A CONDITION FOR EXISTENCE OF MOMENTS OF INFINITELY DIVISIBLE DISTRIBUTIONS

J. M. SHAPIRO

Let $F(x)$ be an infinitely divisible distribution and let $\phi(t)$ be its characteristic function. As is well known according to the formula of Lévy and Khintchine, $\phi(t)$ has the following representation:

$$\phi(t) = \exp\left\{i\gamma t + \int_{-\infty}^{\infty}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)\frac{1+u^2}{u^2}\,dG(u)\right\} \tag{1}$$

where γ is a real constant and $G(u)$ is a bounded nondecreasing function. A simple necessary and sufficient condition for the moments of $F(x)$ to exist is contained in the following Theorem.

THEOREM. *A necessary and sufficient condition for the* $(2k)$*th moment of* $F(x)$ *to be finite is that the* $(2k)$*th moment of* $G(u)$ *be finite where* k *is any positive integer.*

Proof. To prove sufficiency we assume

$$\int_{-\infty}^{\infty} u^{2k}dG(u) < \infty$$

and note that this implies that

$$\alpha(t) \equiv \int_{-\infty}^{\infty}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)\frac{1+u^2}{u^2}\,dG(u)$$

may be differentiated under the integral sign $2k$ times. Hence using (1) it follows that $d^{2k}\phi(t)/dt^{2k}$ exists and is finite. This implies that

$$\int_{-\infty}^{\infty} x^{2k}dF(x) < \infty,$$

(**1**, p. 90), which proves sufficiency.

Now assuming that

$$\int_{-\infty}^{\infty} x^{2k}dF(x) < \infty$$

it follows that $\phi(t)$ has a finite derivative of order $2k$ and thus from (1) we see that $\alpha(t)$ has a finite $(2k)$th derivative. Now

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$$\alpha''(t) = \lim_{h\to 0} \frac{\alpha(t+2h) - 2\alpha(t) + \alpha(t-2h)}{4h^2}$$

so that

$$\alpha''(0) = \lim_{t\to 0} \int_{-\infty}^{\infty} (e^{2itu} - 2 + e^{-2itu}) \frac{1+u^2}{4t^2u^2} dG(u)$$
$$= \lim_{t\to 0} \int_{-\infty}^{\infty} [(e^{itu} - e^{-itu})/2t]^2 \frac{1+u^2}{u^2} dG(u)$$
$$= -\lim_{t\to 0} \int_{-\infty}^{\infty} \left(\frac{\sin tu}{t}\right)^2 \frac{1+u^2}{u^2} dG(u).$$

But by Fatou's Theorem

$$\int_{-\infty}^{\infty} (1+u^2)\, dG(u) \leqslant \lim_{t\to 0} \int_{-\infty}^{\infty} \left(\frac{\sin tu}{t}\right)^2 \frac{1+u^2}{u^2} dG(u)$$

and hence

$$\int_{-\infty}^{\infty} u^2 dG(u) < \infty.$$

But as in the sufficiency proof this implies that we may differentiate $\alpha(t)$ under the integral sign twice and we see

$$\alpha''(t) = -\int_{-\infty}^{\infty} (1+u^2)\, e^{itu} dG(u) \equiv \beta(t).$$

Now $\beta(t)$ has a finite derivative of order $2k-2$ so that using an argument identical to that of (**1**, p. 90) we see that

$$\int_{-\infty}^{\infty} u^{2k} dG(u) < \infty.$$

This proves the Theorem.

It is not difficult to find the relation between the moments of $F(x)$ and those of $G(u)$. In fact if we assume that the $(2k)$th moment of $F(x)$ (or of $G(u)$) is finite it follows that the semi-invariants of $F(x)$ (which are expressible in terms of the moments of $F(x)$)

$$\chi_r = i^{-r}[d^r \log \phi(t)/dt^r]_{t=0}$$

exist for $1 \leqslant r \leqslant 2k$. But

$$\log \phi(t) = i\gamma t + \int_{-\infty}^{\infty} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right) \frac{1+u^2}{u^2} dG(u)$$

and differentiating under the integral sign, since

$$\int_{-\infty}^{\infty} u^{2k} dG(u) < \infty,$$

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we see that

$$\chi_1 = \gamma + \int_{-\infty}^{\infty} u dG(u),$$

$$\chi_r = \int_{-\infty}^{\infty} (u^{r-2} + u^r)\, dG(u), \qquad 2 \leqslant r \leqslant 2k.$$

Reference

1. Harald Cramér, *Mathematical methods of statistics* (Princeton University Press, 1946).

Ohio State University