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EXISTENCE AND CONCENTRATION OF SOLUTION FOR A NON-LOCAL REGIONAL SCHRÖDINGER EQUATION WITH COMPETING POTENTIALS

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Abstract. In this paper, we study the existence and concentration phenomena of solutions for the following non-local regional Schrödinger equation

$$\begin{cases} \epsilon^{2\alpha}(-\Delta)^{\alpha}_{\rho}u + Q(x)u = K(x)|u|^{p-1}u, & \text{in } \mathbb{R}^n,\\ u \in H^{\alpha}(\mathbb{R}^n) \end{cases}$$

where ϵ is a positive parameter, $0 < \alpha < 1$, $1 < p < \frac{n+2\alpha}{n-2\alpha}$, $n > 2\alpha$; $(-\Delta)^{\alpha}_{\rho}$ is a variational version of the regional fractional Laplacian, whose range of scope is a ball with radius $\rho(x) > 0$, ρ, Q, K are competing functions.

1. Introduction. The aim of this paper is to study the existence of ground state solution for a non-linear Schrödinger equation with non-local regional diffusion and competing potentials of the type

$$(P) \qquad \begin{cases} \epsilon^{2\alpha}(-\Delta)^{\alpha}_{\rho}u + Q(x)u = K(x)|u|^{p-1}u, & \text{in } \mathbb{R}^n,\\ u \in H^{\alpha}(\mathbb{R}^n), \end{cases}$$

where $0 < \alpha < 1$, $\epsilon > 0$, $n > 2\alpha$, $Q, K \in C(\mathbb{R}^n, \mathbb{R}^+)$ are bounded and the operator $(-\Delta)^{\alpha}_{\rho}$ is a variational version of the non-local regional fractional Laplacian, with range of scope determined by a positive function $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$, which is defined as

$$\int_{\mathbb{R}^n} (-\Delta)^{\alpha}_{\rho}u(x)\varphi(x)dx = \int_{\mathbb{R}^n}\int_{B(0,\rho(x))} \frac{[u(x+z) - u(z)][\varphi(x+z) - \varphi(x)]}{|z|^{n+2\alpha}}dzdx.$$

Recently, the study of problems involving fractional Schrödinger equations has attracted much attention from many mathematicians. For example, when $(-\Delta)^{\alpha}_{\rho}$ is replaced by $(-\Delta)^{\alpha}$ and $\epsilon = 1$, Cheng [**1**] studied the existence of ground state solution of non-linear fractional Schrödinger equation

$$(-\Delta)^{\alpha}u + V(x)u = |u|^{p-1}u \ \text{ in } \mathbb{R}^n, \tag{1}$$

with unbounded potential. The existence of a ground state of (1) was obtained by using Lagrange multiplier theorem on Nehari manifold. If $V(x) = 1$, Dipierro et al. [**4**] proved existence and symmetry of ground state solutions of (1). Felmer et al. [**5**], studied the same equation with a more general non-linearity $f(x, u)$, they obtained the existence, regularity and qualitative properties of ground states. Secchi [**11**] obtained positive solutions of a more general fractional Schrödinger equation by critical point theory and variational method. When $\epsilon \neq 1$, Chen and Zheng [**2**] showed that when $n = 1, 2, 3$, ϵ is sufficiently small, $\max\{\frac{1}{2}, \frac{n}{4}\} < \alpha < 1$ and Q satisfies some smoothness and boundedness assumptions, the equation (P) has a non-trivial solution u_ϵ concentrated to some single point as $\epsilon \to 0$. In [**3**], Dávila, del Pino and Wei generalized various existence results of (P) with $\alpha = 1$ to the fractional Laplacian. Moreover, we also mention the works by Shang and Zhang [**12, 13**], where it was considered the non-linear fractional Schrödinger equation with competing potentials

$$\epsilon^{2\alpha}(-\Delta)^\alpha u + V(x)u = K(x)|u|^{p-2}u + Q(x)|u|^{q-2}u, \quad x \in \mathbb{R}^n, \tag{2}$$

where $2 < q < p < 2^*_\alpha$. By using perturbative variational method, mountain pass arguments and Nehari manifold method, they analyzed the existence, multiplicity and concentration phenomena of solutions of the equation (2).

On the other hand, research has been done in recent years regarding regional fractional Laplacian, where the scope of the operator is restricted to a variable region near each point. We mention the work by Guan [**8**] and Guan and Ma [**9**] where they studied these operators, their relation with stochastic processes, and the work by Ishii and Nakamura [**10**], where the authors considered the Dirichlet problem for regional fractional Laplacian modelled on the p-Laplacian.

Recently, Felmer and Torres [**6, 7**] established the existence of positive solution for the non-linear Schrödinger equation with non-local regional diffusion

$$\epsilon^{2\alpha}(-\Delta)^\alpha_\rho u + u = f(u), \quad u \in H^\alpha(\mathbb{R}^n), \tag{3}$$

where the operator $(-\Delta)^\alpha_\rho$ is defined as above. Under suitable assumptions on the non-linearity f and the range of scope ρ, they obtained the existence of a ground state solution by mountain pass argument and a comparison method. Furthermore, they analyzed symmetry properties and concentration phenomena of these solutions. These regional operators present various interesting characteristics that make them very attractive from the point of view of mathematical theory of non-local operators. We also mention the recent works by Torres [**14–16**], where existence, multiplicity and symmetry results were considered for related problems.

Motivated by these previous works, in the present paper, we intend to study the existence and concentration behaviour of solutions for (P). We will prove the existence of solutions that concentrate around a global minimum point of the ground state energy function $\xi \mapsto C(\xi)$, where $C(\xi)$ is defined as being the mountain pass level of the energy functional associated with the problem

$$(-\Delta)^\alpha u + Q(\xi)u = K(\xi)|u|^{p-1}u, \quad x \in \mathbb{R}^n,$$

where $\xi \in \mathbb{R}^n$ is regard as a parameter instead of an independent variable. Here, the functions ρ, Q and K satisfy the following conditions:

(H_0) There are positive real numbers Q_∞, K_∞ such that

$$Q_\infty = \lim_{|\xi| \to +\infty} Q(\xi) \quad \text{and} \quad K_\infty = \lim_{|\xi| \to +\infty} K(\xi).$$

(H_1) There are numbers $0 < \rho_0 < \rho_\infty \leq \infty$ such that

$$\rho_0 \leq \rho(\xi) < \rho_\infty, \quad \forall \xi \in \mathbb{R}^n \quad \text{and} \quad \lim_{|\xi| \to \infty} \rho(\xi) = \rho_\infty.$$

(H_2) $Q, K : \mathbb{R}^n \to \mathbb{R}$ are continuous functions satisfying

$$0 < a_1 \leq Q(\xi), K(\xi) \leq a_2 \quad \forall \xi \in \mathbb{R}^n,$$

for some positive constants a_1, a_2.

Before stating our main result, let us introduce more some notations. By considering the change of variable $x \to \epsilon x$, the problem (P) is equivalent to

$$(P') \qquad (-\Delta)^\alpha_{\rho_\epsilon} v + Q(\epsilon x) v = K(\epsilon x)|v|^{p-1} v, \quad x \in \mathbb{R}^n,$$

where $\rho_\epsilon = \frac{1}{\epsilon}\rho(\epsilon x)$. Associated with (P') we have the energy functional $I_{\rho_\epsilon} : H^\alpha(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$I_{\rho_\epsilon}(v) = \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{B(0,\frac{1}{\epsilon}\rho(\epsilon x))} \frac{|v(x+z)-v(x)|^2}{|z|^{n+2\alpha}} + \int_{\mathbb{R}^n} Q(\epsilon x)|v(x)|^2 dx\right) - \frac{1}{p+1}\int_{\mathbb{R}^n} K(\epsilon x)|v(x)|^{p+1} dx,$$

Hereafter, we say that $v \in H^\alpha(\mathbb{R}^n)$ is a weak solution of (P') if v is a critical point of I_{ρ_ϵ}. Moreover, we say that v is a ground state solution of (P') if

$$I'_{\rho_\epsilon}(v) = 0 \quad \text{and} \quad I_{\rho_\epsilon}(v) = C_{\rho_\epsilon},$$

where C_{ρ_ϵ} denotes the mountain pass level associated with I_{ρ_ϵ}.

Now, we are ready to state the main result of this paper:

THEOREM 1.1. *Assume $(H_0) - (H_2)$. Then, if*

$$(C) \qquad \inf_{\xi \in \mathbb{R}^n} C(\xi) < \liminf_{|\xi| \to +\infty} C(\xi),$$

problem (P') has a ground state solution $u_\epsilon \in H^\alpha(\mathbb{R}^n)$ for ϵ small enough. Moreover, for each sequence $\epsilon_m \to 0$, there is a subsequence such that for each $m \in \mathbb{N}$, the solution u_{ϵ_m} concentrates around a minimum point ξ^* of the function $C(\xi)$, in the following sense: given $\delta > 0$, there are $\epsilon_0, R > 0$ such that

$$\int_{B^c(\xi^*, \epsilon_m R)} |u_{\epsilon_m}|^2 dx \leq \epsilon_m^n \delta \quad \text{and} \quad \int_{B(\xi^*, \epsilon_m R)} |u_{\epsilon_m}|^2 dx \geq \epsilon_m^n C, \quad \forall \epsilon_m \leq \epsilon_0,$$

where C is a constant independent of δ and m.

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We would like to point out that the condition (C) is not empty, because it holds by supposing that there is $\xi_0 \in \mathbb{R}^n$ such that

$$(H_3) \qquad \frac{Q(\xi_0)^{\frac{p+1}{p-1}-\frac{n}{2\alpha}}}{K(\xi_0)^{\frac{2}{p-1}}} < \frac{Q_\infty^{\frac{p+1}{p-1}-\frac{n}{2\alpha}}}{K_\infty^{\frac{2}{p-1}}}.$$

For more details, see Corollary 2.1 in Section 3.

2. Preliminary results. The main goal of this section is to study some properties involving the function $\xi \mapsto C(\xi)$, which is the mountain pass level of the functional $J_\xi : H^\alpha(\mathbb{R}^n) \to \mathbb{R}$ given by

$$\begin{aligned} J_\xi(u) = \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n} Q(\xi)|u(x)|^2 dx\right) \\ -\frac{1}{p+1}\int_{\mathbb{R}^n} K(\xi)|u(x)|^{p+1}dx. \end{aligned} \tag{4}$$

By using well-known arguments, $J_\xi \in C^1(H^\alpha(\mathbb{R}^n), \mathbb{R})$ and

$$J'_\xi(u)v = \langle u, v\rangle_\xi - \int_{\mathbb{R}^n} K(\xi)|u(x)|^{p-1}u(x)v(x)dx, \quad \forall\, v \in H^\alpha(\mathbb{R}^n),$$

where

$$\langle u, v\rangle_\xi = \int_{\mathbb{R}^n}\int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n} Q(\xi)uvdx.$$

From this, it is clear that critical points of J_ξ are weak solutions of

$$(-\Delta)^\alpha u + Q(\xi)u = K(\xi)|u|^{p-1}u, \quad x \in \mathbb{R}^n. \tag{5}$$

The same arguments explored in Willem [**17**, Chapter 4] work to prove that

$$0 < C(\xi) = \inf_{u\in\mathcal{N}_\xi} J_\xi(u),$$

where $\mathcal{N}_\xi$ is the Nehari manifold defined by

$$\mathcal{N}_\xi = \{u \in H^\alpha(\mathbb{R}^n) \setminus \{0\} : J'_\xi(u)u = 0\}.$$

Moreover, the characterization below also occur

$$C(\xi) = \inf_{v\in H^\alpha(\mathbb{R}^n)\setminus\{0\}} \max_{t>0} J_\xi(tv) = \inf_{\gamma\in\Gamma_\xi} \max_{t\in[0,1]} J_\xi(\gamma(t)),$$

where

$$\Gamma_\xi = \{\gamma \in C([0,1], H^\alpha(\mathbb{R}^n)) : \quad \gamma(0) = 0, \quad J_\xi(\gamma(1)) < 0\}.$$

By [**5**], we know that (5) has a non-trivial non-negative ground state solution, that is, $C(\xi)$ is the least critical value of J_ξ. Next, we will study the continuity of $C(\xi)$.

LEMMA 2.1. *The function $\xi \to C(\xi)$ is continuous.*

Proof. Let $\{\xi_r\} \subset \mathbb{R}^n$ and $\xi_0 \in \mathbb{R}^n$ verifying

$$\xi_r \to \xi_0 \quad \text{in} \quad \mathbb{R}^n.$$

By using the conditions (H_0) and (H_2), we know that there are $A_1, B_1 > 0$ such that

$$0 < A_1 \leq C(\xi) \leq B_1, \quad \forall \xi \in \mathbb{R}^N,$$

showing that $\{C(\xi_r)\}$ is a bounded sequence. Next, let $v_r \in H^\alpha(\mathbb{R}^n)$ be a function that satisfies

$$J_{\xi_r}(v_r) = C(\xi_r) \quad \text{and} \quad J'_{\xi_r}(v_r) = 0.$$

In the sequel, we will consider two sequences $\{\xi_{r_j}\}$ and $\{\xi_{r_k}\}$ such that

$$C(\xi_{r_j}) \geq C(\xi_0), \quad \forall r_j \tag{I}$$

and

$$C(\xi_{r_k}) \leq C(\xi_0), \quad \forall r_k. \tag{II}$$

Analysis of (I): Using the fact that $\{C(\xi_{r_j})\}$ is bounded, there are $\{\xi_{r_{j_i}}\} \subset \{\xi_{r_j}\}$ and $C_0 > 0$ such that

$$C(\xi_{r_{j_i}}) \to C_0.$$

By using the notations

$$v_i = v_{r_{j_i}} \quad \text{and} \quad \xi_i = \xi_{r_{j_i}},$$

it follows that

$$\xi_i \to \xi_0 \quad \text{and} \quad C(\xi_i) \to C_0.$$

Claim A: $C_0 = C(\xi_0)$.
From (I),

$$\lim_i C(\xi_i) \geq C(\xi_0),$$

and so,

$$C_0 \geq C(\xi_0). \tag{6}$$

Now, we are going to prove that $C_0 \leq C(\xi_0)$. To this end, let $w_0 \in H^\alpha(\mathbb{R}^n)$ be a function satisfying

$$J_{\xi_0}(w_0) = C(\xi_0) \quad \text{and} \quad J'_{\xi_0}(w_0) = 0$$

and $t_i > 0$ be a real number satisfying

$$J_{\xi_i}(t_i w_0) = \max_{t \geq 0} J_{\xi_i}(t w_0).$$

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From definition of $C(\xi_i)$,

$$C(\xi_i) \le J_{\xi_i}(t_i w_0).$$

We claim that $\{t_i\}$ is a bounded sequence. In fact, by definition of t_i we have

$$t_i^2 \|w_0\|_{\xi_i}^2 = t_i^{p+1} \int_{\mathbb{R}^n} K(\xi_i)|w_0(x)|^{p+1} dx. \tag{7}$$

Now for each $i \in \mathbb{N}$, two things can be happen

$$0 < t_i \le 1 \quad \text{or} \quad t_i > 1.$$

We suppose that there is $i_0 > 0$ such that

$$t_i > 1, \quad \forall \ i \ge i_0,$$

otherwise $\{t_i\}$ would be limited. Fixing $\mu \in (2, p+1)$, we derive that

$$\begin{aligned}\int_{\mathbb{R}^n} t_i^{p+1} K(\xi_i)|w_0(x)|^{p+1} dx &\ge \frac{\mu}{p+1} \int_{\mathbb{R}^n} t_i^{p+1} K(\xi_i)|w_0(x)|^{p+1} dx \\ &\ge \frac{\mu}{p+1} \int_{\mathbb{R}^n} t_i^{\mu} K(\xi_i)|w_0(x)|^{p+1} dx.\end{aligned}$$

Consequently,

$$t_i^2 \|w_0\|_{\xi_i}^2 = t_i^{p+1} \int_{\mathbb{R}^n} K(\xi_i)|w_0(x)|^{p+1} dx \ge \frac{\mu}{p+1} \int_{\mathbb{R}^n} t_i^{\mu} K(\xi_i)|w_0(x)|^{p+1} dx,$$

or yet

$$t_i^{\mu-2} \le \frac{(p+1)\|w_0\|_{\xi_i}^2}{\mu \int_{\mathbb{R}^n} K(\xi_i)|w_0(x)|^{p+1} dx} \to \frac{(p+1)\|w_0\|_{\xi_0}^2}{\mu \int_{\mathbb{R}^n} K(\xi_0)|w_0(x)|^{p+1} dx} \quad \text{as} \ \ i \to \infty,$$

which is absurd, because $t_i^{\mu-2} \to +\infty$. Therefore, $\{t_i\}$ be a bounded sequence. Then without loss of generality, we can assume that $t_i \to t_0$. This limit combined with the Lebesgue's Theorem provides

$$\lim_i J_{\xi_i}(t_i w_0) = J_{\xi_0}(t_0 w_0) \le J_{\xi_0}(w_0) = C(\xi_0),$$

leading to

$$C_0 \le C(\xi_0). \tag{8}$$

From (6)–(8),

$$C(\xi_0) = C_0.$$

The above study implies that

$$\lim_i C(\xi_{r_{j_i}}) = C(\xi_0).$$

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Analysis of (II): By the definition of $\{v_r\}$,

$$\left(\frac{1}{2}-\frac{1}{p+1}\right)\|v_r\|_{\xi_r}^2 = J_{\xi_r}(v_r)-\frac{1}{p+1}J'_{\xi_r}(v_r)v_r \le C(\xi_r)+C\|v_r\|_{\xi_r},$$

from where it follows that $\{v_r\}$ is a bounded sequence in $H^\alpha(\mathbb{R}^n)$. Consequently, there is $v_0 \in H^\alpha(\mathbb{R}^n)$ such that

$$v_r \rightharpoonup v_0 \quad \text{in} \quad H^\alpha(\mathbb{R}^n).$$

By using [**6**, Lemma 2.1], we can assume that $v_0 \neq 0$, because for any sequence of the type $\tilde{v}_r(x) = v_r(x+y_r)$ also satisfies

$$J_{\xi_r}(\tilde{v}_r) = C(\xi_r) \quad \text{and} \quad J'_{\xi_r}(\tilde{v}_r) = 0.$$

The above information permits to conclude that v_0 is a non-trivial solution of the problem

$$(-\Delta)^\alpha u + Q(\xi_0)u = K(\xi_0)|u|^{p-1}u \quad \text{in } \mathbb{R}^n. \tag{9}$$

By Fatous' lemma, it is possible to prove that

$$\liminf_r J_{\xi_r}(v_r) \ge J_{\xi_0}(v_0). \tag{10}$$

On the other hand, there is $s_r > 0$ such that

$$C(\xi_r) \le J_{\xi_r}(s_r v_0), \quad \forall r.$$

Thus,

$$\limsup_r J_{\xi_r}(v_r) = \limsup_r C(\xi_r) \le \limsup_r J_{\xi_r}(s_r v_0) = J_{\xi_0}(v_0). \tag{11}$$

From (10)–(11), we get the limit below

$$\lim_r J_{\xi_r}(v_r) = J_{\xi_0}(v_0),$$

which leads to

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v_r(x+z)-v_r(x)|^2}{|z|^{n+2\alpha}}dzdx \to \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v_0(x+z)-v_0(x)|^2}{|z|^{n+2\alpha}}dzdx$$

and

$$\int_{\mathbb{R}^n} Q(\xi_r)|v_r(x)|^2dx \to \int_{\mathbb{R}^n} Q(\xi_{r_0})|v_0(x)|^2dx.$$

Since $v_r \rightharpoonup v_0$ in $H^\alpha(\mathbb{R}^n)$, the above limits ensure that

$$v_r \to v_0 \quad \text{in} \quad H^\alpha(\mathbb{R}^n).$$

On the other hand, as $\{C(\xi_{r_j})\}$ is bounded, there are a subsequence $\{\xi_{r_{j_k}}\} \subset \{\xi_{r_j}\}$ and $C_* > 0$ such that

$$C(\xi_{r_{j_k}}) \to C_*.$$

Setting the notations

$$v_k = v_{r_{j_k}} \quad \text{and} \quad \xi_k = \xi_{r_{j_k}},$$

we have

$$v_k \to v_0, \quad \xi_k \to \xi_0 \quad \text{and} \quad C(\xi_k) \to C_*.$$

In what follows, we denote by $t_k > 0$ the real number that verifies

$$J_{\xi_0}(t_k v_k) = \max_{t \geq 0} J_{\xi_0}(t v_k).$$

Thus, by definition of $C(\xi_0)$,

$$C(\xi_0) \leq J_{\xi_0}(t_k v_k).$$

It is possible to prove that $\{t_k\}$ is a bounded sequence, then without loss of generality, we can assume that $t_k \to t_*$. This limit together with the Lebesgue's Theorem gives

$$\lim_k J_{\xi_0}(t_k v_k) = J_{\xi_0}(t_* v_0) = \lim_k J_{\xi_k}(t_k v_k) \leq \lim_k C(\xi_k) = C_*,$$

implying that

$$C(\xi_0) \leq C_*. \tag{12}$$

On the other hand, from (II),

$$\lim_k C(\xi_k) \leq C(\xi_0),$$

leading to

$$C_* \geq C(\xi_0). \tag{13}$$

From (12)–(13),

$$C_* = C(\xi_0).$$

The above analyze guarantees that

$$\lim_k C(\xi_{n_{j_k}}) = C(\xi_0).$$

From (I) and (II),

$$\lim_r C(\xi_r) = C(\xi_0),$$

showing the lemma. □

In the next lemma, D denotes the mountain level of the functional $J : H^\alpha(\mathbb{R}^n) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |u(x)|^2 dx\right) - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx.$$

LEMMA 2.2. *The function $C(\xi)$ verifies the equality*

$$C(\xi)=\frac{Q(\xi)^{\frac{p+1}{p-1}-\frac{n}{2\alpha}}}{K(\xi)^{\frac{2}{p-1}}}D,\quad \forall\xi\in\mathbb{R}^n. \tag{14}$$

Proof. Let $u\in H^\alpha(\mathbb{R}^n)$ be a function verifying

$$J(u)=D\quad\text{and}\quad J'(u)=0.$$

For each $\xi\in\mathbb{R}^n$ fixed, let $\sigma^{2\alpha}=\frac{1}{Q(\xi)}$ and define

$$w(x)=\left[\frac{Q(\xi)}{K(\xi)}\right]^{\frac{1}{p-1}}u(\frac{x}{\sigma}).$$

A simple change of variable gives

$$\begin{aligned}
J_\xi(w)&=\frac{Q(\xi)}{2}\left(\sigma^{2\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|w(x+z)-w(x)|^2}{|z|^{n+2\alpha}}dzdx+\int_{\mathbb{R}^n}|w|^2dx\right)\\
&-\frac{1}{p+1}\int_{\mathbb{R}^n}K(\xi)|w|^{p+1}dx\\
&=\frac{Q(\xi)^{\frac{p+1}{p-1}}}{K(\xi)^{\frac{2}{p-1}}}\left[\left(\frac{\sigma^{2\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(\frac{x}{\sigma}+\frac{z}{\sigma})-u(\frac{x}{\sigma})|^2}{|z|^{n+2\alpha}}dzdx+\frac{1}{2}\int_{\mathbb{R}^n}|u(\frac{x}{\sigma})|^2dx\right)\right]\\
&-\frac{Q(\xi)^{\frac{p+1}{p-1}}}{K(\xi)^{\frac{2}{p-1}}}\left[\frac{1}{p+1}\int_{\mathbb{R}^n}|u(\frac{x}{\sigma})|^{p+1}dx\right]\\
&=\frac{Q(\xi)^{\frac{p+1}{p-1}-\frac{n}{2\alpha}}}{K(\xi)^{\frac{2}{p-1}}}J(u).
\end{aligned}$$

The same type of argument yields $J'_\xi(w)(w)=0$, from where it follows

$$C(\xi)\le\frac{Q(\xi)^{\frac{p+1}{p-1}-\frac{n}{2\alpha}}}{K(\xi)^{\frac{2}{p-1}}}D,\quad \forall\xi\in\mathbb{R}^n. \tag{15}$$

On the other hand, taking $w\in H^\alpha(\mathbb{R}^n)$ such that

$$J_\xi(w)=C(\xi)\ \text{ and }\ J'_\xi(w)=0.$$

and

$$u(x)=\left[\frac{K(\xi)}{Q(\xi)}\right]^{\frac{1}{p-1}}w(\sigma x),$$

we can show that

$$J(u)\le\frac{K(\xi)^{\frac{2}{p-1}}}{Q(\xi)^{\frac{p+1}{p-1}-\frac{n}{2\alpha}}}J_\xi(w),$$

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that is,

$$\frac{Q(\xi)^{\frac{p+1}{p-1}-\frac{n}{2\alpha}}}{K(\xi)^{\frac{2}{p-1}}} D \leq C(\xi) \ \ \forall \xi \in \mathbb{R}^n. \tag{16}$$

By (15) and (16), we get (14)

□

As a byproduct of the last proof, we have the following corollary

COROLLARY 2.1. *Assume* (H_3). *Then,*

$$\inf_{\xi \in \mathbb{R}^n} C(\xi) < \liminf_{|\xi| \to +\infty} C(\xi) = C(\infty),$$

where $C(\infty)$ *is the mountain pass level of the functionals* $J_\infty : H^\alpha(\mathbb{R}^n) \to \mathbb{R}$ *given by*

$$J_\infty(u) = \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} Q_\infty |u|^2 dx\right) - \frac{1}{p+1}\int_{\mathbb{R}^n} K_\infty |u|^{p+1} dx.$$

3. Ground state solution. By using the studies made in the previous section, we are going to prove that C_{ρ_ϵ} is a critical level for I_{ρ_ϵ} for ϵ small enough, that is, there is $u_\epsilon \in H^\alpha(\mathbb{R}^n)$ satisfying

$$I_{\rho_\epsilon}(u_\epsilon) = C_{\rho_\epsilon} \quad \text{and} \quad I'_{\rho_\epsilon}(u_\epsilon) = 0.$$

The function u_ϵ that verifies the above equality is called a ground state solution of (P').

From now on, we are considering in $H^\alpha(\mathbb{R}^n)$ the following norm

$$\|v\|_{\rho_\epsilon} = \left(\int_{\mathbb{R}^n}\int_{B(0,\rho_\epsilon(x))} \frac{|v(x+z)-v(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} Q(\epsilon x)|v(x)|^2 dx\right)^{\frac{1}{2}},$$

which is equivalent the usual norm of $H^\alpha(\mathbb{R}^n)$, more precisely, there exists a constant $\mathfrak{S} > 0$ independent of ϵ such that

$$\|u\|_{\rho_\epsilon} \leq \|u\| \leq \mathfrak{S}\|u\|_{\rho_\epsilon}, \quad \forall u \in H^\alpha(\mathbb{R}^n). \tag{17}$$

For more details about this subject see [**6**, Proposition 2.1]. This fact combined with the embeddings given in [**6**, Theorem 2.1] ensures that $I_{\rho_\epsilon} \in C^1(H^\alpha(\mathbb{R}^n), \mathbb{R})$ with

$$I'_{\rho_\epsilon}(u)v = \langle u, v\rangle_{\rho_\epsilon} - \int_{\mathbb{R}^n} K(\epsilon x)|u(x)|^{p-1}u(x)v(x)dx, \ \ \forall\, v \in H^\alpha(\mathbb{R}^n),$$

where

$$\langle u, v\rangle_{\rho_\epsilon} = \int_{\mathbb{R}^n}\int_{B(0,\rho_\epsilon(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} Q(\epsilon x) uv dx.$$

Using well-known arguments, it is possible to show that I_{ρ_ϵ} verifies the mountain pass geometry. Then, there is a $(PS)_c$ sequence $\{u_k\} \subset H^\alpha(\mathbb{R}^n)$ such that

$$I_{\rho_\epsilon}(u_k) \to C_{\rho_\epsilon} \quad \text{and} \quad I'_{\rho_\epsilon}(u_k) \to 0, \tag{18}$$

where C_{ρ_ϵ} is the mountain pass level given by

$$C_{\rho_\epsilon} = \inf_{\gamma\in\Gamma_{\rho_\epsilon}} \sup_{t\in[0,1]} I_{\rho_\epsilon}(\gamma(t)) > 0,$$

with

$$\Gamma_{\rho_\epsilon} = \{\gamma \in C([0,1], H^\alpha(\mathbb{R}^n)) : \gamma(0) = 0, \quad I_{\rho_\epsilon}(\gamma(1)) < 0\}.$$

In the sequel, $\mathcal{N}_{\rho_\epsilon}$ denotes the Nehari manifold associated to the functional I_{ρ_ϵ}, that is,

$$\mathcal{N}_{\rho_\epsilon} = \{u \in H^\alpha(\mathbb{R}^n)\backslash\{0\} : \quad I'_{\rho_\epsilon}(u)u = 0\}.$$

It is easy to see that all non-trivial solutions of (P') belongs to $\mathcal{N}_{\rho_\epsilon}$. Moreover, by applying standard arguments, it is possible to prove the equality below

$$C_{\rho_\epsilon} = \inf_{u\in\mathcal{N}_{\rho_\epsilon}} I_{\rho_\epsilon}(u), \tag{19}$$

and the existence of $\beta > 0$, which is independent of ϵ, such that

$$\beta \le \|u\|^2_{\rho_\epsilon}, \quad \forall u \in H^\alpha(\mathbb{R}^n). \tag{20}$$

From (19), if C_{ρ_ϵ} is a critical value of I_{ρ_ϵ} then it is the least energy critical value of I_{ρ_ϵ}.

The next lemma studies the behaviour of function C_{ρ_ϵ} when ϵ goes to 0.

LEMMA 3.1. $\limsup_{\epsilon\to 0} C_{\rho_\epsilon} \le \inf_{\xi\in\mathbb{R}^n} C(\xi)$. *Hence,* $\limsup_{\epsilon\to 0} C_{\rho_\epsilon} < C(\infty)$.

Proof. Fix $\xi_0 \in \mathbb{R}^N$ and $w \in H^\alpha(\mathbb{R}^n)$ with

$$J_{\xi_0}(w) = \max_{t\ge 0} J_{\xi_0}(tw) = C(\xi_0) \quad \text{and} \quad J'_{\xi_0}(w) = 0,$$

where

$$J_{\xi_0}(u) = \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n} Q(\xi_0)|u(x)|^2 dx\right) - \frac{1}{p+1}\int_{\mathbb{R}^n} K(\xi_0)|u|^{p+1}dx.$$

Then, we set $\widehat{w}_\epsilon(x) = w(x - \frac{\xi_0}{\epsilon})$ and $t_\epsilon > 0$ satisfying

$$C_{\rho_\epsilon} \le I_{\rho_\epsilon}(t_\epsilon\widehat{w}_\epsilon) = \max_{t\ge 0} I_{\rho_\epsilon}(t\widehat{w}_\epsilon).$$

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The change of variable $\tilde{x} = x - \frac{\xi_0}{\epsilon}$ gives

$$
\begin{aligned}
I_{\rho_\epsilon}(t_\epsilon \widehat{w}_\epsilon) &= \frac{t_\epsilon^2}{2}\left(\int_{\mathbb{R}^n}\int_{B(0,\frac{1}{\epsilon}\rho(\epsilon x))} \frac{|\widehat{w}_\epsilon(x+z) - \widehat{w}_\epsilon(x)|^2}{|z|^{n+2\alpha}} dxdx + \int_{\mathbb{R}^n} Q(\epsilon x)\widehat{w}_\epsilon^2(x)dx\right)\\
&- \frac{t_\epsilon^{p+1}}{p+1}\int_{\mathbb{R}^n} K(\epsilon x)|\widehat{w}_\epsilon|^{p+1}(x)dx\\
&= \frac{t_\epsilon^2}{2}\left(\int_{\mathbb{R}^n}\int_{B(0,\frac{1}{\epsilon}\rho(\epsilon\tilde{x}+\xi_0))} \frac{|w(\tilde{x}+z) - w(\tilde{x})|^2}{|z|^{n+2\alpha}} dzd\tilde{x} + \int_{\mathbb{R}^n} Q(\epsilon\tilde{x}+\xi_0)w^2(\tilde{x})d\tilde{x}\right)\\
&- \frac{t_\epsilon^{p+1}}{p+1}\int_{\mathbb{R}^n} K(\epsilon\tilde{x}+\xi_0)|w|^{p+1}(\tilde{x})d\tilde{x}.
\end{aligned}
$$

On the other hand, for any sequence $\epsilon_n \to 0$, the equality $I'_{\rho_{\epsilon_n}}(t_{\epsilon_n}\widehat{w}_{\epsilon_n})(t_{\epsilon_n}\widehat{w}_{\epsilon_n}) = 0$ yields $\{t_{\epsilon_n}\}$ is bounded. Thus, we can assume that

$$t_{\epsilon_n} \to t_* > 0,$$

for some $t_* > 0$. Thereby, taking the limit of $n \to +\infty$, we can infer that

$$J'_{\xi_0}(t_* w)(t_* w) = 0.$$

On the other hand, we know that $J'_{\xi_0}(w)(w) = 0$, then we must have

$$t_* = 1.$$

From this,

$$I_{\rho_{\epsilon_n}}(t_{\epsilon_n}\widehat{w}_{\epsilon_n}) \to J_{\xi_0}(w) = C(\xi_0) \ \text{ as } \ \epsilon \to 0.$$

As the point $\xi_0 \in \mathbb{R}^n$ is arbitrary, the lemma is proved. □

THEOREM 3.1. *For $\epsilon > 0$ small enough, the problem (P') has a ground state solution.*

Proof. In what follows, $\{u_k\} \subset H^\alpha(\mathbb{R}^N)$ is a sequence satisfying

$$I_{\rho_\epsilon}(u_k) \to C_{\rho_\epsilon} \quad \text{and} \quad I'_{\rho_\epsilon}(u_k) \to 0.$$

If $u_k \rightharpoonup 0$ in $H^\alpha(\mathbb{R}^N)$, then

$$u_k \to 0 \ \text{ in } \ L^p_{loc}(\mathbb{R}^n) \ \text{ for } \ p \in [2, 2^*_\alpha). \tag{21}$$

By (H_0), we can take $\delta, R > 0$ such that

$$Q_\infty - \delta \le Q(x) \le Q_\infty + \delta \ \text{ and } \ K_\infty - \delta \le K(x) \le K_\infty + \delta, \tag{22}$$

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for all $|x| \geq R$. Then, for all $t \geq 0$,

$$\begin{aligned} I_{\rho_\epsilon}(tu_k) &= I^{\delta}_{\epsilon,\infty}(tu_k) + \frac{t^2}{2}\int_{\mathbb{R}^n}[Q(x) - Q_\infty + \delta]|u_k(x)|^2 dx \\ &+ \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^n}[K_\infty + \delta - K(x)]|u_k(x)|^{p+1}dx \\ &\geq I^{\delta}_{\epsilon,\infty}(tu_k) + \frac{t^2}{2}\int_{B(0,\frac{R}{\epsilon})}[Q(x) - Q_\infty + \delta]|u_k(x)|^2 dx \\ &+ \frac{t^{p+1}}{p+1}\int_{B(0,\frac{R}{\epsilon})}[K_\infty + \delta - K(x)]|u_k(x)|^{p+1}dx, \end{aligned}$$

where

$$\begin{aligned} I^{\delta}_{\epsilon,\infty}(u) = \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{B(0,\frac{1}{\epsilon}\rho(\epsilon x))}\frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}}dxdx + \int_{\mathbb{R}^n}(Q_\infty - \delta)|u(x)|^2 dx\right) \\ - \frac{1}{p+1}\int_{\mathbb{R}^n}(K_\infty + \delta)|u(x)|^{p+1}dx. \end{aligned}$$

In the sequel, we fix $\tau_k > 0$ satisfying

$$I^{\delta}_{\epsilon,\infty}(\tau_k u_k) \geq C(\frac{\rho(\epsilon x)}{\epsilon}, Q_\infty - \delta, K_\infty + \delta),$$

where

$$C(\frac{\rho(\epsilon x)}{\epsilon}, Q_\infty - \delta, K_\infty + \delta) = \inf_{v\in H^\alpha(\mathbb{R}^n)\setminus\{0\}}\sup_{t\geq 0} I^{\delta}_{\epsilon,\infty}(tv).$$

Since $Q(x) - Q_\infty + \delta$, $K_\infty + \delta - K(x)$ are continuous in $B(0, \frac{R}{\epsilon})$, then there exists positive constants C_Q, C_k, such that

$$\begin{aligned} C_{\rho_\epsilon} &\geq C(\frac{\rho(\epsilon x)}{\epsilon}, Q_\infty - \delta, K_\infty + \delta) + \frac{\tau_k^2}{2}\int_{B(0,\frac{R}{\epsilon})}[Q(x) - Q_\infty + \delta]|u_k(x)|^2 dx \\ &+ \frac{\tau_k^{p+1}}{p+1}\int_{B(0,\frac{R}{\epsilon})}[K_\infty + \delta - K(x)]|u_k(x)|^{p+1}dx \\ &\geq C(\frac{\rho(\epsilon x)}{\epsilon}, Q_\infty - \delta, K_\infty + \delta) + \frac{\tau_k^2 C_Q}{2}\int_{B(0,\frac{R}{\epsilon})}|u_k(x)|^2 \\ &+ \frac{\tau_k^{p+1}C_K}{p+1}\int_{B(0,\frac{R}{\epsilon})}|u_k(x)|^{p+1}dx. \end{aligned}$$

Then by (21), taking the limit as $k \to \infty$, and after $\delta \to 0$, we find

$$C_{\rho_\epsilon} \geq C(\frac{\rho(\epsilon x)}{\epsilon}, Q_\infty, K_\infty), \tag{23}$$

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where $C(\frac{\rho(\epsilon x)}{\epsilon}, Q_\infty, K_\infty)$ designates the mountain pass level of the functional

$$I^0_{\epsilon,\infty}(u) = \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{B(0,\frac{1}{\epsilon}\rho(\epsilon x))}\frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n} Q_\infty |u|^2 dx\right) - \frac{1}{p+1}\int_{\mathbb{R}^n} K_\infty |u|^{p+1}dx.$$

Now note that

$$I^0_{\epsilon,\infty}(tu) = J_\infty(tu) - \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(0,\frac{1}{\epsilon}\rho(\epsilon x))}\frac{|tu(x+z)-tu(x)|^2}{|z|^{n+2\alpha}}dzdx,$$

for $t \geq 0$, and we estimate the second term on the right. First, we see that for any $\epsilon > 0$ and $\bar{t}$, there exists $R > 0$ such that

$$\int_{B^c(0,\frac{R}{\epsilon})}\int_{B^c(0,\frac{1}{\epsilon}\rho(\epsilon x))}\frac{|tu(x+z)-tu(x)|^2}{|z|^{n+2\alpha}}dzdx \leq \epsilon, \tag{24}$$

for all $t \in [0, \bar{t}]$. In fact, by our assumption, for any $M > 0$, exists $R > 0$ such that, for $|x| > \frac{R}{\epsilon}$ we have that $\rho(\epsilon x) > M$. From here, using Fubini's theorem we have

$$\begin{aligned}
&\int_{B^c(0,\frac{R}{\epsilon})}\int_{B^c(0,\frac{1}{\epsilon}\rho(\epsilon x))}\frac{|tu(x+z)-tu(x)|^2}{|z|^{n+2\alpha}}dzdx \\
&\leq \int_{B^c(0,\frac{M}{\epsilon})}\int_{B^c(0,\frac{R}{\epsilon})}\frac{|tu(x+z)-tu(x)|^2}{|z|^{n+2\alpha}}dxdz \\
&\leq \int_{B^c(0,\frac{M}{\epsilon})}\int_{\mathbb{R}^n}\frac{|tu(x+z)-tu(x)|^2}{|z|^{n+2\alpha}}dxdz \\
&\leq \frac{2\bar{t}^2|S^{n-1}|}{\alpha M^{2\alpha}}\|u\|^2_{L^2(\mathbb{R}^n)}\epsilon^{2\alpha},
\end{aligned}$$

from were we conclude (24) choosing $R > 0$ large enough. From now on we fix $R > 0$ large enough. Next, we prove that

$$\lim_{\epsilon\to 0}\int_{B(0,\frac{R}{\epsilon})}\int_{B^c(0,\frac{1}{\epsilon}\rho(\epsilon x))}\frac{|tu(x+z)-tu(x)|^2}{|z|^{n+2\alpha}}dzdx = 0, \tag{25}$$

for all $t \in [0, \bar{t}]$. In fact, by (H_1) there exists $\rho_0 > 0$ such that $\rho(\epsilon x) \geq \rho_0$ for all $x \in \mathbb{R}^n$, so that

$$\begin{aligned}
&\int_{B(0,\frac{R}{\epsilon})}\int_{B^c(0,\frac{1}{\epsilon}\rho(\epsilon x))}\frac{|tu(x+z)-tu(x)|^2}{|z|^{n+2\alpha}}dzdx \\
&\leq \int_{B^c(0,\frac{\rho_0}{\epsilon})}\int_{B(0,\frac{R}{\epsilon})}\frac{|tu(x+z)-tu(x)|^2}{|z|^{n+2\alpha}}dxdz \leq \frac{2\bar{t}^2|S^{n-1}|}{\alpha\rho_0^{2\alpha}}\|u\|^2_{L^2(B(0,\frac{R}{\epsilon}))}\epsilon^{2\alpha},
\end{aligned} \tag{26}$$

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and we obtain (25) by (26). Thus, by (24) and (26)

$$I^0_{\epsilon,\infty}(tu) \geq J_\infty(tu) - \epsilon - \int_{B(0,\frac{R}{\epsilon})}\int_{B^c(0,\frac{1}{\epsilon}\rho(\epsilon x))} \frac{|tu(x+z)-tu(x)|^2}{|z|^{n+2\alpha}}dzdx.$$

Now let $\tilde{u} \in H^\alpha(\mathbb{R}^n)$ such that $I^0_{\epsilon,\infty}(\tilde{u}) = C(\frac{\rho(\epsilon x)}{\epsilon}, Q_\infty, K_\infty)$, then, if we choose $t = t^*$ such that $J_\infty(t^*\tilde{u}) = \max_{t\geq 0} J_\infty(t\tilde{u})$ then we see that

$$C(\frac{\rho(\epsilon x)}{\epsilon}, Q_\infty, K_\infty) \geq C(\infty) - \epsilon,$$

then

$$\liminf_{\epsilon\to 0} C(\frac{\rho(\epsilon x)}{\epsilon}, Q_\infty, K_\infty) \geq C(\infty).$$

Therefore, if there is sequence $\epsilon_n \to 0$ such that the $(\mathrm{PS})_{C_{\rho_{\epsilon_n}}}$ sequence has weak limit equal to zero, we must have

$$C_{\rho_{\epsilon_n}} \geq C(\frac{\rho(\epsilon_n x)}{\epsilon_n}, Q_\infty, K_\infty), \quad \forall n \in \mathbb{N},$$

leading to

$$\liminf_{n\to+\infty} C_{\rho_{\epsilon_n}} \geq C(\infty),$$

which contradicts Lemma 3.1. This proves that the weak limit is non-trivial for $\epsilon > 0$ small enough and standard arguments show that its energy is equal to C_{ρ_ϵ}, showing the desired result. □

4. Concentration of the solutions u_ϵ.

LEMMA 4.1. *If u_ϵ is the ground state solution of (P') obtained in the last section, then there exists a family $\{y_\epsilon\} \subset \mathbb{R}^n$ and positive constants R and β_1 such that*

$$\liminf_{\epsilon\to 0^+} \int_{B(y_\epsilon,R)} |u_\epsilon|^2\, dx \geq \beta_1 > 0. \tag{27}$$

Proof. First of all, by (H_1) and (H_2),

$$I_{\rho_\epsilon}(v) \geq I_*(v) = \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{B(0,\rho_0)} \frac{|v(x+z)-v(x)|^2}{|z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n} a_1|v|^2dx\right) - \frac{1}{p+1}\int_{\mathbb{R}^n} a_2|u|^{p+1}dx, \quad \forall v \in H^\alpha(\mathbb{R}^n).$$

Since there exists unique $t_\epsilon > 0$ such that

$$t_\epsilon u_\epsilon \in \mathcal{N}_* = \{v \in H^\alpha(\mathbb{R}^n)\setminus\{0\} : \;\; I'_*(v)v = 0\},$$

it follows that

$$0 < C(\rho_0, a_1, a_2) = \inf_{v\in\mathcal{N}_*} I_*(v) \leq I_*(t_\epsilon u_\epsilon) \leq I_{\rho_\epsilon}(t_\epsilon u_\epsilon) \leq I_{\rho_\epsilon}(u_\epsilon) = C_{\rho_\epsilon}. \tag{28}$$

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Now, arguing by contradiction, if (27) does not hold, it would exist a sequence $u_k = u_{\epsilon_k}$ such that

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^n} \int_{B(y,R)} |u_k|^2 dx = 0.$$

By [**6**, Lemma 2.1], $v_k \to 0$ in $L^q(\mathbb{R}^n)$ for any $2 < q < 2^*_\alpha$. However, this is impossible, because by (28)

$$\begin{aligned} 0 < C(\rho_0, a_1, a_2) \le C_{\rho_\epsilon} &= I_{\rho_\epsilon}(v_\epsilon) - \frac{1}{2} I'_{\rho_\epsilon}(v_\epsilon)v_\epsilon \\ &= \frac{p-1}{2(p+1)} \int_{\mathbb{R}^n} K(\epsilon x)|v_\epsilon|^{p+1} dx \\ &\le \frac{p-1}{2(p+1)} \int_{\mathbb{R}^n} a_2 |v_\epsilon|^{p+1} dx \to 0, \quad \text{as } k \to \infty. \end{aligned}$$

□

From now on, we set

$$w_\epsilon(x) = u_\epsilon(x + y_\epsilon). \tag{29}$$

Then, by (27),

$$\liminf_{\epsilon\to 0^+} \int_{B(0,R)} |w_\epsilon|^2 dx \ge \beta_1 > 0. \tag{30}$$

To continue, we consider the rescaled scope function $\overline{\rho}_\epsilon$ defined by

$$\bar{\rho}_\epsilon(x) = \frac{1}{\epsilon}\rho(\epsilon x + \epsilon y_\epsilon).$$

Using this function, it follows that w_ϵ is a solution of the equation

$$(-\Delta)^{\alpha}_{\overline{\rho}_\epsilon} w_\epsilon(x) + Q(\epsilon x + \epsilon y_\epsilon) w_\epsilon(x) = K(\epsilon x + \epsilon y_\epsilon)|w_\epsilon(x)|^{p-1} w_\epsilon(x), \quad \text{in } \mathbb{R}^n. \tag{31}$$

LEMMA 4.2. *The sequence $\{\epsilon y_\epsilon\}$ is bounded. Moreover, if $\epsilon_m y_{\epsilon_m} \to \xi^*$, then*

$$C(\xi^*) = \inf_{\xi\in\mathbb{R}^n} C(\xi).$$

Proof. Suppose by contradiction that $|\epsilon_m y_{\epsilon_m}| \to \infty$ and consider the function w_{ϵ_m} given in (29), which satisfies (31). Since $\{C_{\rho_{\epsilon_m}}\}$ is bounded, the sequence $\{w_m\}$ is also bounded in $H^\alpha(\mathbb{R}^n)$. Then, $w_m \rightharpoonup w$ in $H^\alpha(\mathbb{R}^n)$, and $w \neq 0$ by Lemma 4.1. Now, by (31), we have the equality

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{B(0, \frac{1}{\epsilon_m}\rho(\epsilon_m x + \epsilon_m y_{\epsilon_m}))} \frac{[w_m(x+z) - w_m(x)][w(x+z) - w(x)]}{|z|^{n+2\alpha}} dz dx \\ &\quad + \int_{\mathbb{R}^n} Q(\epsilon_m x + \epsilon_m y_{\epsilon_m}) w_m w dx = \int_{\mathbb{R}^n} K(\epsilon_m x + \epsilon_m y_{\epsilon_m}) |w_m|^{p-1} w_m w dx. \end{aligned}$$

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This equality combines with Fatou's Lemma to give

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|w(x+z)-w(x)|^2}{|z|^{n+2\alpha}}dzdx+\int_{\mathbb{R}^n}Q_\infty|w|^2dx\le\int_{\mathbb{R}^n}K_\infty|w|^{p+1}dx. \tag{32}$$

Let $\theta > 0$ such that

$$J_\infty(\theta w)=\max_{t\ge 0}J_\infty(tw).$$

From (32), $\theta\in(0,1]$. Thus,

$$\begin{aligned}C(\infty)&\le J_\infty(\theta w)-\frac{1}{2}J'_\infty(\theta w)\theta w=\left(\frac{1}{2}-\frac{1}{p+1}\right)\theta^{p+1}\int_{\mathbb{R}^n}K_\infty|w(x)|^{p+1}dx\\ &\le\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}K_\infty|w(x)|^{p+1}dx\\ &\le\left(\frac{1}{2}-\frac{1}{p+1}\right)\liminf_{m\to\infty}\int_{\mathbb{R}^n}K(\epsilon_m x+\epsilon_m y_{\epsilon_m})|w_m(x)|^{p+1}dx\\ &=\liminf_{m\to\infty}C_{\rho_{\epsilon_m}}<C(\infty),\end{aligned}$$

which is a contradiction, showing that $\{\epsilon y_\epsilon\}$ is bounded. Hence, there exists a subsequence of $\{\epsilon y_\epsilon\}$ such that $\epsilon_m y_{\epsilon_m}\to\xi^*$.

Repeating the above arguments for the function

$$w_m(x)=v_{\epsilon_m}(x+y_{\epsilon_m})=u_{\epsilon_m}(\epsilon_m x+\epsilon_m y_{\epsilon_m}),$$

we have that this function satisfies the equation (31), and again $\{w_m\}$ is bounded in $H^\alpha(\mathbb{R}^n)$. Then, $w_m\rightharpoonup w$ in $H^\alpha(\mathbb{R}^n)$ and w satisfies the equation below

$$(-\Delta)^\alpha w+Q(\xi^*)w=K(\xi^*)|w|^{p-1}w,\quad x\in\mathbb{R}^n, \tag{33}$$

in the weak sense. Furthermore, associated with (33), we have the energy functional

$$\begin{aligned}J_{\xi^*}(u)=&\frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}dzdx+\int_{\mathbb{R}^n}Q(\xi^*)|u(x)|^2dx\right)\\ &-\frac{1}{p+1}\int_{\mathbb{R}^n}K(\xi^*)|u(x)|^{p+1}dx.\end{aligned}$$

Using w as a test function in (31) and taking the limit of $m\to+\infty$, we get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|w(x+z)-w(x)|^2}{|z|^{n+2\alpha}}dzdx+\int_{\mathbb{R}^n}Q(\xi^*)|w|^2dx\le\int_{\mathbb{R}^n}K(\xi^*)|w|^{p+1}dx,$$

which implies that there exists $\theta \in (0, 1]$ such that $J_{\xi^*}(\theta w) = \max_{t\geq 0} J_{\xi^*}(tw)$. So, by Lemma 3.1,

$$\begin{aligned} C(\xi^*) \leq J_{\xi^*}(\theta w) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\theta^{p+1}\int_{\mathbb{R}^n} K(\xi^*)|w(x)|^{p+1}dx \\ &\leq \left(\frac{1}{2} - \frac{1}{p+1}\right)\liminf_{m\to\infty}\int_{\mathbb{R}^n} K(\epsilon_m x + \epsilon_m y_{\epsilon_m})|w_m(x)|^{p+1}dx \\ &= \liminf_{m\to\infty}[I_{\rho_{\epsilon_m}}(v_{\epsilon_m}) - I'_{\rho_{\epsilon_m}}(v_{\epsilon_m})v_{\epsilon_m}] \\ &= \liminf_{m\to\infty} C_{\rho_{\epsilon_m}} \leq \limsup_{m\to\infty} C_{\rho_{\epsilon_m}} \leq \inf_{\xi\in\mathbb{R}^n} C(\xi), \end{aligned}$$

showing that $C(\xi^*) = \inf_{\xi\in\mathbb{R}^n} C(\xi)$. □

Now we prove the convergence of w_ϵ as $\epsilon \to 0$.

LEMMA 4.3. *For every sequence $\{\epsilon_m\}$ there is a subsequence, we keep calling the same, so that $w_{\epsilon_m} = w_m \to w$ in $H^\alpha(\mathbb{R}^n)$.*

Proof. Since w is a solution of (33), from Lemma 3.1,

$$\begin{aligned} \inf_{\xi\in\mathbb{R}^n} C(\xi) = C(\xi^*) \leq J_{\xi^*}(w) &= J_{\xi^*}(w) - \frac{1}{2}J'_{\xi^*}(w)w \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} K(\xi^*)|w|^{p+1}dx \\ &\leq \left(\frac{1}{2} - \frac{1}{p+1}\right)\liminf_{m\to\infty}\int_{\mathbb{R}^n} K(\epsilon_m x + \epsilon_m y_{\epsilon_m})|w_m|^{p+1}dx \\ &\leq \left(\frac{1}{2} - \frac{1}{p+1}\right)\limsup_{m\to\infty}\int_{\mathbb{R}^n} K(\epsilon_m x + \epsilon_m y_{\epsilon_m})|w_m|^{p+1}dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\limsup_{m\to\infty}\int_{\mathbb{R}^n} K(\epsilon_m x)|v_m|^{p+1}dx \\ &\leq \limsup_{m\to\infty}\left(I_{\rho_{\epsilon_m}}(v_m) - \frac{1}{p+1}I'_{\rho_{\epsilon_m}}(v_m)v_m\right) \\ &= \limsup_{m\to\infty} C_{\overline{\rho}_{\epsilon_m}} \leq \inf_{\xi\in\mathbb{R}^n} C(\xi). \end{aligned}$$

The above inequalities lead to

$$\lim_{m\to\infty}\int_{\mathbb{R}^n} K(\epsilon_m x + \epsilon_m y_{\epsilon_m})|w_m|^{p+1}dx = \int_{\mathbb{R}^n} K(\xi^*)|w|^{p+1}dx.$$

Consequently,

$$(a)\ \lim_{m\to\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|w_m(x+z) - w_m(x)|^2}{|z|^{n+2\alpha}}dzdx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|w(x+z) - w(x)|^2}{|z|^{n+2\alpha}}dzdx$$

$$(b)\ \lim_{m\to\infty}\int_{\mathbb{R}^n} Q(\epsilon_m x + \epsilon_m y_{\epsilon_m})|w_m(x)|^2dx = \int_{\mathbb{R}^n} Q(\xi^*)|w(x)|^2dx.$$

From (b), given $\delta > 0$ there exists $R > 0$ such that

$$\int_{|x|\geq R} Q(\epsilon_m x + \epsilon_m y_{\epsilon_m})|w_m(x)|^2 dx \leq \delta.$$

This together with (H_2) gives

$$\int_{|x|\geq R} |w_m(x)|^2 dx \leq \frac{\delta}{a_1}. \tag{34}$$

On the other hand,

$$\lim_{m\to\infty} \int_{|x|\leq R} |w_m(x)|^2 dx = \int_{|x|\leq R} |w(x)|^2 dx. \tag{35}$$

From (34) and (35), $w_m \to w$ in $L^2(\mathbb{R}^n)$. From this, given $\delta > 0$ there are $\epsilon_0, R > 0$ such that

$$\int_{B^c(x^*,\epsilon_m R)} |u_{\epsilon_m}|^2 \, dx \leq \epsilon_m^n \delta \quad \text{and} \quad \int_{B(x^*,\epsilon_m R)} |u_{\epsilon_m}|^2 \, dx \geq \epsilon_m^n C, \quad \forall \epsilon_m \leq \epsilon_0,$$

where C is a constant independent of δ and m, showing the concentration of the solutions $\{u_{\epsilon_n}\}$. □

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REFERENCES

1. M. Cheng, Bound state for the fractional Schrödinger equation with unbounded potential, *J. Math. Phys.* **53** (2012), 043507.

2. G. Chen and Y. Zheng, Concentration phenomenon for fractional nonlinear Schrödinger equations, *Comm. Pure Appl. Anal.* **13**(6) (2014), 2359–2376.

3. J. Dávila, M. Del Pino and J. Wei, Concentrating standing waves for the fractional nonlinear Schrödinger equation, *J. Differ. Equ.* **256** (2014), 858–892.

4. S. Dipierro, G. Palatucci and E. Valdinoci Valdinoci, Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian, *Matematiche* **68** (2013), 201–216.

5. P. Felmer, A. Quaas and J. Tan, Positive solutions of nonlinear Schrödinger equation with the fractional laplacian, *Proc. Edinburgh: Sect. A Math.* **142**(6) (2012), 1237–1262.

6. P. Felmer and C. Torres, Non-linear Schrödinger equation with non-local regional diffusion, *Calc. Var. Partial Diff. Equ.* **54** (2015), 75–98.

7. P. Felmer and C. Torres, Radial symmetry of ground states for a regional fractional nonlinear Schrödinger equation, *Comm. Pure Appl. Anal.* **13** (2014), 2395–2406.

8. Q.-Y. Guan, Integration by parts formula for regional fractional Laplacian, *Commun. Math. Phys.* **266** (2006), 289–329.

9. Q.-Y. Guan and Z. M. Ma, The reflected α-symmetric stable processes and regional fractional Laplacian. *Probab. Theory Relat. Fields* **134** (2006), 649–694.

10. H. Ishii and G. Nakamura, A class of integral equations and approximation of p-Laplace equations, *Calc. Var.* **37** (2010), 485–522.

11. S. Secchi, Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^n$, *J. Math. Phys.* **54** (2013), 031501.

12. X. Shang and J. Zhang, Concentrating solutions of nonlinear fractional Schrödinger equation with potentials, *J. Differ. Equ.* **258** (2015), 1106–1128.

13. X. Shang and J. Zhang, Existence and multiplicity solutions of fractional Schrödinger equation with competing potential functions, *Complex Variables Elliptic Equ.* **61** (2016), 1435–1463.

14. C. Torres, Symmetric ground state solution for a non-linear Schrödinger equation with non-local regional diffusion, *Complex Variables Elliptic Equ.*, http://dx.doi.org/10.1080/17476933.2016.1178730 (2016)

15. C. Torres, Multiplicity and symmetry results for a nonlinear Schrödinger equation with non-local regional diffusion, *Math. Meth. Appl. Sci.* **39** (2016), 2808–2820.

16. C. Torres, Nonlinear Dirichlet problem with non local regional diffusion, *Fract. Cal. Appl. Anal.* **19**(2) (2016), 379–393.

17. M. Willem, *Minimax theorems* (Birkhäuser, Boston, Basel, Berlin, 1996).