

$$j\colon K_1^G\Big(k[x_1^{\pm 1},\dots,x_n^{\pm 1}]\Big) \longrightarrow K_1^G\Big(k((x_1))[x_2^{\pm 1},\dots,x_n^{\pm 1}]\Big)$$

is injective. The latter follows from the injectivity of the composition

$$\begin{aligned}
j_1\colon K_1^G\big(k[x_1^{\pm 1},\dots,x_n^{\pm 1}]\big) \xrightarrow{j} K_1^G\big(k((x_1))[x_2^{\pm 1},\dots,x_n^{\pm 1}]\big)& \\
\longrightarrow K_1^G\big(k[x_2^{\pm 1},\dots,x_n^{\pm 1}]((x_1))\big)&,
\end{aligned}$$

which we proceed to establish.

The group G is loop reductive by [GP3, Corollary 6.3], since it has a maximal torus. We apply Lemma 4.1 to G, and we use Notation 4.2. Consider the following commutative diagram. Here j_1, j_2 are the natural maps, and the isomorphism g_1 and the map g_2 exist by Lemma 4.1:

$$\begin{array}{ccc}
K_1^G\big(k[x_1^{\pm1},\dots,x_n^{\pm1}]\big) & \xrightarrow{j_1} & K_1^G\big(k[x_2^{\pm1},\dots,x_n^{\pm1}]((x_1))\big) \\
\downarrow{\scriptstyle f_1: x_i\mapsto z_i} & & \downarrow{\scriptstyle f_2: x_i\mapsto z_i}\ \cong \\
K_1^{G_z}\big(k[z_1^{\pm1},\dots,z_n^{\pm1},t_1^{\pm1},\dots,t_n^{\pm1}]\big) & & K_1^{G_z}\big(k[z_2^{\pm1},\dots,z_n^{\pm1}]((z_1))\big) \\
\downarrow{\scriptstyle g_1: z_i\mapsto w_i t_i^d}\ \cong & & \downarrow{\scriptstyle g_2: z_i\mapsto w_i t_i^d} \\
K_1^{G_w}\big(k[w_1^{\pm1},\dots,w_n^{\pm1},t_1^{\pm1},\dots,t_n^{\pm1}]\big) & \xrightarrow{j_2} & K_1^{G_w}\big(k[w_1^{\pm1},\dots,w_n^{\pm1},t_2^{\pm1},\dots,t_n^{\pm1}]((t_1))\big)
\end{array}$$

In order to show that j_1 is injective, it is enough to show that f_1 and j_2 are injective. The map f_1 is injective, since it has a retraction that sends z_i to x_i and t_i to 1. Set

$$A = k[w_1^{\pm1},\dots,w_n^{\pm1},t_2^{\pm1},\dots,t_n^{\pm1}].$$

By Lemma 4.5(ii) we have $K_1^{G_w}(A[t_1]) = K_1^{G_w}(A)$; therefore, by Corollary 3.4 the map j_2 is injective. Therefore, the map j_1 is injective.

To finish the proof of the theorem, it remains to note that, if G is a semisimple group, the map

$$K_1^G\big(k[x_1^{\pm1},\dots,x_n^{\pm1}]\big) \longrightarrow K_1^G\big(k((x_1))\cdots((x_n))\big)$$

is surjective by Corollary 2.13. ■

5 Application to Lie Tori

Throughout this section, we assume that k is an algebraically closed field of characteristic 0. We fix a compatible set of primitive m-th roots of unity $\xi_m \in k$, $m \geq 1$.

Let G be an adjoint simple algebraic group over k (a Chevalley group), and $L = \operatorname{Lie}(G)$ the corresponding simple Lie algebra over k. It is well known that

$$\operatorname{Aut}_k(L) \cong \operatorname{Aut}_k(G) \cong G \ltimes N,$$

where N is the finite group of automorphisms of the Dynkin diagram of the root system of L and G. Fix two integers $n \geq 0$, $m \geq 1$ and let

$$\sigma = (\sigma_1,\dots,\sigma_n)$$

be an n-tuple of pairwise commuting elements of order m in $\operatorname{Aut}_k(L)$. Such an n-tuple determines a $\mathbb{Z}^n$-grading on L with

$$L_{i_1\cdots i_n} = \big\{ x \in L \mid \sigma_j(x) = \xi_m^{i_j} x,\ 1 \leq j \leq n \big\}.$$

Set $R = k[x_1^{\pm1},\dots,x_n^{\pm1}]$, and let $\widetilde{R} = k[x_1^{\pm\frac{1}{m}},\dots,x_n^{\pm\frac{1}{m}}]$, $m \geq 1$, be another copy of R, considered as an R-algebra via the natural embedding $R \subseteq \widetilde{R}$. Then $\widetilde{R}/R$ is a Galois ring extension with the Galois group $\operatorname{Gal}(\widetilde{R}/R) \cong (\mathbb{Z}/m\mathbb{Z})^n$.

Definition 5.1 The *multiloop Lie algebra* $\mathcal{L}(L,\sigma)$ is the $\mathbb{Z}^n$-graded k-Lie subalgebra

$$\mathcal{L}(L,\sigma) = \bigoplus_{(i_1,\dots,i_n)\in\mathbb{Z}^n} L_{i_1\cdots i_n} \otimes x_1^{\frac{i_1}{m}} \cdots x_n^{\frac{i_n}{m}}$$

of the k-Lie algebra $L \otimes_k \widetilde{R}$.

Note that, considered as an R-Lie algebra, the algebra $\mathcal{L}(L,\sigma)$ is an $\widetilde{R}/R$-twisted form of the R-Lie algebra $L \otimes_k R$. Indeed,

$$\mathcal{L}(L,\sigma) \otimes_R \widetilde{R} \cong (L \otimes_k R) \otimes_R \widetilde{R}.$$

Let Δ be a finite root system in the sense of [Bou] together with the 0-vector, which we include following the tradition in the theory of extended affine Lie algebras. We set $\Delta^\times = \Delta \setminus \{0\}$, $Q = \mathbb{Z}\Delta$, and

$$\Delta^\times_{\text{ind}} = \{\alpha \in \Delta^\times \mid \tfrac{1}{2}\alpha \notin \Delta\}.$$

The importance of multiloop Lie algebras stems from the fact that they provide explicit realizations for a class of infinite-dimensional Lie algebras over k called Lie tori. This was shown by B. Allison, S. Berman, J. Faulkner and A. Pianzola in [ABFP].

Definition 5.2 ([ABFP, Def. 1.1.6]) A *Lie Λ-torus of type Δ* is a $Q \times \Lambda$-graded Lie algebra $\mathcal{L} = \bigoplus_{(\alpha,\lambda)\in Q\times\Lambda} \mathcal{L}^\lambda_\alpha$ over k satisfying

(i) $\mathcal{L}^\lambda_\alpha = 0$ for all $\alpha \in Q \setminus \Delta$ and all $\lambda \in \Lambda$;
(ii) $\mathcal{L}^0_\alpha \neq 0$ for all $\alpha \in \Delta^\times_{\text{ind}}$;
(iii) Λ is generated by the set of all $\lambda \in \Lambda$ such that $\mathcal{L}^\lambda_\alpha \neq 0$ for some $\alpha \in \Delta$;
(iv) for all $(\alpha,\lambda) \in \Delta^\times \times \Lambda$ such that $\mathcal{L}^\lambda_\alpha \neq 0$, there exist elements $e^\lambda_\alpha \in \mathcal{L}^\lambda_\alpha$ and $f^\lambda_\alpha \in \mathcal{L}^{-\lambda}_{-\alpha}$ satisfying

$$\mathcal{L}^\lambda_\alpha = ke^\lambda_\alpha, \quad \mathcal{L}^{-\lambda}_{-\alpha} = kf^\lambda_\alpha, \quad \text{and} \quad [[e^\lambda_\alpha, f^\lambda_\alpha], x] = \langle\beta, \alpha^\vee\rangle x$$

for all $x \in \mathcal{L}^\mu_\beta$, $(\beta,\mu) \in \Delta \times \Lambda$;
(v) $\mathcal{L}$ is generated as a k-Lie algebra by the subspaces $\mathcal{L}^\lambda_\alpha$, $(\alpha,\lambda) \in \Delta^\times \times \Lambda$.

If $\Lambda = \mathbb{Z}^n$, then n is called the *nullity* of $\mathcal{L}$.

In what follows we will always assume that $\Lambda = \mathbb{Z}^n$.

By [ABFP, Lemma 1.3.5 and Prop. 1.4.2], if a centerless Lie torus $\mathcal{L}$ with $\Lambda \cong \mathbb{Z}^n$ is finitely generated over its centroid (fgc), then the centroid is isomorphic as a k-algebra to

$$k[\mathbb{Z}^n] \cong k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = R.$$

Note that, according to an announced result of E. Neher [N, Theorem 7(b)], all Lie tori are fgc, except for just one class of Lie tori of type A_n called quantum tori; see [ABFP, Remark 1.4.3].

If a centerless Lie torus $\mathcal{L}$ is fgc, the Realization theorem [ABFP, Theorem 3.3.1] asserts that $\mathcal{L}$ as a Lie algebra over its centroid R is $\mathbb{Z}^n$-graded isomorphic to a multiloop algebra $\mathcal{L}(L,\sigma)$. In particular, the Lie torus $\mathcal{L}$ is a $\widetilde{R}/R$-twisted form of a split simple Lie algebra $L \otimes_k R$. Consequently, the group scheme of R-equivariant automorphisms $\mathrm{Aut}_R(\mathcal{L})$ is a twisted form of $\mathrm{Aut}_R(L \otimes_k R)$, and $\mathrm{Aut}_R(\mathcal{L})^\circ$ is an adjoint simple reductive group over R. Moreover,

$$\mathrm{Lie}(\mathrm{Aut}_R(\mathcal{L})^\circ) \cong \mathcal{L}$$

as Lie algebras over R, *e.g.*, [GP1, Prop. 4.10].

Proof of Theorem 1.3. First we show that the adjoint simple reductive group $G = \mathrm{Aut}_R(\mathcal{L})^\circ$ over R contains a closed R-subgroup $S \cong (\mathbf{G}_{\mathrm{m},R})^r$, where $r = \operatorname{rank}\Delta$. Indeed, the Lie algebra $\mathcal{L}$ over R is Q-graded, where $Q = \mathbb{Z}\Delta$. This grading naturally determines a closed subgroup $S \cong (\mathbf{G}_{\mathrm{m},R})^r$ of $\mathrm{Aut}_R(\mathcal{L})$, where $r = \operatorname{rank}\Delta$. Namely, let $\Pi \subseteq \Delta$ be a system of simple roots, $|\Pi| = r$. For any simple root $\alpha \in \Pi$, any commutative R-algebra R', and any $c \in (R')^\times = \mathbf{G}_\mathrm{m}(R')$, there is a unique automorphism $t_\alpha(c)$ of $\mathcal{L} \otimes_R R'$ such that, for any $\lambda \in \mathbb{Z}^n$, one has

$$t_\alpha(c)(e_\alpha^\lambda) = ce_\alpha^\lambda, \qquad t_\alpha(c)(f_\alpha^\lambda) = c^{-1}f_\alpha^\lambda, \qquad \text{and}$$

$$t_\alpha(c)(e_\beta^\lambda) = e_\beta^\lambda, \qquad t_\alpha(c)(f_\beta^\lambda) = f_\beta^\lambda \qquad \text{for all } \beta \in \Pi,\ \beta \neq \alpha.$$

Clearly, $S \subseteq \mathrm{Aut}_R(\mathcal{L})^\circ$.

Conversely, the grading induced by the adjoint action of S on $\mathrm{Lie}(\mathrm{Aut}_R(\mathcal{L})^\circ) \cong \mathcal{L}$ is exactly the initial Q-grading. The system of simple roots $\Pi \subseteq \Delta$ determines a decomposition $\Delta = \Delta^+ \cup \Delta^- \cup \{0\}$, and by Lemma 2.8 there exist two opposite parabolic R-subgroups $P^+ = U_{\Delta^+\cup\{0\}}$, $P^- = U_{\Delta^-\cup\{0\}}$ of G, and their unipotent radicals are of the form U_{Δ^+} and U_{Δ^-}, respectively. Since $\operatorname{Spec} R$ is connected, the relative roots and relative roots subschemes with respect to $P^\pm$ are defined over $\operatorname{Spec} R$. By Lemma 3.13(iv) the groups $U_{\Delta^\pm}$ are generated by the root elements $X_\alpha(v)$, $\alpha \in \Delta^\pm$, $v \in \mathrm{Lie}(G)_\alpha$. By Example 3.12 we can identify $X_\alpha(v)$ with $\exp(\mathrm{ad}_v)$. Therefore, we have

$$E_{P^+}(R) = \big\langle U_{\Delta^+}(R), U_{\Delta^-}(R)\big\rangle = E_{exp}(\mathcal{L}).$$

Since $\operatorname{rank}\Delta \geq 2$, the group G contains $(\mathbf{G}_{\mathrm{m},R})^2$. It also contains a maximal R-torus, since by [GP2, p. 532] the group G is loop reductive. It remains to apply Theorem 1.2. ∎

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