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ELEMENTS OF SPECTRAL THEORY FOR GENERALIZED DERIVATIONS II: THE SEMI-FREDHOLM DOMAIN

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1. Introduction. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ denote infinite dimensional Hilbert spaces and let $\mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ denote the space of all bounded linear operators from $\mathscr{H}_2$ to $\mathscr{H}_1$. For A in $\mathscr{L}(\mathscr{H}_1)$ and B in $\mathscr{L}(\mathscr{H}_2)$, let τ_{AB} denote the operator on $\mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ defined by $\tau_{AB}(X) = AX - XB$. The purpose of this note is to characterize the semi-Fredholm domain of τ_{AB} (Corollary 3.16). Section 3 also contains formulas for $\operatorname{ind}(\tau_{AB} - \lambda)$. These results depend in part on a decomposition theorem for Hilbert space operators corresponding to certain "singular points" of the semi-Fredholm domain (Theorem 2.2). Section 4 contains a particularly simple formula for $\operatorname{ind}(\tau_{AB} - \lambda)$ (in terms of spectral and algebraic invariants of A and B) for the case when $\tau_{AB} - \lambda$ is Fredholm (Theorem 4.2). This result is used to prove that $\operatorname{ind}(\tau_{BA}) = -\operatorname{ind}(\tau_{AB})$ (Corollary 4.3). We also prove that when A and B are bi-quasi-triangular, then the semi-Fredholm domain of τ_{AB} contains no points corresponding to non-zero indices.

Note that if τ_{AB} is semi-Fredholm, then so is the operator $T = \tau_{AB} + S + K$, where $K \in \mathscr{L}(\mathscr{L}(\mathscr{H}_2, \mathscr{H}_1))$ is any compact operator and $S \in \mathscr{L}(\mathscr{L}(\mathscr{H}_2, \mathscr{H}_1))$ is any operator of sufficiently small norm; moreover, in this case, $\operatorname{ind}(T) = \operatorname{ind}(\tau_{AB})$ (see [**17**, Theorems 5.17, 5.26, Chapter IV]). Our results thus yield data that is pertinent to the study of the operator equation $AX - XB + S(X) + K(X) = Y$ (cf. [**18**] [**24**]). (An example of such a compact operator K is provided by C. K. Fong and A. Sourour [**14**]: if $\{A_i\}_{1 \leqq i \leqq n}$ and $\{B_i\}_{1 \leqq i \leqq n}$ denote sequences of compact operators on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, then $K(X) = \sum_{i=1}^{n} A_i X B_i$ defines a compact operator on $\mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ [**14**, Theorem 2].)

Before stating our principal result in detail, we recall some terminology. Let $\mathscr{X}$ denote an infinite dimensional complex Banach space and let $\mathscr{L}(\mathscr{X})$ denote the algebra of all bounded linear operators on $\mathscr{X}$. For $T \in \mathscr{L}(\mathscr{X})$, let $\sigma(T)$, $\sigma_l(T)$, $\sigma_r(T)$ denote, respectively, the spectrum, left spectrum, and right spectrum of T. Let $\rho(T) = \mathbf{C}\backslash\sigma(T)$, $\rho_l(T) = \mathbf{C}\backslash\sigma_l(T)$, and $\rho_r(T) = \mathbf{C}\backslash\sigma_r(T)$ denote, respectively, the resolvent, left

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resolvent, and right resolvent sets of T. For $\mathscr{X} = \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ and $T = \tau_{AB}$, Rosenblum's Theorem states that

$$\sigma(\tau_{AB}) = \sigma(A) - \sigma(B) \equiv \{\alpha - \beta: \alpha \in \sigma(A), \beta \in \sigma(B)\};$$

thus τ_{AB} is invertible if and only if A and B have disjoint spectra [**24**].

Various results in the literature show that the fine structure of $\sigma(\tau_{AB})$ depends on separation properties of $\sigma(A)$ and $\sigma(B)$. Let $\sigma_\pi(T)$ and $\sigma_\delta(T)$ denote, respectively, the approximate point spectrum and the approximate defect spectrum of $T \in \mathscr{L}(\mathscr{X})$, i.e.,

$$\sigma_\delta(T) = \{\lambda \in \sigma(T): T - \lambda \text{ is not surjective}\} \quad [\mathbf{7}].$$

(For T in $\mathscr{L}(\mathscr{H})$ ($\mathscr{H}$ a Hilbert space), $\sigma_l(T) = \sigma_\pi(T)$ and $\sigma_r(T) = \sigma_\delta(T)$.) In [**7**], C. Davis and P. Rosenthal proved that

$$\sigma_\pi(\tau_{AB}) = \sigma_l(A) - \sigma_r(B) \quad \text{and} \quad \sigma_\delta(\tau_{AB}) = \sigma_r(A) - \sigma_l(B).$$

Thus τ_{AB} is bounded below (resp. surjective) if and only if $\sigma_l(A) \cap \sigma_r(B) = \emptyset$ (resp. $\sigma_r(A) \cap \sigma_l(B) = \emptyset$); additionally, $\sigma_\delta(\tau_{AB}) = \sigma_r(\tau_{AB})$ and $\sigma_\pi(\tau_{AB}) = \sigma_l(\tau_{AB})$ [**10**]. Moreover, for the case $\mathscr{H}_1 = \mathscr{H}_2$, the range of τ_{AB} is dense in $\mathscr{L}(\mathscr{H}_1)$ if and only if $\sigma_{re}(A) \cap \sigma_{le}(B) = \emptyset$ and there exists no nonzero trace class operator $X \in \mathscr{L}(\mathscr{H}_1)$ such that $BX = XA$ [**11**] (see below for notation).

We next recall some results on semi-Fredholm operators and essential spectra. For a Banach space operator $T \in \mathscr{L}(\mathscr{X})$, let $\ker(T)$ and $\mathscr{R}(T)$ denote the kernel and range of T. Let

$$\text{nul}(T) = \dim(\ker(T)) \quad \text{and} \quad \text{def}(T) = \dim(\mathscr{X}/\mathscr{R}(T)^-)$$

(where $\mathscr{R}(T)^-$ denotes the norm closure of the range of T in $\mathscr{X}$) [**17**]. An operator T is *semi-Fredholm* if $\mathscr{R}(T)$ is closed and either $\text{nul}(T) < \infty$ or $\text{def}(T) < \infty$; in this case the index of T is defined by $\text{ind}(T) = \text{nul}(T) - \text{def}(T)$. If $\text{nul}(T)$ and $\text{def}(T)$ are both finite, then T is *Fredholm*. Let $\sigma_e(T)$ denote the Fredholm essential spectrum of T, i.e.,

$$\sigma_e(T) = \{\lambda \in \mathbf{C}: T - \lambda \text{ is not Fredholm}\}.$$

In [**12**] it was proved that

$$\sigma_e(\tau_{AB}) = (\sigma_e(A) - \sigma(B)) \cup (\sigma(A) - \sigma_e(B));$$

thus τ_{AB} is Fredholm if and only if

$$\sigma_e(A) \cap \sigma(B) = \sigma(A) \cap \sigma_e(B) = \emptyset.$$

Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{K}(\mathscr{H})$ denote the ideal of all compact operators in $\mathscr{L}(\mathscr{H})$. Let $\mathscr{C} = \mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ denote the Calkin algebra and let $\pi: \mathscr{L}(\mathscr{H}) \to \mathscr{C}$ denote the canonical projection; we denote $\pi(T)$ by $\tilde{T}$. Thus, for $T \in \mathscr{L}(\mathscr{H})$, $\sigma_e(T) = \sigma(\tilde{T})$. Let $\sigma_{le}(T) =$

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$\sigma_l(\tilde{T})$ and $\sigma_{re}(T) = \sigma_r(\tilde{T})$. If $\rho_{le}(T) = \mathbf{C}\backslash\sigma_{le}(T)$, then

$$\begin{aligned}&\rho_{le}(T)\\ &= \{\lambda \in \mathbf{C}: T - \lambda \text{ is semi-Fredholm and } \operatorname{ind}(T) < +\infty\}\\ &= \{\lambda \in \mathbf{C}: \mathscr{R}(T - \lambda) \text{ is closed and } \operatorname{nul}(T - \lambda) < \infty\}\end{aligned}$$

[**20**, Proposition 1.20]. Similarly, if $\rho_{re}(T) = \mathbf{C}\backslash\sigma_{re}(T)$, then

$$\begin{aligned}&\rho_{re}(T)\\ &= \{\lambda \in \mathbf{C}: T - \lambda \text{ is semi-Fredholm and } \operatorname{ind}(T - \lambda) > -\infty\}\\ &= \{\lambda \in \mathbf{C}: \mathscr{R}(T - \lambda) \text{ is closed and } \operatorname{def}(T - \lambda) < \infty\}\end{aligned}$$

[**20**, Proposition 1.20]. Let

$$\rho_{SF}(T) = \{\lambda \in \mathbf{C}: T - \lambda \text{ is semi-Fredholm}\},$$

the semi-Fredholm domain of T; thus

$$\rho_{SF}(T) = \rho_{le}(T) \cup \rho_{re}(T).$$

A *hole* H in $\sigma_e(T)$ is a bounded component of $\mathbf{C}\backslash\sigma_e(T)$ [**20**, page 2]; if $\lambda \in H \cap \sigma(T)$ and λ is not isolated in $\sigma(T)$, then $H \subset \sigma(T)$. If $\lambda \in H \cap \sigma(T)$ and λ is isolated in $\sigma(T)$, then $\operatorname{ind}(T - \lambda) = 0$ (see [**20**, page 4]). A *pseudohole* H in $\sigma_e(T)$ is a component of $\sigma_e(T)\backslash\sigma_{le}(T)$ or $\sigma_e(T)\backslash\sigma_{re}(T)$ [**20**, page 2]; thus $\operatorname{ind}(T - \lambda) = -\infty$ for all $\lambda \in H$ or $\operatorname{ind}(T - \lambda) = +\infty$ for all $\lambda \in H$, respectively; in particular, $H \subset \sigma(T)$. Note that if λ is an isolated point of $\sigma(T)$ and $T - \lambda$ is semi-Fredholm, then $T - \lambda$ is Fredholm and $\operatorname{ind}(T - \lambda) = 0$.

Let $\rho_{SF}(\tau_{AB})$ denote the semi-Fredholm domain of τ_{AB}, i.e.,

$$\rho_{SF}(\tau_{AB}) = \{\lambda \in \mathbf{C}: \tau_{AB} - \lambda \text{ is semi-Fredholm}\}.$$

Our principal result is that

$$\begin{aligned}&\sigma_{SF}(\tau_{AB}) \equiv \mathbf{C}\backslash\rho_{SF}(\tau_{AB})\\ &= [(\sigma_{le}(A) - \sigma_r(B)) \cup (\sigma_l(A) - \sigma_{re}(B))]\\ &\ \cap [(\sigma_{re}(A) - \sigma_l(B)) \cup (\sigma_r(A) - \sigma_{le}(B))]\end{aligned}$$

(Corollary 3.16). Thus τ_{AB} is semi-Fredholm if and only if

$$\sigma_{re}(A) \cap \sigma_l(B) = \sigma_r(A) \cap \sigma_{le}(B) = \emptyset$$

or

$$\sigma_{le}(A) \cap \sigma_r(B) = \sigma_l(A) \cap \sigma_{re}(B) = \emptyset$$

(Corollary 3.15).

We conclude this section by recording several results from [**12**] that will be cited frequently in the sequel. For $T \in \mathscr{L}(\mathscr{H})$, let

$$\mathscr{U}(T) = \{U^*TU: UU^* = U^*U = 1\},$$

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the unitary orbit of T. The first result is essentially due to C. Apostol [**2**, Lemma 2.2].

LEMMA 1.1. [**12**, Lemma 2.10] i) *If* $T \in \mathscr{L}(\mathscr{H})$ *and* $0 \in \sigma_l(T)$, *then either* $\mathrm{nul}(T) > 0$ *or* $0 \in \sigma_{le}(T)$; *if* $0 \in \sigma_{le}(T)$, *then there exists* $S \in \mathscr{U}(T)^-$ *such that* $\mathrm{nul}(S) = \infty$. ii) *If* $0 \in \sigma_r(T)$, *then either* $\mathrm{nul}(T^*) > 0$ *or* $0 \in \sigma_{re}(T)$; *if* $0 \in \sigma_{re}(T)$, *then there exists* $S \in \mathscr{U}(T)^-$ *such that* $\mathrm{nul}(S^*) = \infty$.

For $S, T \in \mathscr{L}(\mathscr{H})$, S and T are *approximately similar* ($S \sim_a T$) if there exists a sequence of invertible operators $\{X_n\} \subset \mathscr{L}(\mathscr{H})$ such that

$$\sup_n \|X_n\| < \infty, \sup_n \|X_n^{-1}\| < \infty, \quad \text{and} \quad X_n^{-1}TX_n \to S \quad [\mathbf{15}].$$

LEMMA 1.2. [**12**, Proposition 2.8] *If* $A' \sim_a A$ *and* $B' \sim_a B$, *then* τ_{AB} *is semi-Fredholm if and only if* $\tau_{A'B'}$ *is semi-Fredholm, and in this case*

$$\mathrm{nul}(\tau_{AB}) = \mathrm{nul}(\tau_{A'B'}),$$
$$\mathrm{def}(\tau_{AB}) = \mathrm{def}(\tau_{A'B'}), \quad \textit{and}$$
$$\mathrm{ind}(\tau_{AB}) = \mathrm{ind}(\tau_{A'B'}).$$

LEMMA 1.3. [**12**, Lemma 2.9] $\mathrm{nul}(\tau_{AB}) = \mathrm{nul}(\tau_{B^*A^*})$, $\mathrm{def}(\tau_{AB}) = \mathrm{def}(\tau_{B^*A^*})$. τ_{AB} *is semi-Fredholm if and only if* $\tau_{B^*A^*}$ *is, in which case* $\mathrm{ind}(\tau_{AB}) = \mathrm{ind}(\tau_{B^*A^*})$.

2. Spectral decomposition. In this section we obtain a decomposition for Hilbert space operators corresponding to certain isolated points of the left or right spectrum. We begin by recalling the Riesz Decomposition Theorem [**22**, Theorem, p. 421]. For T in $\mathscr{L}(\mathscr{H})$, suppose $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are disjoint nonempty closed subsets of $\sigma(T)$. Then there exist nontrivial complementary closed T-invariant subspaces $\mathscr{M}_1$ and $\mathscr{M}_2$ such that

$$\sigma(T|\mathscr{M}_1) = \sigma_1 \quad \text{and} \quad \sigma(T|\mathscr{M}_2) = \sigma_2.$$

Moreover, T is similar to an operator of the form $T_1 \oplus T_2$, where $\sigma(T_1) = \sigma_1$ and $\sigma(T_2) = \sigma_2$ [**12**, Lemma 2.14]. For the case when $\sigma_1 = \{\lambda\}$ ($\lambda \in \mathbf{C}$) and $T - \lambda$ is Fredholm, we have the following result.

LEMMA 2.1. *Let* T *be in* $\mathscr{L}(\mathscr{H})$ *and let* $\lambda \in \mathbf{C}$ *be an isolated point of* $\sigma(T)$ *such that* $T - \lambda$ *is Fredholm. Then there exists an orthogonal decomposition* $\mathscr{H} = \mathscr{M}_1 \oplus \mathscr{M}_2$ *and operators* $T_i \in \mathscr{L}(\mathscr{M}_i)$ $(i = 1, 2)$ *such that:*

1) $\dim \mathscr{M}_1 < \infty$;
2) $\sigma(T_1) = \{\lambda\}$;
3) $\lambda \notin \sigma(T_2)$;
4) *T is similar to* $T_1 \oplus T_2$.

Proof. The existence of $\mathcal{M}_1, \mathcal{M}_2, T_1$, and T_2 satisfying properties 2), 3), and 4) follows directly from the preceding discussion. If $\mathcal{M}_1$ is infinite dimensional, then

$$\lambda \in \sigma_e(T_1) \subset \sigma_e(T_1 \oplus T_2) = \sigma_e(T),$$

which contradicts the assumption that $T - \lambda$ is Fredholm. Thus $\mathcal{M}_1$ is finite dimensional and the proof is complete.

Let $\mathcal{H}_1$ denote a finite dimensional Hilbert space and let $\mathcal{H}_2$ denote an arbitrary infinite dimensional Hilbert space. Let $N \in \mathcal{L}(\mathcal{H}_1)$ be nilpotent and let $S \in \mathcal{L}(\mathcal{H}_2)$ be left invertible. (Note that $S - \lambda$ is left invertible for $|\lambda|$ sufficiently small.) Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and let $T = N \oplus S$ in $\mathcal{L}(\mathcal{H})$; T clearly satisfies the following properties: 1) $\tilde{T}$ is left invertible, i.e., $\mathcal{R}(T)$ is closed and $\operatorname{nul}(T) < \infty$; 2) 0 is an isolated point of $\sigma_l(T)$. We next prove that (up to similarity) each operator satisfying 1) and 2) may be decomposed as just described; thus the following result is the analogue of Lemma 2.1 for left spectra.

At the conclusion of the section we give an alternate proof of this result using a decomposition theorem of C. Apostol [**1**]. The proof given below is considerably simpler than the proof of Apostol's more general result.

In the remainder of this section, $\mathcal{H}$ denotes an arbitrary infinite dimensional Hilbert space.

THEOREM 2.2. *Let T be in $\mathcal{L}(\mathcal{H})$ and let $\lambda \in \mathbf{C}$ be an isolated point of $\sigma_l(T)$ such that $\tilde{T} - \lambda$ is left invertible (i.e., $T - \lambda$ is semi-Fredholm and* $\operatorname{ind}(T - \lambda) < +\infty$). *Then there exists an orthogonal decomposition $\mathcal{H} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and operators $T_i \in \mathcal{L}(\mathcal{M}_i)$ $(i = 1, 2)$ such that:*

1) $\dim(\mathcal{M}_1) < \infty$;
2) $\sigma(T_1) = \{\lambda\}$;
3) $\lambda \notin \sigma_l(T_2)$;
4) *T is similar to $T_1 \oplus T_2$.*

Before proving Theorem 2.2 we introduce some additional notation. For T in $\mathcal{L}(\mathcal{H})$, let

$$\mathcal{M}(T) = \{x \in \mathcal{H}: \|T^n x\|^{1/n} \to 0\}.$$

$\mathcal{M}(T)$ is a (not necessarily closed) linear submanifold of $\mathcal{H}$ that is invariant for T; moreover, T is quasinilpotent (i.e., $\sigma(T) = \{0\}$) if and only if $\mathcal{M}(T) = \mathcal{H}$ [**6**, Lemma, page 28]. Suppose that λ is an isolated point of $\sigma(T)$ such that $T - \lambda$ is Fredholm. Then $\mathcal{M}_\lambda \equiv \mathcal{M}(T - \lambda)$ coincides with the Riesz subspace for T corresponding to $\{\lambda\}$ [**22**, Theorem ff., page 424]. In particular, $\mathcal{M}_\lambda$ is closed, and since $T - \lambda$ is Fredholm, then $\dim(\mathcal{M}_\lambda) < \infty$. It follows that $(T - \lambda)| M_\lambda$ is nilpotent, so for some $n > 0$,

$$\mathcal{M}(T - \lambda) = \{x \in \mathcal{H}: (T - \lambda)^n x = 0\}.$$

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Proof of Theorem 2.2. Without loss of generality we may assume that $\lambda = 0$. If 0 is an isolated point of $\sigma(T)$, then since T is semi-Fredholm, it follows that T is Fredholm (with $\operatorname{ind}(T) = 0$). In this case the result follows from Lemma 2.1.

We may thus assume that 0 is not isolated in $\sigma(T)$. Thus, since $0 \notin \sigma_{le}(T)$, it follows that there exists a hole or pseudohole H in $\sigma_e(T)$ such that $0 \in H \subset \sigma(T)$. Moreover, since 0 is isolated in $\sigma_l(T)$, there exists $\delta > 0$ such that $T - \alpha$ is left invertible but not invertible for $0 < |\alpha| < \delta$. (Thus $\operatorname{ind}(T - \alpha) < 0$ for each $\alpha \in H$.)

Let $\mathcal{M} = \mathcal{M}(T)^-$. $\mathcal{M}$ is T-invariant, and since $0 \in \sigma_l(T)\backslash\sigma_{le}(T)$, then $\operatorname{nul}(T) > 0$ and so $\mathcal{M} \supset \ker(T) \neq \{0\}$. We seek to prove that $\mathcal{M}$ is finite dimensional (from which it will also follow that $\mathcal{M} \neq \mathcal{H}$). Assume to the contrary that $\mathcal{M}$ is infinite dimensional. Relative to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, the operator matrix of T is of the form $\begin{pmatrix} N & A \\ O & S \end{pmatrix}$, where $N \in \mathcal{L}(\mathcal{M})$. (At this point in the proof we allow the possibility that $\mathcal{M}^\perp = \{0\}$, in which case $T = N$; in the sequel we will conclude that this case cannot actually arise.) Since $\mathcal{M}(T) \subset \mathcal{M}$, a matrix calculation shows that $\mathcal{M}(N) = \mathcal{M}(T)$. Thus $\mathcal{M}(N)^- = \mathcal{M}$, and it follows that if $\mathcal{N}$ is a reducing subspace for N, then $\mathcal{M}(N|\mathcal{N})^- = \mathcal{N}$.

For $0 < |\alpha| < \delta$, $T - \alpha$ has a left inverse L_α, and an operator matrix calculation of $L_\alpha(T - \alpha) = 1$ readily implies that $N - \alpha$ has a left inverse in $\mathcal{L}(\mathcal{M})$. Suppose there exists α, $0 < |\alpha| < \delta$, such that $\alpha \in \sigma(N)$. Thus $\alpha \in \sigma(N)\backslash\sigma_l(N)$ and so there exists $x_0 \in \mathcal{M}$, $x_0 \neq 0$, such that $(N - \alpha)^* x_0 = 0$. Let $\mathcal{N}$ denote the closed reducing subspace for N generated by $\{p(N, N^*)x_0 \colon p(t, s)$ is a noncommutative polynomial in t and s with complex coefficients which have rational real and imaginary parts$\}$; let $N_1 = N|\mathcal{N}$. Then $\mathcal{M}(N_1)^- = \mathcal{N}$, and since $\mathcal{N}$ is separable, it follows from [**3**, Theorem] that there exists a (compact) quasinilpotent operator $K \in \mathcal{L}(\mathcal{N})$ and a quasiaffinity $X \in \mathcal{L}(\mathcal{N})$ such that $N_1X = XK$. Now $(N_1 - \alpha)X = X(K - \alpha)$, and thus

$$(K - \alpha)^* X^* x_0 = X^*(N_1 - \alpha)^* x_0 = 0.$$

Since K is quasinilpotent, $(K - \alpha)^*$ is invertible, and since X^* is injective, it follows that $x_0 = 0$, which is a contradiction.

Thus $\{\alpha \in \mathbf{C} \colon 0 < |\alpha| < \delta\} \subset \rho(N)$; since $\operatorname{nul}(N) = \operatorname{nul}(T) > 0$, it follows that 0 is an isolated point of $\sigma(N)$. Since $\tilde{T}$ is left invertible, a matrix calculation implies that $\tilde{N}$ is left invertible, and since 0 is isolated in $\sigma(N)$, it follows that N is Fredholm (with $\operatorname{ind}(N) = 0$). Now $\mathcal{M}(N)$ is the (closed) Riesz subspace for N corresponding to $\{0\} \subset \sigma(N)$, and thus

$$\mathcal{M} = \mathcal{M}(T)^- = \mathcal{M}(N)^- = \mathcal{M}(N) = \mathcal{M}(T).$$

Since $\mathcal{M}$ is infinite dimensional, so is $\mathcal{M}(N)$, and since $\sigma(N|\mathcal{M}(N)) =$

$\{0\}$, it follows that $0 \in \sigma_e(N)$. Thus N is not Fredholm, which is a contradiction. We therefore conclude that $\mathscr{M}$ is finite dimensional and, in particular, $\mathscr{M}^\perp \neq \{0\}$. Since $\dim(\mathscr{M}) < \infty$, and $\sigma(N) = \{0\}$, it also follows that N is nilpotent.

We next prove that S is left invertible in $\mathscr{L}(\mathscr{M}^\perp)$. Since T is semi-Fredholm and $\mathscr{M}$ is finite dimensional, it follows that $0_{\mathscr{M}} \oplus S$ is semi-Fredholm, whence $\mathscr{R}(S)$ is closed. It thus suffices to prove that $\operatorname{nul}(S) = 0$. If $\ker(S) \neq \{0\}$, let $\mathscr{M}_1 = \ker(S)$ and $\mathscr{M}_2 = \mathscr{M}^\perp \ominus \mathscr{M}_1$. With respect to the decomposition

$$\mathscr{H} = \mathscr{M} \oplus \mathscr{M}_1 \oplus \mathscr{M}_2,$$

the operator of T is of the form

$$\begin{pmatrix} N & A_1 & A_2 \\ 0 & 0 & V \\ 0 & 0 & W \end{pmatrix}.$$

Let $k > 0$ be such that $N^k = 0$; then

$$\begin{pmatrix} N & A_1 \\ 0 & 0 \end{pmatrix}^{k+1} = 0,$$

and thus the matrix of T^{k+1} is of the form

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & W^{k+1} \end{pmatrix}.$$

If $x \in \mathscr{M}_1$, $x \neq 0$, then $T^{k+1}x = 0$, whence

$$x \in \mathscr{M}_1 \cap \mathscr{M} = \{0\},$$

and this contradiction implies that S is left invertible.

Since $\sigma(N) = \{0\}$ and $0 \notin \sigma_l(S)$, then $\sigma_r(N) \cap \sigma_l(S) = \emptyset$. It follows from [**7**, Theorem 5] that there exists $X \in \mathscr{L}(\mathscr{M}^\perp, \mathscr{M})$ such that $NX - XS = -A$. Let J denote the operator on $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp$ whose matrix is $\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}$. Then $J^{-1} = \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix}$ and a calculation shows that $J^{-1}TJ = N \oplus S$. The proof is now complete (with $\mathscr{M}_1 = \mathscr{M}$, $M_2 = \mathscr{M}^\perp$, $T_1 = N$, $T_2 = S$).

COROLLARY 2.3. *Let T be in $\mathscr{L}(\mathscr{H})$ and let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\sigma_l(T) \backslash \sigma_{le}(T)$ such that λ_i is isolated in $\sigma_l(T)$, $i = 1, \ldots, n$. Then there exists an orthogonal decomposition $\mathscr{H} = \mathscr{M}_1 \oplus \ldots \oplus \mathscr{M}_{n+1}$ and operators $T_i \in \mathscr{L}(\mathscr{M}_i)$ $(1 \leqq i \leqq n+1)$ such that:*

1) *$\mathscr{M}_i$ is finite dimensional $(1 \leqq i \leqq n)$;*
2) *$\sigma(T_i) = \{\lambda_i\}$ $(1 \leqq i \leqq n)$;*
3) *$\sigma_l(T_{n+1}) \cap \{\lambda_1, \ldots, \lambda_n\} = \emptyset$;*
4) *T is similar to $T_1 \oplus \ldots \oplus T_{n+1}$.*

Proof. Theorem 2.2 implies the result for $n = 1$. Assuming the result for $n - 1$, there exists an orthogonal decomposition

$$\mathscr{H} = \mathscr{M}_1 \oplus \ldots \oplus \mathscr{M}_{n-1} \oplus \mathscr{M}_n'$$

and operators $T_i \in \mathscr{L}(\mathscr{M}_i)$ $(1 \leqq i \leqq n-1)$ and $T' \in \mathscr{L}(\mathscr{M}_n')$ such that $\mathscr{M}_i$ is finite dimensional $(1 \leqq i \leqq n-1)$,

$$\sigma(T_i) = \{\lambda_i\} \quad (1 \leqq i \leqq n-1),$$
$$\sigma_l(T') \cap \{\lambda_1, \ldots, \lambda_{n-1}\} = \emptyset,$$

and T is similar to $T_1 \oplus \ldots \oplus T_{n-1} \oplus T'$.

Since $\lambda_n \in \sigma_l(T) \backslash \sigma_{le}(T)$ and $\lambda_n \neq \lambda_i$ for $i \neq n$, it follows that

$$\lambda_n \in \sigma_l(T') \backslash \sigma_{le}(T').$$

Moreover, since $\sigma_l(T') \subset \sigma_l(T)$, then λ_n is isolated in $\sigma_l(T')$. Theorem 2.2 implies that there exists an orthogonal decomposition $\mathscr{M}_n' = \mathscr{M}_n \oplus \mathscr{M}_{n+1}$, and operators $T_i \in \mathscr{L}(\mathscr{M}_i)$, $i = n, n+1$, such that $\mathscr{M}_n$ is finite dimensional, $\sigma(T_n) = \{\lambda_n\}$, $\lambda_n \notin \sigma_l(T_{n+1})$, and T' is similar to $T_n \oplus T_{n+1}$. Since $\sigma_l(T_{n+1}) \subset \sigma_l(T')$, then

$$\{\lambda_1, \ldots, \lambda_{n-1}\} \cap \sigma_l(T_{n+1}) = \emptyset;$$

moreover, T is similar to $T_1 \oplus \ldots \oplus T_{n+1}$, so the proof is complete.

COROLLARY 2.4. *Let T be in $\mathscr{L}(\mathscr{H})$ and let $\lambda_1, \ldots, \lambda_n$ be distinct points of $\sigma_r(T) \backslash \sigma_{re}(T)$ such that λ_i is isolated in $\sigma_r(T)$ $(1 \leqq i \leqq n)$. Then there exists an orthogonal decomposition $\mathscr{H} = \mathscr{M}_1 \oplus \ldots \oplus \mathscr{M}_{n+1}$ and operators $T_i \in \mathscr{L}(\mathscr{M}_i)$ $(1 \leqq i \leqq n+1)$ such that:*

1) *$\mathscr{M}_i$ is finite dimensional $(1 \leqq i \leqq n)$;*
2) *$\sigma(T_i) = \{\lambda_i\}$ $(1 \leqq i \leqq n)$;*
3) *$\sigma_r(T_{n+1}) \cap \{\lambda_1, \ldots, \lambda_n\} = \emptyset$;*
4) *T is similar to $T_1 \oplus \ldots \oplus T_{n+1}$.*

Proof. Apply Corollary 2.3 to T^* and $\bar{\lambda}_1, \ldots, \bar{\lambda}_n$, and then take adjoints.

COROLLARY 2.5. *Let T be in $\mathscr{L}(\mathscr{H})$ and let $\alpha_1, \ldots, \alpha_n$ be distinct points in $\sigma_r(T) \backslash \sigma_{re}(T)$ such that α_i is isolated in $\sigma_r(T)$ $(1 \leqq i \leqq n)$; moreover, let $\beta_1, \ldots, \beta_m$ be distinct points in $\sigma_l(T) \backslash \sigma_{le}(T)$ such that β_j is isolated in $\sigma_l(T)$ $(1 \leqq j \leqq m)$. Assume that $\alpha_i \neq \beta_j$, $1 \leqq i \leqq n$, $1 \leqq j \leqq m$. Then there exists an orthogonal decomposition $\mathscr{H} = \mathscr{M}_1 \oplus \ldots \oplus \mathscr{M}_{n+m+1}$ and operators $T_i \in \mathscr{L}(\mathscr{M}_i)$ $(1 \leqq i \leqq n+m+1)$ such that:*

1) *$\mathscr{M}_i$ is finite dimensional $(1 \leqq i \leqq n+m)$;*
2) *$\sigma(T_i) = \{\alpha_i\}$ $(1 \leqq i \leqq n)$; $\sigma_r(T_{n+m+1}) \cap \{\alpha_1, \ldots, \alpha_n\} = \emptyset$;*
3) *$\sigma(T_i) = \{\beta_j\}$ $(n+1 \leqq j \leqq n+m)$; $\sigma_l(T_{n+m+1}) \cap \{\beta_1, \ldots, \beta_m\} = \emptyset$;*
4) *T is similar to $T_1 \oplus \ldots \oplus T_{n+m+1}$.*

Proof. Apply Corollary 2.3 and Corollary 2.4 successively.

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We conclude this section by presenting another proof of Theorem 2.2 based on results of [**1**].

Let T be in $\mathcal{L}(\mathcal{H})$. A complex number μ is said to be a *T-singular point* if the function $\lambda \to P_{\ker(T-\lambda)}$ is discontinuous at μ; μ is said to be *T-regular* if μ is not T-singular. Let

$$\rho_{SF}{}^s(T) = \{\mu \in \rho_{SF}(T) : \mu \text{ is } T\text{-singular}\}$$

and let

$$\rho_{SF}{}^r(T) = \rho_{SF}(T) \backslash \rho_{SF}{}^s(T) = \{\mu \in \rho_{SF}(T) : \mu \text{ is } T\text{-regular}\}.$$

For $\mu \in \rho_{SF}(T)$, $\mu \in \rho_{SF}{}^r(T)$ if and only if the function

$$\lambda \to \min \{\text{nul } (T - \lambda), \text{nul } ((T - \lambda)^*)\}$$

is continuous at μ [**1**, Proposition 2.6]. In [**1**], C. Apostol proved the following result.

THEOREM 2.6. ([**1**, Theorem 3.3]) *Let T be in $\mathcal{L}(\mathcal{H})$ and let $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ be a finite subset of $\rho_{SF}{}^s(T)$. Then there exist complementary closed T-invariant subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ such that:*

1) *$\dim (\mathcal{N}_1) = \sum_{i=1}^n \dim (\mathcal{M}(T - \lambda_i))$ $(< \infty)$;*

2) *$\sigma(T|\mathcal{N}_1) = \sigma$;*

3) *$\dim (\mathcal{M}((T - \lambda_i)|\mathcal{N}_1)) = \dim (\mathcal{M}(T - \lambda_i))$ $(1 \leqq i \leqq n)$;*

4) *$\rho_{SF}{}^r(T|\mathcal{N}_2) = \rho_{SF}{}^r(T) \cup \sigma$; in particular, $\lambda_1, \ldots, \lambda_n$ are regular points of $\rho_{SF}(T|\mathcal{N}_2)$.*

To prove Theorem 2.2, suppose that $\lambda \in \mathbf{C}$ is an isolated point of $\sigma_l(T)$ and $\widetilde{T - \lambda}$ is left invertible; clearly $\lambda \in \rho_{SF}{}^s(T)$. From Theorem 2.6, there exist complementary closed T-invariant subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ such that

$$\dim (\mathcal{N}_1) = \dim (\mathcal{M}(T - \lambda)) < \infty,$$
$$\sigma(T|\mathcal{N}_1) = \{\lambda\},$$

and λ is regular for $T|\mathcal{N}_2$. Since λ is isolated in $\sigma_l(T)$, it follows that $T|\mathcal{N}_2$ is left invertible. Let J denote the (bounded) idempotent whose range is $\mathcal{N}_1$ and null space is $\mathcal{N}_2$. There exists an invertible operator X and an orthogonal projection P such that $J = X^{-1}PX$. Since $T|\mathcal{N}_1$ is similar to $S_1 \equiv XTX^{-1}|P\mathcal{H}$ and $T|\mathcal{N}_2$ is similar to $S_2 = XTX^{-1}|(1 - P)\mathcal{H}$, the result follows.

3. The semi-Fredholm domain of τ_{AB}. In this section we characterize the semi-Fredholm domain of τ_{AB} and give formulas for ind $(\tau_{AB} - \lambda)$ $(\lambda \in \rho_{SF}(\tau_{AB}))$. Unless otherwise noted, $\mathcal{H}_1$ and $\mathcal{H}_2$ are arbitrary infinite dimensional Hilbert spaces, $A \in \mathcal{L}(\mathcal{H}_1)$, and $B \in$

$\mathcal{L}(\mathcal{H}_2)$. The first two lemmas are essentially excerpts from the proof of [**12**, Theorem 3.1], which we include for the sake of completeness.

LEMMA 3.1. *If* $(\sigma_{le}(A) \cap \sigma_r(B)) \cup (\sigma_l(A) \cap \sigma_{re}(B)) \neq \emptyset$ *and* τ_{AB} *is semi-Fredholm, then* nul $(\tau_{AB}) = \infty$.

Proof. Suppose $\lambda \in (\sigma_{le}(A) \cap \sigma_r(B)) \cup (\sigma_l(A) \cap \sigma_{re}(B))$; since $\tau_{AB} = \tau_{A-\lambda, B-\lambda}$, we may assume that $\lambda = 0$. Lemma 1.1 implies that there exist $A' \in \mathcal{U}(A)^-$ and $B' \in \mathcal{U}(B)^-$ such that nul $(A') > 0$, nul $(B'^*) > 0$, and nul $(A') = \infty$ or nul $(B'^*) = \infty$. Since τ_{AB} is semi-Fredholm, then $\tau_{A'B'}$ is semi-Fredholm and nul $(\tau_{A'B'})$ = nul (τ_{AB}) (Lemma 1.2). Relative to the decomposition

$$\mathcal{H}_1 = \ker(A') \oplus (\ker(A'))^\perp \text{ and } \mathcal{H}_2 = \ker(B'^*) \oplus \mathcal{R}(B')^-,$$

the operator matrices of A', B', and $X \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ are of the form

$$A' = \begin{pmatrix} 0 & A_{12} \\ 0 & A_{22} \end{pmatrix}, B' = \begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \end{pmatrix}, \quad \text{and } X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

A matrix calculation shows that if X_{12}, X_{21}, and X_{22} are zero operators, then $A'X - XB' = 0$. Thus

$$\text{nul}(\tau_{AB}) = \text{nul}(\tau_{A'B'}) \geqq \dim(\mathcal{L}(\ker(B'^*), \ker(A'))).$$

Since ker (A') and ker (B'^*) are nontrivial and at least one is infinite dimensional, then $\mathcal{L}$ (ker (B'^*), ker (A')) is infinite dimensional and the result follows.

LEMMA 3.2. *If* $(\sigma_{re}(A) \cap \sigma_l(B)) \cup (\sigma_r(A) \cap \sigma_{le}(B)) \neq \emptyset$ *and* τ_{AB} *is semi-Fredholm, then* def $(\tau_{AB}) = \infty$.

Proof. As in the proof of Lemma 3.1, and by virtue of Lemma 1.1 and Lemma 1.2, we may assume $0 \in (\sigma_{re}(A) \cap \sigma_l(B)) \cup (\sigma_r(A) \cap \sigma_{le}(B))$, nul $(A^*) > 0$, nul $(B) > 0$, and nul $(A^*) = \infty$ or nul $(B) = \infty$. With respect to the decompositions $\mathcal{H}_1$ = ker $(A^*) \oplus \mathcal{R}(A)^-$ and $\mathcal{H}_2$ = ker $(B) \oplus (\ker(B))^\perp$, the matrices of A, B, and $X \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ are of the form

$$\begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} 0 & B_{12} \\ 0 & B_{22} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

The matrix of $AX - XB$ is of the form $\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$. Thus

$$\text{def}(\tau_{AB}) \geqq \dim(\mathcal{L}(\ker(B), \ker(A^*))) = \infty.$$

COROLLARY 3.3. *If*

$$(\sigma_{le}(A) \cap \sigma_r(B)) \cup (\sigma_l(A) \cap \sigma_{re}(B)) \neq \emptyset$$

and

$$(\sigma_{re}(A) \cap \sigma_l(B)) \cup (\sigma_r(A) \cap \sigma_{le}(B)) \neq \emptyset,$$

then τ_{AB} *is not semi-Fredholm.*

Proof. The result follows from Lemma 3.1 and Lemma 3.2.

COROLLARY 3.4.

$$\sigma \equiv [(\sigma_{le}(A) - \sigma_r(B)) \cup (\sigma_l(A) - \sigma_{re}(B))] \cap [(\sigma_{re}(A) - \sigma_l(B)) \cup (\sigma_r(A) - \sigma_{le}(B))] \subset \sigma_{SF}(\tau_{AB}).$$

Proof. If z is in σ, then $z = \alpha - \beta$, where $\alpha \in \sigma_{le}(A)$, $\beta \in \sigma_r(B)$ or $\alpha \in \sigma_l(A)$, $\beta \in \sigma_{re}(B)$. Now

$$\tau_{AB} - z = \tau_{A-\alpha, B-\beta} \quad \text{and}$$
$$0 \in (\sigma_{le}(A - \alpha) \cap \sigma_r(B - \beta)) \cup (\sigma_l(A - \alpha) \cap \sigma_{re}(B - \beta)).$$

Thus if $\tau_{AB} - z$ is semi-Fredholm, it follows from Lemma 3.1 that

$$\text{nul } (\tau_{AB} - z) = \infty.$$

Similarly, $z = \gamma - \lambda$, where $\gamma \in \sigma_{re}(A)$, $\lambda \in \sigma_l(B)$ or $\gamma \in \sigma_r(A)$, $\lambda \in \sigma_{le}(B)$. Since

$$\tau_{AB} - z = \tau_{A-\gamma, B-\lambda} \quad \text{and}$$
$$0 \in (\sigma_{re}(A - \gamma) \cap \sigma_l(B - \lambda)) \cup (\sigma_r(A - \gamma) \cap \sigma_{le}(B - \lambda)),$$

Lemma 3.2 implies that if $\tau_{AB} - z$ is semi-Fredholm, then

$$\text{def } (\tau_{AB} - z) = \infty.$$

Thus if $\tau_{AB} - z$ is semi-Fredholm, then

$$\text{nul } (\tau_{AB} - z) = \text{def } (\tau_{AB} - z) = \infty,$$

a contradiction which implies that $z \in \sigma_{SF}(\tau_{AB})$.

In the sequel we will prove the converse of Corollary 3.4, thereby characterizing $\sigma_{SF}(\tau_{AB})$. We require several preliminary results, the first of which will prove useful in calculating ind $(\tau_{AB} - z)$.

LEMMA 3.5. *Let* $\mathscr{H}_1$ *be a finite dimensional Hilbert space and let* $\mathscr{H}$ *be an infinite dimensional Hilbert space. For* $A \in \mathscr{L}(\mathscr{H})$ *define* $\tau_A \in \mathscr{L}(\mathscr{L}(\mathscr{H}_1, \mathscr{H}))$ *by* $\tau_A(X) = AX$. *If A is semi-Fredholm, then so is* τ_A, *and*

$$\text{nul } (\tau_A) = \dim (\mathscr{H}_1) \text{ nul } (A),$$
$$\text{def } (\tau_A) = \dim (\mathscr{H}_1) \text{ def } (A), \textit{ and}$$
$$\text{ind } (\tau_A) = \dim (\mathscr{H}_1) \text{ ind } (A).$$

Proof. Since A has closed range, so does τ_A [**12**, Lemma 3.6]. Note that

$$\ker (\tau_A) = \{X \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}) : \mathscr{R}(X) \subset \ker (A)\} \cong \mathscr{L}(\mathscr{H}_1, \ker (A)),$$

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and thus

$$\text{nul}\ (\tau_A) = \dim\ (\mathcal{L}(\mathcal{H}_1, \ker\ (A))) = \dim\ (\mathcal{H}_1)\ \text{nul}\ (A).$$

We next show that

$$\mathcal{R}(\tau_A) = \{Y \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}) : \mathcal{R}(Y) \subset \mathcal{R}(A)\} \cong \mathcal{L}(\mathcal{H}_1, \mathcal{R}(A)).$$

Indeed, if $Y = \tau_A(X) = AX$, then clearly $\mathcal{R}(Y) = \mathcal{R}(AX) \subset \mathcal{R}(A)$. Conversely, suppose $Y \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $\mathcal{R}(Y) \subset \mathcal{R}(A)$; by a straightforward modification of a factorization theorem of R. G. Douglas [**8**, Theorem 1], there exists $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ such that $Y = AX \in \mathcal{R}(\tau_A)$.

Thus $\mathcal{R}(\tau_A) \cong \mathcal{L}(\mathcal{H}_1, \mathcal{R}(A))$ and we claim that

$$\mathcal{Q} \equiv \mathcal{L}(\mathcal{H}_1, \mathcal{H})/\mathcal{L}(\mathcal{H}_1, \mathcal{R}(A)) \cong \mathcal{L}(\mathcal{H}_1, \mathcal{R}(A)^\perp).$$

Let P denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(A)$. For $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$, let $[X]$ denote the image of X in $\mathcal{Q}$; thus

$$[X] = [(1-P)X] + [PX] = [(1-P)X].$$

Define f: $\mathcal{Q} \to \mathcal{L}(\mathcal{H}_1, \mathcal{R}(A)^\perp)$ by $f([X]) = (1-P)X$. It follows readily that f is a well defined linear isomorphism onto $\mathcal{L}(\mathcal{H}_1, \mathcal{R}(A)^\perp)$, and thus

$$\text{def}\ (\tau_A) = \dim\ (\mathcal{Q}) = \dim\ (\mathcal{L}(\mathcal{H}_1, \mathcal{R}(A)^\perp)) = \dim\ (\mathcal{H}_1)\ \text{def}\ (A).$$

Finally,

$$\text{ind}\ (\tau_A) = \text{nul}\ (\tau_A) - \text{def}\ (\tau_A) = \dim\ (\mathcal{H}_1)\ \text{ind}\ (A).$$

Let $\mathcal{H}$ denote an infinite dimensional Hilbert space. A theorem of [**21**, Theorem 1] implies that if $T \in \mathcal{L}(\mathcal{H})$, $\lambda \in$ bdry $(\sigma(T))$, and λ is not an isolated point of $\sigma(T)$, then $\lambda \in \sigma_{le}(T) \cap \sigma_{re}(T)$. We next obtain an analogue of this result for boundary points of the left or right spectrum.

LEMMA 3.6. i) *If* $T \in \mathcal{L}(\mathcal{H})$, $\lambda \in$ bdry $(\sigma_r(T))$, *and* λ *is not an isolated point of* $\sigma_r(T)$, *then* $\lambda \in \sigma_{le}(T) \cap \sigma_{re}(T)$;

ii) *If* $T \in \mathcal{L}(\mathcal{H})$, $\lambda \in$ bdry $(\sigma_l(T))$, *and* λ *is not an isolated point of* $\sigma_l(T)$, *then* $\lambda \in \sigma_{le}(T) \cap \sigma_{re}(T)$.

Proof. i) Suppose $\lambda \in$ bdry $(\sigma_r(T))$, λ is not isolated in $\sigma_r(T)$, but $T - \lambda$ is left or right invertible, i.e., $T - \lambda$ is semi-Fredholm. Then [**17**, Theorem 5.31, Chapter IV] implies that there exists $\delta > 0$ such that $\mathcal{R}(T-\beta)$ is closed and nul $(T-\beta)$, def $(T-\beta)$ are constant for $0 < |\beta - \lambda| < \delta$. Since $\lambda \in$ bdry $(\sigma_r(T))$, there exists $\beta_0 \in \mathbf{C}$, $0 < |\beta_0 - \lambda| < \delta$, such that $T - \beta_0$ is right invertible, and thus def $(T - \beta_0) = 0$. It follows that def $(T-\beta) = 0$ for all β such that $0 < |\beta - \lambda| < \delta$, i.e.,

$$\{\beta \in \mathbf{C} : 0 < |\beta - \lambda| < \delta\} \subset \rho_r(T).$$

Since λ is not isolated in $\sigma_r(T)$, we have a contradiction and the proof is complete.

ii) Apply i) to T^*.

Remark. Let U denote a unilateral shift of multiplicity one, let K denote a compact operator with infinite spectrum, and let $T = U^* \oplus K$. It follows readily that $0 \in$ bdry $(\sigma_r(T))$ and 0 is not isolated in $\sigma_r(T)$. Since $0 \in$ int $(\sigma(T))$, the conclusion that $0 \in \sigma_{le}(T) \cap \sigma_{re}(T)$ does not follow from [**21**, Theorem 1].

PROPOSITION 3.7. *If $\sigma_{re}(A) \cap \sigma_l(B) = \sigma_r(A) \cap \sigma_{le}(B) = \emptyset$, then $K \equiv \sigma_r(A) \cap \sigma_l(B)$ is finite (or empty), and if $\lambda \in K$, then λ is isolated in $\sigma_r(A)$ or $\sigma_l(B)$.*

Proof. We first show that K is finite. Suppose to the contrary that K is infinite, let λ be a nonisolated point of bdry (K), and note that

$$\text{bdry } (K) \subset \text{bdry } (\sigma_r(A)) \cup \text{bdry } (\sigma_1(B)).$$

Suppose $\lambda \in$ bdry $(\sigma_r(A))$; since λ is not isolated in K, λ is not isolated in $\sigma_r(A)$ and thus Lemma 3.6 i) implies that $\lambda \in \sigma_{re}(A)$. Thus

$$\lambda \in K \cap \sigma_{re}(A) \subset \sigma_l(B) \cap \sigma_{re}(A),$$

which is a contradiction. If $\lambda \in$ bdry $(\sigma_l(B))$, then since λ is not isolated in K $(\subset \sigma_l(B))$, Lemma 3.6 ii) implies that

$$\lambda \in K \cap \sigma_{le}(B) \subset \sigma_r(A) \cap \sigma_{le}(B),$$

which is also a contradiction. Thus K is finite.

Note that

$$K = \sigma_r(A) \cap \sigma_l(B) = (\sigma_r(A)\backslash\sigma_{re}(A)) \cap (\sigma_l(B)\backslash\sigma_{le}(B)).$$

Let $\lambda \in \sigma_r(A) \cap \sigma_l(B)$. If λ is not isolated in $\sigma_r(A)$, then there exists a sequence of distinct points $\{\lambda_n\} \subset \sigma_r(A)$ such that $\lambda_n \to \lambda$. If λ is not isolated in $\sigma_l(B)$, then there exists a sequence of distinct points $\{\beta_n\} \subset \sigma_l(B)$ such that $\beta_n \to \lambda$. Since K is finite, we may assume that $A - \beta_n$ is right invertible for each n. Since $\beta_n \to \lambda$, $\{\beta_n\} \subset \rho_r(A)$, and $\lambda \in \sigma_r(A)$, then $\lambda \in$ bdry $(\sigma_r(A))$. Moreover, since $\{\lambda_n\} \subset \sigma_r(A)$ and $\lambda_n \to \lambda$, λ is not isolated in $\sigma_r(A)$. Thus Lemma 3.6 i) implies that

$$\lambda \in \sigma_{re}(A) \cap K \subset \sigma_{re}(A) \cap \sigma_l(B),$$

which is a contradiction. It follows that λ is isolated in $\sigma_r(A)$ or $\sigma_l(B)$ and the proof is complete.

COROLLARY 3.8. *If $\sigma_{le}(A) \cap \sigma_r(B) = \sigma_l(A) \cap \sigma_{re}(B) = \emptyset$, then $K \equiv \sigma_l(A) \cap \sigma_r(B)$ is finite (or empty), and if $\lambda \in K$, then λ is isolated in $\sigma_l(A)$ or $\sigma_r(B)$.*

Proof. Apply Proposition 3.7 to A^* and B^* and then take adjoints.

THEOREM 3.9. *If* $\sigma_{re}(A) \cap \sigma_l(B) = \sigma_r(A) \cap \sigma_{le}(B) = \emptyset$, *then* τ_{AB} *is semi-Fredholm and* ind $(\tau_{AB}) > -\infty$.

Proof. Let $K = \sigma_r(A) \cap \sigma_l(B)$; if $K = \emptyset$, then τ_{AB} is surjective [**7**, Theorem 5], so clearly τ_{AB} is semi-Fredholm and

$$\text{ind } (\tau_{AB}) = \text{nul } (\tau_{AB}) \geqq 0 > -\infty.$$

We may thus assume that $K \neq \emptyset$. Proposition 3.7 implies that K is finite and admits the following decomposition. The distinct points of K are of the form

$$\{\alpha_1, \ldots, \alpha_n\} \cup \{\beta_1, \ldots, \beta_p\}$$

and satisfy the following properties:

1) $\alpha_i \in (\sigma_r(A)\backslash\sigma_{re}(A)) \cap (\sigma_l(B)\backslash\sigma_{le}(B))$

and α_i is isolated in $\sigma_r(A)$ $(1 \leqq i \leqq n)$;

2) $\beta_j \in (\sigma_r(A)\backslash\sigma_{re}(A)) \cap (\sigma_l(B)\backslash\sigma_{le}(B))$,

β_j is not isolated in $\sigma_r(A)$, and β_j is isolated in $\sigma_l(B)$ $(1 \leqq j \leqq p)$. (In the sequel we assume that both α_i's and β_j's are present; if, instead, K consists entirely of α_i's or entirely of β_j's, then it is necessary to make certain obvious modifications in the following argument.)

Since $\alpha_i \in \sigma_r(A)\backslash\sigma_{re}(A)$ and α_i is isolated in $\sigma_r(A)$ $(1 \leqq i \leqq n)$, Corollary 2.4 implies that there exists an orthogonal decomposition $\mathscr{H}_1 = \mathscr{M}_1 \oplus \ldots \oplus \mathscr{M}_{n+1}$ and operators $A_i \in \mathscr{L}(\mathscr{M}_i)$ $(1 \leqq i \leqq n + 1)$ such that:

1) $\mathscr{M}_i$ is finite dimensional $(1 \leqq i \leqq n)$;
2) $\sigma(A_i) = \{\alpha_i\}$ $(1 \leqq i \leqq n)$;
3) $\sigma_r(A_{n+1}) \cap \{\alpha_1, \ldots, \alpha_n\} = \emptyset$;
4) A is similar to $A' = A_1 \oplus \ldots \oplus A_{n+1}$.

Similarly, since $\beta_j \in \sigma_l(B)\backslash\sigma_{le}(B)$ and β_j is isolated in $\sigma_l(B)$ $(1 \leqq j \leqq p)$, then Corollary 2.3 implies that there exists an orthogonal decomposition $\mathscr{H}_2 = \mathscr{K}_1 \oplus \ldots \oplus \mathscr{K}_{p+1}$ and operators $B_j \in \mathscr{L}(\mathscr{K}_j)$ $(1 \leqq j \leqq p + 1)$ such that:

1′) $\mathscr{K}_j$ is finite dimensional $(1 \leqq j \leqq p)$;
2′) $\sigma(B_j) = \{\beta_j\}$ $(1 \leqq j \leqq p)$;
3′) $\sigma_l(B_{p+1}) \cap \{\beta_1, \ldots, \beta_p\} = \emptyset$;
4′) B is similar to $B' = B_1 \oplus \ldots \oplus B_{p+1}$.

From Lemma 1.2 it suffices to prove that $\tau_{A'B'}$ is semi-Fredholm and that ind $(\tau_{A'B'}) > -\infty$. Let $X = (X_{ij})_{1 \leqq i \leqq n+1, 1 \leqq j \leqq p+1}$ denote the operator matrix of an operator $X \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ relative to the above decompositions of $\mathscr{H}_2$ and $\mathscr{H}_1$. A matrix calculation shows that the row i, column j entry of the matrix of $A'X - XB'$ is equal to $A_iX_{ij} - X_{ij}B_j$

$(1 \leqq i \leqq n + 1, 1 \leqq j \leqq p + 1)$; thus the ij entry of $\tau_{A'B'}(X)$ is equal to $\tau_{A_iB_j}(X_{ij})$.

A straightforward matrix calculation now shows that if each $\tau_{A_iB_j}$ is semi-Fredholm (as an operator on $\mathscr{L}(\mathscr{K}_j, \mathscr{M}_i)$), and if ind $(\tau_{A_iB_j}) > -\infty$, then $\tau_{A'B'}$ is semi-Fredholm and ind $(\tau_{A'B'}) > -\infty$; in this case,

$$\text{nul } (\tau_{A'B'}) = \sum_i \sum_j \text{nul } (\tau_{A_iB_j}), \text{def } (\tau_{A'B'}) = \sum_i \sum_j \text{def } (\tau_{A_iB_j}),$$

$$\text{and ind } (\tau_{A'B'}) = \sum_i \sum_j \text{ind } (\tau_{A_iB_j}).$$

If $1 \leqq i \leqq n$ and $1 \leqq j \leqq p$, then

$$\sigma(A_i) \cap \sigma(B_j) = \{\alpha_i\} \cap \{B_j\} = \emptyset,$$

so $\tau_{A_iB_j}$ is invertible [**24**]. Moreover, since

$$\sigma_r(A') \cap \sigma_l(B') = \sigma_r(A) \cap \sigma_l(B) = K,$$

then 3), 4), 3′) and 4′) imply that

$$\sigma_r(A_{n+1}) \cap \sigma_l(B_{p+1}) = \emptyset,$$

and it follows from [**7**, Theorem 5] that $\tau_{A_{n+1}B_{p+1}}$ is surjective. (Hence ind $(\tau_{A_{n+1}B_{p+1}}) > -\infty$.) It thus suffices to consider the operators $\tau_{A_iB_{p+1}}$ $(1 \leqq i \leqq n)$ and $\tau_{A_{n+1}B_j}$ $(1 \leqq j \leqq p)$.

For $1 \leqq j \leqq p$, $\beta_j \in K \subset \sigma_r(A)\backslash\sigma_{re}(A)$, and thus $\beta_j \notin \sigma_{re}(A_{n+1})$, i.e., $\mathscr{R}(A_{n+1} - \beta_j)$ is closed and def $(A_{n+1} - \beta_j) < \infty$. Lemma 3.5 implies that

$$\tau_{A_{n+1}-\beta_j, 0_{\mathscr{K}_j}} \in \mathscr{L}(\mathscr{L}(\mathscr{K}_j, \mathscr{M}_{n+1}))$$

is semi-Fredholm and that

$$\text{ind } (\tau_{A_{n+1}-\beta_j, 0_{\mathscr{K}_j}}) = \dim (\mathscr{K}_j) \text{ ind } (A_{n+1} - \beta_j) > -\infty.$$

Thus there exists $\delta > 0$ such that if

$$S \in \mathscr{L}(\mathscr{L}(\mathscr{K}_j, \mathscr{M}_{n+1})) \quad \text{and} \quad \|S - \tau_{A_{n+1}-\beta_j, 0_{\mathscr{K}_j}}\| < \delta,$$

then S is semi-Fredholm and

$$\text{ind } (S) = \text{ind } (\tau_{A_{n+1}-\beta_j, 0_{\mathscr{K}_j}}) > -\infty$$

[**17**, Theorem 5.17, Chapter IV]. Since $\sigma(B_j) = \{\beta_j\}$, $B_j - \beta_j$ is nilpotent; thus there exists an invertible operator $X_j \in \mathscr{L}(\mathscr{K}_j)$ such that

$$\|X_j^{-1}(B_j - \beta_j)X_j\| < \delta.$$

It follows that $S = \tau_{A_{n+1}-\beta_j, X_j^{-1}(B_j-\beta_j)X_j}$ is semi-Fredholm with ind $(S) > -\infty$, and Lemma 1.2 implies that

$$\tau_{A_{n+1}, B_j} = \tau_{A_{n+1}-\beta_j, B_j-\beta_j}$$

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is semi-Fredholm with

$$\text{ind }(\tau_{A_{n+1},B_j}) > -\infty.$$

Finally we consider the operators $\tau_{A_i,B_{p+1}}$ $(1 \leqq i \leqq n)$. Since $\alpha_i \notin \sigma_{le}(B)$, $\alpha_i \notin \sigma_{le}(B_{p+1})$, and thus $\bar{\alpha}_i \notin \sigma_{re}(B_{p+1}{}^*)$. Lemma 3.5 implies that

$$\tau_{(B_{p+1}-\alpha_i)^*,0_{\mathcal{M}_i}} \in \mathcal{L}(\mathcal{L}(\mathcal{M}_i, \mathcal{K}_{p+1}))$$

is semi-Fredholm and that

$$\text{ind }(\tau_{(B_{p+1}-\alpha_i)^*,0_{\mathcal{M}_i}}) = \dim(\mathcal{M}_i) \text{ ind }((B_{p+1} - \alpha_i)^*) > -\infty.$$

Since $(A_i - \alpha_i)^*$ is nilpotent, the above argument (for τ_{A_{n+1},B_j}) implies that $\tau_{(B_{p+1}-\alpha_i)^*,(A_i-\alpha_i)^*}$ is semi-Fredholm with index not equal to $-\infty$. Lemma 1.3 now implies that

$$\tau_{A_iB_{p+1}} = \tau_{A_i-\alpha_i,B_{p+1}-\alpha_i}$$

is semi-Fredholm and that

$$\text{ind }(\tau_{A_iB_{p+1}}) > -\infty.$$

The proof is now complete.

COROLLARY 3.10. *$\mathcal{R}(\tau_{AB})$ is closed and* def $(\tau_{AB}) < \infty$ *(i.e., τ_{AB} is semi-Fredholm with* ind $(\tau_{AB}) > -\infty$*) if and only if*

$$\sigma_r(A) \cap \sigma_{le}(B) = \sigma_{re}(A) \cap \sigma_l(B) = \emptyset.$$

Proof. Suppose $\sigma_r(A) \cap \sigma_{le}(B) \neq \emptyset$ or $\sigma_{re}(A) \cap \sigma_l(B) \neq \emptyset$. If τ_{AB} is not semi-Fredholm, then the result is clear. If τ_{AB} is semi-Fredholm, then Lemma 3.2 implies that def $(\tau_{AB}) = \infty$, and so ind $(\tau_{AB}) = -\infty$. The converse follows from Theorem 3.9.

COROLLARY 3.11. *Suppose τ_{AB} is semi-Fredholm with* ind $(\tau_{AB}) > -\infty$. *Let A, B, $\{\alpha_i\}_{1\leqq i\leqq n}$, $\{\beta_j\}_{1\leqq j\leqq p}$, A_{n+1}, B_{p+1} be as in Theorem 3.9 and its proof. Then*

$$\text{ind }(\tau_{AB}) = \sum_{j=1}^{p} \dim(\mathcal{M}(B - \beta_j)) \text{ ind }(A - \beta_j)$$

$$+ \sum_{i=1}^{n} \dim(\mathcal{M}((A - \alpha_i)^*)) \text{ ind }((B - \alpha_i)^*) + \text{nul }(\tau_{A_{n+1}B_{p+1}}).$$

Proof. Let $1 \leqq j \leqq p$; 3′) of Theorem 3.9 implies that $\beta_j \in \rho_l(B_{p+1})$, and since $\sigma(B_j) = \{\beta_j\}$ (2′), it follows that $\mathcal{K}_j = \mathcal{M}(B' - \beta_j)$. For $1 \leqq i \leqq n$, $\alpha_i \in \rho_r(A_{n+1})$, and so $\bar{\alpha}_i \in \rho_l(A_{n+1}{}^*)$. Since $\sigma(A_i) = \{\alpha_i\}$, it follows that

$$\mathcal{M}_i = \mathcal{M}((A' - \alpha_i)^*) \quad (1 \leqq i \leqq n).$$

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Thus

$$\dim(\mathscr{K}_j) = \dim(\mathscr{M}(B' - \beta_j)) = \dim(\mathscr{M}(B - \beta_j))$$

and

$$\dim(\mathscr{M}_i) = \dim(\mathscr{M}((A' - \alpha_i)^*)) = \dim(\mathscr{M}((A - \alpha_i)^*)).$$

Since $\mathscr{M}_i$ $(1 \leqq i \leqq n)$ and $\mathscr{K}_j$ $(1 \leqq j \leqq p)$ are finite dimensional, we have

$$\operatorname{ind}(A_{n+1} - \beta_j) = \operatorname{ind}(A' - \beta_j) = \operatorname{ind}(A - \beta_j) \ (1 \leqq j \leqq p)$$

and

$$\operatorname{ind}(B_{p+1} - \alpha_i) = \operatorname{ind}(B' - \alpha_i) = \operatorname{ind}(B - \alpha_i) \ (1 \leqq i \leqq n).$$

The proof of Theorem 3.9 now shows that

$$\begin{aligned}\operatorname{ind}(\tau_{AB}) &= \sum_i \sum_j \operatorname{ind}(\tau_{A_iB_j}) \\ &= \sum_{j=1}^{p} \operatorname{ind}(\tau_{A_{n+1}B_j}) + \sum_{i=1}^{n} \operatorname{ind}(\tau_{A_iB_{p+1}}) + \operatorname{ind}(\tau_{A_{n+1},B_{p+1}}) \\ &= \sum_{j=1}^{p} \dim(\mathscr{K}_j) \operatorname{ind}(A_{n+1} - \beta_j) \\ &+ \sum_{i=1}^{n} \dim(\mathscr{M}_i) \operatorname{ind}((B_{p+1} - \alpha_i)^*) + \operatorname{nul}(\tau_{A_{n+1}B_{p+1}}) \\ &= \sum_{j=1}^{p} \dim(\mathscr{M}(B - \beta_j)) \operatorname{ind}(A - \beta_j) \\ &+ \sum_{i=1}^{n} \dim(\mathscr{M}((A - \alpha_i)^*)) \operatorname{ind}((B - \alpha_i)^*) + \operatorname{nul}(\tau_{A_{n+1},B_{p+1}}).\end{aligned}$$

Remark. The problem of evaluating nul $(\tau_{A_{n+1}B_{p+1}})$ appears to be as general as the problem of evaluating nul (τ_{AB}) for an arbitrary pair of operators (A, B) satisfying $\sigma_r(A) \cap \sigma_l(B) = \emptyset$. We know of no method of evaluating nul (τ_{AB}) in this generality. This difficulty in evaluating ind (τ_{AB}) will disappear in the case when τ_{AB} is Fredholm (Theorem 4.2).

THEOREM 3.12. *If* $\sigma_{le}(A) \cap \sigma_r(B) = \sigma_l(A) \cap \sigma_{re}(B) = \emptyset$, *then* τ_{AB} *is semi-Fredholm and* ind $(\tau_{AB}) < +\infty$.

Proof. The pattern of the proof is similar to that of Theorem 3.9, so we will sketch the outline but omit certain details. If $\sigma_l(A) \cap \sigma_r(B) = \emptyset$, then τ_{AB} is bounded below [**7**, Theorem 4], so we may assume that $K \equiv \sigma_l(A) \cap \sigma_r(B) \neq \emptyset$. Corollary 3.8 implies that K is finite and admits a decomposition

$$K = \{\alpha_1, \ldots, \alpha_n\} \cup \{\beta_1, \ldots, \beta_p\}$$

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where the α_i's and β_j's are distinct and satisfy the following properties:

1) $\alpha_i \in (\sigma_l(A)\backslash\sigma_{le}(A)) \cap (\sigma_r(B)\backslash\sigma_{re}(B))$ and α_i is isolated in $\sigma_l(A)$ $(1 \leqq i \leqq n)$;

2) $\beta_j \in (\sigma_l(A)\backslash\sigma_{le}(A)) \cap (\sigma_r(B)\backslash\sigma_{re}(B))$, β_j is not isolated in $\sigma_l(A)$, and β_j is isolated in $\sigma_r(B)$ $(1 \leqq j \leqq p)$.

Corollary 2.3 implies that there exists an orthogonal decomposition $\mathscr{H}_1 = \mathscr{M}_1 \oplus \ldots \oplus \mathscr{M}_{n+1}$ and operators $A_i \in \mathscr{L}(\mathscr{M}_i)$ $(1 \leqq i \leqq n+1)$ such that:

1) $\mathscr{M}_i$ is finite dimensional $(1 \leqq i \leqq n)$;

2) $\sigma(A_i) = \{\alpha_i\}$ $(1 \leqq i \leqq n)$;

3) $\sigma_l(A_{n+1}) \cap \{\alpha_i, \ldots, \alpha_n\} = \emptyset$;

4) A is similar to $A' = A_1 \oplus \ldots \oplus A_{n+1}$.

From 1) – 3) it follows that

$$\mathscr{M}_i = \mathscr{M}(A' - \alpha_i) = \ker\, ((A' - \alpha_i)^{n_i})$$

(for a sufficiently large integer n_i), and thus, from 4),

$$\dim\, (\mathscr{M}_i) = \dim\, (\mathscr{M}(A' - \alpha_i)) = \dim\, (\mathscr{M}(A - \alpha_i)) < \infty.$$

Similarly, Corollary 2.4 implies that there is an orthogonal decomposition $\mathscr{H}_2 = \mathscr{K}_1 \oplus \ldots \oplus \mathscr{K}_{p+1}$ and operators $B_j \in \mathscr{L}(\mathscr{K}_j)$ $(1 \leqq j \leqq p+1)$ such that:

1′) $\mathscr{K}_j$ is finite dimensional $(1 \leqq j \leqq p)$;

2′) $\sigma(B_j) = \{\beta_j\}$ $(1 \leqq j \leqq p)$;

3′) $\sigma_r(B_{p+1}) \cap \{\beta_1, \ldots, \beta_p\} = \emptyset$;

4′) B is similar to $B' = B_1 \oplus \ldots \oplus B_{p+1}$.

It follows from 1′) – 3′) that $\mathscr{K}_j = \mathscr{M}((B' - \beta_j)^*)$, and 4′) implies that

$$\dim\, (\mathscr{K}_j) = \dim\, (\mathscr{M}((B' - \beta_j)^*)) = \dim\, (\mathscr{M}((B - \beta_j)^*)) < \infty.$$

As in the proof of Theorem 3.9, it suffices to prove that each $\tau_{A_iB_j}$ (acting on $\mathscr{L}(\mathscr{K}_j, \mathscr{M}_i)$) is semi-Fredholm with ind $(\tau_{A_iB_j}) < +\infty$. Clearly, $\tau_{A_iB_j}$ is invertible for $1 \leqq i \leqq n$ and $1 \leqq j \leqq p$. Moreover,

$$\sigma_l(A_{n+1}) \cap \sigma_r(B_{p+1}) = \emptyset,$$

so $\tau_{A_{n+1}B_{p+1}}$ is bounded below [**7**, Theorem 4]. For $1 \leqq j \leqq p$, it follows as in the proof of Theorem 3.9 (using Lemma 3.5) that $\tau_{A_{n+1}B_j}$ is semi-Fredholm and

$$\begin{aligned}\text{ind}\, (\tau_{A_{n+1}B_j}) = \text{ind}\, (\tau_{A_{n+1}-\beta_j, B_j-\beta_j}) &= \text{ind}\, (\tau_{A_{n+1}-\beta_j, 0_{\mathscr{K}_j}}) \\ &= \dim\, (\mathscr{K}_j)\ \text{ind}\, (A_{n+1} - \beta_j) \\ &= \dim\, (\mathscr{M}((B - \beta_j)^*))\ \text{ind}\, (A - \beta_j) < +\infty.\end{aligned}$$

Similarly, for $1 \leqq i \leqq n$, $\tau_{A_i, B_{p+1}}$ is semi-Fredholm and

$$\begin{aligned} \text{ind } (\tau_{A_i B_{p+1}}) &= \text{ind } (\tau_{B_{p+1}^* A_i^*}) = \text{ind } (\tau_{(B_{p+1}-\alpha_j)^*, (A_i - \alpha_i)^*}) \\ &= \text{ind } (\tau_{(B_{p+1}-\alpha_i)^*, 0_{\mathscr{M}_i}}) = \dim (\mathscr{M}_i) \text{ ind } ((B_{p+1} - \alpha_i)^*) \\ &= \dim (\mathscr{M}(A - \alpha_i)) \text{ ind } ((B - \alpha_i)^*) < +\infty. \end{aligned}$$

The proof is complete.

COROLLARY 3.13. *$\mathscr{R}(\tau_{AB})$ is closed and* nul $(\tau_{AB}) < \infty$ *(i.e., τ_{AB} is semi-Fredholm with* ind $(\tau_{AB}) < +\infty$ *) if and only if*

$$\sigma_l(A) \cap \sigma_{re}(B) = \sigma_{le}(A) \cap \sigma_r(B) = \emptyset.$$

Proof. The result follows from Lemma 3.1 and Theorem 3.12.

The proof of the following result (using the calculations in Theorem 3.12) is similar to that of Corollary 3.11 and will be omitted.

COROLLARY 3.14. *With the notation of Theorem* 3.12, *if τ_{AB} is semi-Fredholm and* ind $(\tau_{AB}) < +\infty$, *then*

$$\text{ind } (\tau_{AB}) = \sum_{j=1}^{p} \dim (\mathscr{M}(B - \beta_j)^*)) \text{ ind } (A - \beta_j)$$
$$+ \sum_{i=1}^{n} \dim (\mathscr{M}(A - \alpha_i)) \text{ ind } ((B - \alpha_i)^*) - \text{def } (\tau_{A_{n+1}} \beta_{p+1}).$$

We next prove the principal results of this section.

COROLLARY 3.15. *τ_{AB} is semi-Fredholm if and only if*
i) $\sigma_{re}(A) \cap \sigma_l(B) = \sigma_r(A) \cap \sigma_{le}(B) = \emptyset$, *or*
ii) $\sigma_{le}(A) \cap \sigma_r(B) = \sigma_l(A) \cap \sigma_{re}(B) = \emptyset$.

Proof. If i) or ii) hold, then the conclusion that τ_{AB} is semi-Fredholm follows from Theorem 3.9 or Theorem 3.12 respectively. The converse is the contrapositive of Corollary 3.3.

COROLLARY 3.16.

$$\begin{aligned} \sigma_{SF}(\tau_{AB}) = \sigma \equiv &[(\sigma_{le}(A) - \sigma_r(B)) \cup (\sigma_l(A) - \sigma_{re}(B))] \\ &\cap [(\sigma_{re}(A) - \sigma_l(B)) \cup (\sigma_r(A) - \sigma_{le}(B))]. \end{aligned}$$

Proof. Corollary 3.4 states that $\sigma \subset \sigma_{SF}(\tau_{AB})$. To prove the reverse inclusion, it suffices to let $z \in \mathbf{C} \backslash \sigma$ and to show that $\tau_{AB} - z = \tau_{A-z,B}$ is semi-Fredholm. From Corollary 3.15, it suffices to verify that

$$\sigma_{re}(A - z) \cap \sigma_l(B) = \sigma_r(A - z) \cap \sigma_{le}(B) = \emptyset,$$

or

$$\sigma_{le}(A - z) \cap \sigma_r(B) = \sigma_l(A - z) \cap \sigma_{re}(B) = \emptyset.$$

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Assuming the contrary, we will illustrate the case when

$$\sigma_{re}(A - z) \cap \sigma_l(B) \neq \emptyset \quad \text{and} \quad \sigma_l(A - z) \cap \sigma_{re}(B) \neq \emptyset;$$

the other cases are proved similarly and will be omitted. Let

$$\beta \in \sigma_{re}(A - z) \cap \sigma_l(B);$$

then $\beta = \alpha - z$ for some $\alpha \in \sigma_{re}(A)$, and so

$$z = \alpha - \beta \in \sigma_{re}(A) - \sigma_l(B).$$

Let $\gamma \in \sigma_l(A - z) \cap \sigma_{re}(B)$; then $\gamma = \alpha' - z$ for some $\alpha' \in \sigma_l(A)$ and thus $z = \alpha' - \gamma \in \sigma_l(A) - \sigma_{re}(B)$. Now

$$z \in (\sigma_l(A) - \sigma_{re}(B)) \cap (\sigma_{re}(A) - \sigma_l(B)) \subset \sigma,$$

which is a contradiction.

COROLLARY 3.17. *τ_{AB} is semi-Fredholm if and only if τ_{BA} is semi-Fredholm.*

Proof. The result follows immediately from Corollary 3.15.

Remark. In [**12**] we studied the problem of characterizing the case when τ_{AB} has closed range. For the case $A = B$, the characterization is due to C. Apostol [**2**], and [**11**] and [**12**] contain diverse partial results for the general case. (Note for the case $A = B$ that while τ_{AA} may have closed range, it is not semi-Fredholm, since bdry $(\sigma_e(A)) \subset \sigma_{re}(A) \cap \sigma_{le}(A)$ (Corollary 3.15). The results of the present section characterize when $\mathscr{R}(\tau_{AB})$ is closed under the added hypothesis that nul (τ_{AB}) or def (τ_{AB}) is finite. However, the general problem of range closure remains unsolved.

4. The case when τ_{AB} is Fredholm. In this section we consider the case when τ_{AB} is Fredholm; we give a formula for the index in this case and obtain some applications. As before, $\mathscr{H}_1$ and $\mathscr{H}_2$ denote arbitrary infinite dimensional Hilbert spaces, $A \in \mathscr{L}(\mathscr{H}_1)$, and $B \in \mathscr{L}(\mathscr{H}_2)$. Recall the following result of [**12**].

THEOREM 4.1. [**12**, Theorem 3.1] i) *τ_{AB} is Fredholm if and only if $\sigma(A) \cap \sigma_e(B) = \sigma_e(A) \cap \sigma(B) = \emptyset$;*
ii) $\sigma_e(\tau_{AB}) = (\sigma(A) - \sigma_e(B)) \cup (\sigma_e(A) - \sigma(B))$.

Suppose that τ_{AB} is Fredholm but not invertible; then $K \equiv \sigma(A) \cap \sigma(B)$ is nonempty [**24**]. The proof of [**12**, Theorem 3.1] implies that K is finite and admits a decomposition

$$K = \{\alpha_1, \ldots, \alpha_n\} \cup \{\beta_1, \ldots, \beta_p\},$$

where the α_i's and β_j's are distinct and satisfy the following properties:

(4.1) $\quad \alpha_i \in (\sigma(A)\backslash\sigma_e(A)) \cap (\sigma(B)\backslash\sigma_e(B))$

and α_i is isolated in $\sigma(A)$ $(1 \leqq i \leqq n)$;

(4.2) $\quad \beta_j \in (\sigma(A)\backslash\sigma_e(A)) \cap (\sigma(B)\backslash\sigma_e(B)),$

β_j is not isolated in $\sigma(A)$, and β_j is isolated in $\sigma(B)$ $(1 \leqq j \leqq p)$.

As in the proof of [**12**, Theorem 3.1], there exists an orthogonal decomposition $\mathscr{H}_1 = \mathscr{M}_1 \oplus \ldots \oplus \mathscr{M}_{n+1}$ and operators $A_i \in \mathscr{L}(\mathscr{M}_i)$ $(1 \leqq i \leqq n + 1)$ such that:

1) $\mathscr{M}_i$ is finite dimensional $(1 \leqq i \leqq n)$;

2) $\sigma(A_i) = \{\alpha_i\}$ $(1 \leqq i \leqq n)$;

3) $\sigma(A_{n+1}) \cap \{\alpha_1, \ldots, \alpha_n\} = \emptyset$;

4) A is similar to $A' = A_1 \oplus \ldots \oplus A_{n+1}$.

Here $\mathscr{M}_i$ is the (finite dimensional) Riesz subspace for A' corresponding to the isolated point $\alpha_i \in \sigma(A')\backslash\sigma_e(A')$. Thus 2) and 3) imply that $\mathscr{M}_i = \mathscr{M}(A' - \alpha_i)$; moreover 4) implies that

$$\dim (\mathscr{M}_i) = \dim (\mathscr{M}(A' - \alpha_i)) = \dim (\mathscr{M}(A - \alpha_i)).$$

Similarly, there exists an orthogonal decomposition

$$\mathscr{H}_2 = \mathscr{K}_1 \oplus \ldots \oplus \mathscr{K}_{p+1}$$

and operators $B_j \in \mathscr{L}(\mathscr{K}_j)$ $(1 \leqq j \leqq p + 1)$ such that:

1′) $\mathscr{K}_j$ is finite dimensional $(1 \leqq j \leqq p)$;

2′) $\sigma(B_j) = \{\beta_j\}$ $(1 \leqq j \leqq p)$;

3′) $\sigma(B_{p+1}) \cap \{\beta_1, \ldots, \beta_p\} = \emptyset$;

4′) B is similar to $B' = B_1 \oplus \ldots \oplus B_{p+1}$.

Thus $\mathscr{K}_j = \mathscr{M}(B' - \beta_j)$ and $\dim (\mathscr{K}_j) = \dim (\mathscr{M}(B - \beta_j))$ $(1 \leqq j \leqq p)$.

As in the proofs in Section 3, to calculate ind (τ_{AB}), it suffices to prove that each $\tau_{A_iB_j}$ is Fredholm, for then

$$\text{ind } (\tau_{AB}) = \sum_i \sum_j \text{ind } (\tau_{A_iB_j}).$$

Clearly $\sigma(A_i) \cap \sigma(B_j) = \emptyset$ for $1 \leqq i \leqq n$ and $1 \leqq j \leqq p$; moreover, 3), 3′), and the definition of K imply that

$$\sigma(A_{n+1}) \cap (B_{p+1}) = \emptyset.$$

Thus $\tau_{A_iB_j}$ $(1 \leqq i \leqq n)$ $(1 \leqq j \leqq p)$ and $\tau_{A_{n+1}B_{p+1}}$ are invertible [**24**]. It follows that

$$\text{ind } (\tau_{AB}) = \sum_{j=1}^{p} \text{ind } (\tau_{A_{n+1}B_j}) + \sum_{i=1}^{n} \text{ind } (\tau_{A_iB_{p+1}}).$$

As in the proof of [**12**, Theorem 3.1] (or Theorem 3.9 above), for $1 \leqq j \leqq p$ we have

$$\begin{aligned}\text{ind } (\tau_{A_{n+1}B_j}) &= \text{ind } (\tau_{A_{n+1}-\beta_j, B_j-\beta_j}) = \text{ind } (\tau_{A_{n+1}-\beta_j, 0_{\mathscr{K}_j}}) \\ &= \dim (\mathscr{K}_j) \text{ ind } (A_{n+1} - \beta_j) = \dim (\mathscr{M}(B - \beta_j)) \text{ ind } (A - \beta_j).\end{aligned}$$

Similarly, for $1 \leqq i \leqq n$,

$$\begin{aligned}\text{ind}\,(\tau_{A_i B_{p+1}}) &= \text{ind}\,(\tau_{A_i-\alpha_i, B_{p+1}-\alpha_i}) = \text{ind}\,(\tau_{(B_{p+1}-\alpha_i)^*,(A_i-\alpha_i)^*}) \\ &= \text{ind}\,(\tau_{(B_{p+1}-\alpha_i)^*,0_{\mathscr{M}_i}}) = \dim\,(\mathscr{M}_i)\ \text{ind}\,((B_{p+1}-\alpha_i)^*) \\ &= \dim\,(\mathscr{M}(A-\alpha_i))\ \text{ind}\,((B-\alpha_i)^*).\end{aligned}$$

The preceding discussion yields the following result.

THEOREM 4.2. *If τ_{AB} is Fredholm, either τ_{AB} is invertible* ($\sigma(A) \cap \sigma(B) = \emptyset$) *or*

$$\sigma(A) \cap \sigma(B) = \{\alpha_1, \ldots, \alpha_n\} \cup \{\beta_1, \ldots, \beta_p\}$$

where the α_i's satisfy (4.1) *and the β_j's satisfy* (4.2). (*Either the α_i's or the β_j's may be absent.*) *In this case,*

$$\text{ind}\,(\tau_{AB}) = \sum_{j=1}^{p} \dim\,(\mathscr{M}(B-\beta_j))\ \text{ind}\,(A-\beta_j) + \sum_{i=1}^{n} \dim\,(\mathscr{M}(A-\alpha_i))\ \text{ind}\,((B-\alpha_i)^*).$$

There are certain formal similarities between $(\tau_{AB})^*$ and $-\tau_{BA}$. Recall that a Banach space operator $T \in \mathscr{L}(\mathscr{X})$ satisfies $\sigma(T^*) = \sigma(T)$; in particular, T is bounded below if and only if $T^* \in \mathscr{L}(\mathscr{X}^*)$ is surjective, and T is surjective if and only if T^* is bounded below [**23**, Theorems 4.12-4.15]. Moreover, T is semi-Fredholm if and only if T^* is semi-Fredholm, in which case ind $(T^*) = -$ind (T) [**17**, Corollary 5.14, Chapter IV]. Rosenblum's Theorem [**24**] implies that $\sigma(-\tau_{BA}) = \sigma((\tau_{AB})^*)$, and it follows from [**7**] that τ_{AB} is bounded below (resp. surjective) if and only if $-\tau_{BA}$ is surjective (resp. bounded below). Corollary 3.16 and Theorem 4.1 imply that

$$\sigma_{SF}(-\tau_{BA}) = \sigma_{SF}((\tau_{AB})^*) \quad \text{and} \quad \sigma_e(-\tau_{BA}) = \sigma_e((\tau_{AB})^*).$$

We next show that $-\tau_{BA}$ has the same Fredholm index as $(\tau_{AB})^*$.

COROLLARY 4.3. *τ_{AB} is Fredholm if and only if τ_{BA} is Fredholm, in which case* ind $(\tau_{BA}) = -$ind (τ_{AB}).

Proof. It follows directly from Theorem 4.1 that τ_{AB} is Fredholm if and only if $\tau_{A^*B^*}$ is Fredholm, and clearly τ_{AB} is invertible if and only if $\tau_{A^*B^*}$ is invertible. Assume that τ_{AB} is Fredholm but not invertible and let $\{\alpha_i\}_{1\leqq i\leqq n}$ and $\{\beta_j\}_{1\leqq j\leqq p}$ be as in (4.1) and (4.2) respectively. Then

$$\sigma(A^*) \cap \sigma(B^*) = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\} \cup \{\bar{\beta}_1, \ldots, \bar{\beta}_p\},$$

and (4.1) and (4.2) imply the following properties:

i) $\bar{\alpha}_i \in (\sigma(A^*)\backslash\sigma_e(A^*)) \cap (\sigma(B^*)\backslash\sigma_e(B^*))$

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and $\bar{\alpha}_i$ is isolated in $\sigma(A^*)$ $(1 \leqq i \leqq n)$;

ii) $\bar{\beta}_j \in (\sigma(A^*)\backslash\sigma_e(A^*)) \cap (\sigma(B^*)\backslash\sigma_e(B^*))$,

$\bar{\beta}_j$ is not isolated in $\sigma(A^*)$, and $\bar{\beta}_j$ is isolated in $\sigma(B^*)$.

Theorem 4.2 and i) and ii) imply that

$$\text{ind }(\tau_{A^*B^*}) = \sum_{j=1}^{p} \dim (\mathscr{M}(B^* - \bar{\beta}_j)) \text{ ind } (A^* - \bar{\beta}_j) + \sum_{i=1}^{n} \dim (\mathscr{M}(A^* - \alpha_i)) \text{ ind } (B - \alpha_i).$$

Since α_i is isolated in $\sigma(A)$ and $A - \alpha_i$ is Fredholm, it follows from Lemma 2.1 that

$$\dim (\mathscr{M}(A^* - \bar{\alpha}_i)) = \dim (\mathscr{M}(A - \alpha_i));$$

similarly,

$$\dim (\mathscr{M}(B^* - \bar{\beta}_j)) = \dim (\mathscr{M}(B - \beta_j)).$$

Thus

$$\begin{aligned}\text{ind }(\tau_{A^*B^*}) &= \sum_{j=1} \dim (\mathscr{M}(B - \beta_j)) \text{ ind } (A^* - \bar{\beta}_j) \\ &\quad + \sum_{i=1}^{n} \dim (\mathscr{M}(A - \alpha_i)) \text{ ind } (B - \alpha_i) \\ &= - \sum_{j=1}^{p} \dim (\mathscr{M}(B - \beta_j)) \text{ ind } (A - \beta_j) \\ &\quad - \sum_{i=1}^{n} \dim (\mathscr{M}(A - \alpha_i)) \text{ ind } ((B - \alpha_i)^*) = - \text{ ind } (\tau_{AB}).\end{aligned}$$

The result now follows from an application of Lemma 1.3.

Recall that an operator T in $\mathscr{L}(\mathscr{H})$ ($\mathscr{H}$ separable) is *bi-quasitriangular* if T and T^* are quasitriangular in the sense of [**16**]. It follows from [**5**] that T is bi-quasitriangular if and only if ind $(T - \lambda) = 0$ for each $\lambda \in \rho_{SF}(T)$; in particular, if T is bi-quasitriangular, then $\sigma_{le}(T) = \sigma_{re}(T) = \sigma_e(T)$ and $\sigma_l(T) = \sigma_r(T) = \sigma(T)$. Note that each normal operator in $\mathscr{L}(\mathscr{H})$ is bi-quasitriangular [**16**], as is each quasitriangular hyponormal operator. C. K. Fong [**13**] proved that if A and B are normal, then τ_{AB} is normal in the Banach space sense. For the case when A and B are bi-quasitriangular, the following results illustrate the resemblance of the spectral properties of τ_{AB} to those of bi-quasitriangular Hilbert space operators.

THEOREM 4.4. *Let A and B be bi-quasitriangular operators in $\mathscr{L}(\mathscr{H})$. The following are equivalent:*

1) τ_{AB} *is semi-Fredholm;*
2) τ_{AB} *is Fredholm with* ind $(\tau_{AB}) = 0$*;*
3) τ_{AB} *is Fredholm;*
4) $\sigma_e(A) \cap \sigma(B) = \sigma(A) \cap \sigma_e(B) = \emptyset$.

Proof. It follows from Theorem 4.1 that 4) $\Rightarrow$ 3) $\Rightarrow$ 1). Further, since A and B are bi-quasitriangular, then Corollary 3.15 implies that 1) $\Rightarrow$ 4). Assume that τ_{AB} is Fredholm. Since A and B are bi-quasitriangular, then ind $(A - z) = 0$ for all $z \in \rho_{SF}(A)$ and ind $(B - z) = 0$ for each $z \in \rho_{SF}(B)$. Thus Theorem 4.2 implies that ind $(\tau_{AB}) = 0$, and so 3) $\Rightarrow$ 2); since the converse is obvious, the proof is complete.

COROLLARY 4.5. *If A and B are bi-quasitriangular, then* $\sigma_{SF}(\tau_{AB}) = \sigma_e(\tau_{AB})$ *and* ind $(\tau_{AB} - z) = 0$ *for* $z \in \rho_{SF}(\tau_{AB})$.

Proof. Let $z \in \sigma_e(\tau_{AB})$; thus $\tau_{A-z,B} = \tau_{AB} - z$ is not Fredholm and $A - z$ and B are bi-quasitriangular. Theorem 4.4 implies that $\tau_{AB} - z$ is not semi-Fredholm, so it follows that

$$\sigma_{SF}(\tau_{AB}) = \sigma_e(\tau_{AB}).$$

Next, suppose $\tau_{AB} - z = \tau_{A-z,B}$ is semi-Fredholm; since $A - z$ and B are bi-quasitriangular, Theorem 4.4 implies that ind $(\tau_{AB} - z) = 0$.

We conclude by discussing another similarity between the operators τ_{AB} and Hilbert space operators. By a theorem of C. Olsen [**19**, Theorem 2.4], each polynomially compact operator is a compact perturbation of an algebraic operator; in particular, if $T \in \mathscr{L}(\mathscr{H})$, $\tilde{T}^k = 0$, and $\tilde{T}^{k-1} \neq 0$, then there exists a compact operator K such that $(T + K)^k = 0$ and $(T + K)^{k-1} \neq 0$. An operator T is *essentially quasinilpotent* if $\tilde{T}$ is quasinilpotent, i.e., $\sigma_e(T) = \{0\}$. It follows from [**25**, Theorem 4] that if T is essentially quasinilpotent, then there exists a compact operator K such that $T + K$ is quasinilpotent. These two results for Hilbert space operators have strong analogues for τ_{AB}. In [**14**, Proposition 4] it is proved that the following conditions are equivalent: i) ${\tau_{AB}}^k$ is compact for some $k > 0$; ii) ${\tau_{AB}}^k = 0$ for some $k > 0$; iii) there exists a scalar $\alpha \in \mathbf{C}$ such that $A - \alpha$ and $B - \alpha$ are nilpotent. We next give a parallel result for the case when τ_{AB} is essentially quasinilpotent.

COROLLARY 4.6. *The following are equivalent:*
i) $\sigma(A - \alpha) = \sigma(B - \alpha) = \{0\}$ *for some* $\alpha \in \mathbf{C}$*;*
ii) $\sigma(\tau_{AB}) = \{0\}$*;*
iii) τ_{AB} *is essentially quasinilpotent, i.e.,* $\sigma_e(\tau_{AB}) = \{0\}$.

Proof. The equivalence of i) and ii) is an immediate consequence of Rosenblum's Theorem [**24**]. Clearly ii) $\Rightarrow$ iii), so it suffices to prove iii) $\Rightarrow$ i). If $\sigma_e(\tau_{AB}) = \{0\}$, then it follows from Theorem 4.1 ii) that there exist scalars $\alpha, \beta \in \mathbf{C}$ such that $\sigma(A) = \sigma_e(B) = \{\alpha\}$ and $\sigma_e(A) = \sigma(B) = \{\beta\}$. Thus $\alpha = \beta$, so $\sigma(A) = \sigma(B) = \{\alpha\}$ and i) follows.

Remark. Since completing an earlier version of this paper, we learned of several additional results that we mention below:

1) By combining Corollary 3.13 with Theorem 4.1, we obtain the following spectral condition for the case when τ_{AB} is semi-Fredholm with ind $(\tau_{AB}) = -\infty$; this occurs if and only if

$$\sigma_l(A) \cap \sigma_{re}(B) = \sigma_{le}(A) \cap \sigma_r(B) = \emptyset \quad \text{and}$$
$$(\sigma_e(A) \cap \sigma(B)) \cup (\sigma(A) \cap \sigma_e(B)) \neq \emptyset.$$

Similarly, Corollary 3.10 and Theorem 4.1 give a spectral condition for the case when ind $(\tau_{AB}) = +\infty$. These results, together with Theorem 4.2, completely describe ind $(\tau_{AB} - z)$ $(z \in \rho_{SF}(\tau_{AB}))$.

2) In a forthcoming sequel we show that if $\mathscr{J}$ is any norm ideal in $\mathscr{L}(\mathscr{H})$, then $\rho_{SF}(\tau|\mathscr{J}) = \rho_{SF}(\tau)$ and $\tau|\mathscr{J}$ and τ have the same index functions. In particular, if $\mathscr{J}$ is the ideal of all Hilbert-Schmidt operators in $\mathscr{L}(\mathscr{H})$, endowed with its Hilbert space structure, and if A and B^* are quasitriangular, then τ_{AB} is a quasitriangular operator on $\mathscr{J}$.

3) Let $T \in \mathscr{L}(\mathscr{H})$. Note that $K \subset \mathbf{C}$ is an isolated subset of $\sigma_l(T) \backslash \sigma_{le}(T)$ if and only if K is isolated in $\sigma_l(T)$ and $K \cap \sigma_{le}(T) = \emptyset$. Corollary 2.3 implies that if K is a finite isolated subset of $\sigma_l(T) \backslash \sigma_{le}(T)$, then T has a left-spectral decomposition relative to K. This result is best possible in the sense that every isolated subset of $\sigma_l(T) \backslash \sigma_{le}(T)$ is necessarily finite. Indeed, if K is an infinite isolated subset of $\sigma_l(T)$, then there exists $\lambda \in$ bdry (K) such that λ is non-isolated in K; Lemma 3.6 ii) implies that $\lambda \in \sigma_{le}(T)$. We note also that if K is an isolated subset of $\sigma_l(T)$ but $K \cap \sigma_{le}(T) \neq \emptyset$, then there need not exist a left-spectral decomposition of T corresponding to K.

Added in proof. In a personal communication to the author, T. Ichinose has remarked that the description of $\rho_{SF}(\tau)$ and of ind $(\tau - \lambda)$ can also be obtained using results from the theory of tensor products and cross spaces. Although we have found no account of this approach in the literature, we believe it may be carried out along the following lines. Let $\mathscr{X} \hat{\otimes}_\alpha \mathscr{Y}$ denote the completion of the tensor product of the Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ with respect to a quasi-uniform reasonable cross norm α on $\mathscr{X} \otimes \mathscr{Y}$. In [Trans. Amer. Math. Soc. *235* (1978), 75–113] and [Trans. Amer. Math. Soc. *237* (1978), 223–254] T. Ichinose characterized the semi-Fredholm domain and index function of $p(A \otimes 1, 1 \otimes B)$, where A acts on $\mathscr{X}$, B acts on $\mathscr{Y}$, and $p(z, w)$ is in a suitable class of polynomials including $p(z, w) = z - w$ and $p(z, w) = zw$. In the Hilbert space case, $\mathscr{H} \hat{\otimes}_\alpha \mathscr{H}$ corresponds to the (minimal) norm ideal $\mathscr{J}_\alpha \equiv$

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$\mathscr{H} \otimes_\alpha \overline{\mathscr{H}}$ of R. Schatten [Ann. of Math. Stud., no. 26, Princeton Univ. Press, 1950] (see also R. Schatten [Norm Ideals of Completely Continuous Operators, Springer-Verlag, 1960, page 73 (footnote)]). Under this correspondence, $A \otimes 1 - 1 \otimes B$ corresponds to $\tau(B', A')|\mathscr{J}_\alpha$, where A' and B' are certain operators closely related to A and B. (For the case when α is the Hilbert-Schmidt norm, $A \otimes 1 - 1 \otimes B$ is replaced by $A \otimes B$, and $\tau(A, B)$ is replaced by $\mathscr{S}(X) = AXB$, this correspondence is described by A. Brown and C. Pearcy [Proc. Amer. Math. Soc. *17* (1966), 162–166], who use it to determine $\sigma(A \otimes B)$ from results of [**18**].)

The results of T. Ichinose cited above thus characterize $\rho_{SF}(\tau(B', A')|\mathscr{J}_\alpha)$ and ind $((\tau(B', A') - \lambda)|\mathscr{J}_\alpha)$. Apparently, the relationships between A and A' and B and B' can then be used to determine the semi-Fredholm domain and index function of $\tau(A, B)|\mathscr{J}_\alpha$. If one applies this procedure when α is the usual operator norm and $\mathscr{J}_\alpha = \mathscr{K}(\mathscr{H})$, one may then recover results including Corollary 3.16 and Theorem 4.2 by using the duality

$$\tau(A, B) = (\tau(A, B)|\mathscr{K}(\mathscr{H}))^{**}$$

(see [**9**]). On the other hand, since the ideals $\mathscr{J}_\alpha$ are minimal norm ideals, it appears that this approach does not yield our results about $\mathscr{T}|\mathscr{J}$ for arbitrary norm ideals (including the nonseparable norm ideals of I. C. Gohberg and M. G. Krein [Introduction to the Theory of Linear Nonselfadjoint Operators] Amer. Math. Soc., 1969), which are closely based on the techniques of the present paper. These results (mentioned in ii) above) appear in [Trans. Amer. Math. Soc. *267* (1981), 112–124] and in "Spectral properties of elementary operators" (preprint), as do corresponding results for the operators $\mathscr{S}$ and $\mathscr{S}|\mathscr{J}$. As indicated above, the results of T. Ichinose can also be used to derive the semi-Fredholm domain and index functions of $\mathscr{S}$ and $\mathscr{S}|\mathscr{J}_\alpha$.

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