THE EFFECT OF C_{22} ON ORBIT ENERGY AND ANGULAR MOMENTUM

D.J. SCHEERES

Department of Aerospace Engineering & Engineering Mechanics,
Iowa State University, Ames, IA 50011-3231, U.S.A., E-mail: scheeres@iastate.edu

Abstract. The effect of the C_{22} gravity field term on a particle is evaluated analytically over one orbit to find the change in orbit energy and angular momentum as an explicit function of the orbital inclination, argument of pericenter, longitude of the ascending node, orbit parameter and eccentricity. Changes in orbit energy and angular momentum are shown to be proportional to a family of integrals which can be parameterized in terms of eccentricity and non-dimensional pericenter radius.

1. Introduction

A hallmark effect of particle dynamics close to distended bodies in uniform rotation are the large changes in orbit energy and angular momentum which can occur over one pericenter passage. These changes can be large enough to eject the particle from the body onto a hyperbolic orbit, capture a passing hyperbolic orbit, or cause the particle to impact the surface. Previous studies have established that the C_{22} gravity term of the body, commonly termed the ellipticity, is the main contributor to these effects (Scheeres, 1995; Scheeres *et al.*, 1996; 1998b). This paper investigates the effect of this gravity term, taken in isolation, on an otherwise unperturbed orbit. In particular it investigates the change in orbit energy and angular momentum over one orbit about the central body. The problem of the C_{22} gravity term alone is highly idealistic, but an understanding of its effect is important and can be used to analyze the general and qualitative properties of motion about distended bodies in uniform rotation. This paper does not concern itself with a complete characterization of this problem – such as the computation of periodic orbits, equilibrium points, and zero-velocity curves – as such characterizations have already been performed in detail for a variety of specific bodies (Scheeres, 1994; 1995; Scheeres *et al.*, 1996; 1998a).

2. Perturbation Model

The perturbing function for a central body with C_{22} gravity coefficient acting on a particle is:

$$U_{22} = \frac{3\mu}{r^3} R_o^2 C_{22} \cos^2 \delta \cos(2\lambda) \tag{1}$$

where μ is the gravitational parameter, r is the particle radius, δ is the body-fixed latitude of the particle, λ is the body-fixed longitude of the particle, and R_o is the normalizing radius for the body. We assume that the body is in uniform rotation about its largest moment of inertia with rotation rate ω_T.

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The latitude and longitude of the particle in the body-fixed frame can be computed from the osculating orbit elements as:

$$\sin\delta = \sin i \sin u \tag{2}$$

$$\tan\lambda = \frac{\sin(\Omega - \omega_T t)\cos u + \cos(\Omega - \omega_T t)\sin u \cos i}{\cos(\Omega - \omega_T t)\cos u - \sin(\Omega - \omega_T t)\sin u \cos i} \tag{3}$$

where t is the time, $u = \omega + f$ and the orbit elements a, e, i, ω, Ω, and f all have their usual definitions.

To make the problem dimensionless we introduce a new independent parameter, τ, and a length scale, r_s:

$$\tau = \omega_T t \tag{4}$$

$$r_s = \left(\frac{\mu}{\omega_T^2}\right)^{1/3} \tag{5}$$

The parameter τ corresponds to the rotational phase of the body and the length scale r_s corresponds to the radius of a circular 1:1 synchronous orbit with no C_{22} coefficient present. Introducing these scale factors defines the non-dimensional perturbing function:

$$U_{22} = \frac{3}{r^3}\tilde{C}_{22}\cos^2\delta\cos(2\lambda) \tag{6}$$

where all free parameters have been compressed into the one non-dimensional term:

$$\tilde{C}_{22} = \left(\frac{\mu}{\omega_T^2}\right)^{-2/3} R_o^2 C_{22} \tag{7}$$

Values of the scaling parameter, rotation period and $\tilde{C}_{22}$ for a few select asteroids are shown in Table I.

TABLE I
Scaling radius, rotation period, and normalized C_{22} for some select asteroids.

Body	$(\mu/\omega^2)^{1/3}$ (km)	T (hours)	$\tilde{C}_{22}$ (-)
Ida	27.0	4.633	0.044
Eros	18.4	5.27	0.052
Castalia	0.8	4.07	0.047

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3. Jacobi Integral

We note that this dynamical problem has a Jacobi integral defined in the body-fixed coordinate system. The derivation of this integral is analogous to the derivation of the integral in the restricted 3-body problem (see, for example, Brouwer and Clemence, 1961, pg 252) and is not developed here. The Jacobi integral for this dimensionless system has the form:

$$J = \frac{1}{2}v^2 - \frac{1}{2}r_{eq}^2 - \frac{1}{r} - U_{22} \tag{8}$$

where v is the particle speed in the body-fixed frame and r_{eq} is the particle radius projected into the equatorial plane.

This integral can be re-expressed in terms of the Keplerian energy and angular momentum, as is commonly done when deriving the Tisserand criterion in the restricted 3-body problem (Brouwer and Clemence, 1961, pg 256). Doing so yields the simplified integral:

$$J = C - H - U_{22} \tag{9}$$

where C is the energy of the particle as measured with respect to the central body (treated as a point mass) and H is the angular momentum of the particle, projected onto the rotation axis of the body. Equation 9 will play an important role later in our analysis.

4. Choice of Variables and Problem Restriction

This paper concentrates on changes in orbit energy and angular momentum (i.e., orbit elements a, e and i) and does not consider secular changes in mean anomaly, argument of pericenter and longitude of the ascending node. This is justified in comparison to the C_{20} problem where the secular rates of these angles are quite large (relative to their change due to C_{22}). Given this it is only necessary to focus on a subset of the classical canonical orbit elements and their attendant differential equations. The equations of motion of interest are then a modified set of canonical elements (Brouwer and Clemence, 1961, pg 290):

$$C = -1/(2a)\,;\, G = \sqrt{a(1-e^2)}\,;\, H = G\cos i \tag{10}$$

$$C' = \frac{\partial R}{\partial \tau}\,;\, G' = \frac{\partial R}{\partial \omega}\,;\, H' = \frac{\partial R}{\partial \Omega} \tag{11}$$

where R is the perturbing function and the elements C, G, and H represent the Keplerian energy, angular momentum magnitude, and angular momentum projected onto the central body rotation axis, respectively.

5. Changes in C, G, and H over One Orbit

5.1. Method of Evaluation

To evaluate the change in these elements over one orbit we use the first iteration of Picard's method of successive approximations (for a rigorous discussion of this method see Moulton, 1958). Specifically, given a dynamical system of the form $X' = F(X, \tau)$, the first iteration of Picard's method yields:

$$X_1(\tau_2) = X_o + \int_{\tau_1}^{\tau_2} F(X_o, \tau) d\tau \tag{12}$$

where X_o is assumed constant over the interval (representing the unperturbed orbit elements), and the variation of F with τ includes both the central body rotation and the true anomaly of the particle orbit. Noting that $X_1(\tau_1) = X_o$ we immediately derive the result:

$$\Delta X_1 = \int_{\tau_1}^{\tau_2} F(X_o, \tau) d\tau \tag{13}$$

The limits of integration, τ_1 and τ_2, are $t_o - T/2$ and $t_o + T/2$, respectively, for elliptic orbits (t_o being the time of pericenter passage and T being the orbital period) and $t_o - \infty$ and $t_o + \infty$, respectively, for parabolic or hyperbolic orbits. We take $t_o = 0$ in general.

5.2. Application of Method

The perturbing function (Equation 6) can be re-expressed as:

$$\begin{aligned} U_{22} = \frac{3\tilde{C}_{22}}{r^3} \Big[& \frac{1}{2} \sin^2 i \left\{ \cos 2\Omega \cos 2\tau + \sin 2\Omega \sin 2\tau \right\} \\ & + \cos^4(i/2) \left\{ \cos 2(\omega + \Omega) \cos 2(f - \tau) - \sin 2(\omega + \Omega) \sin 2(f - \tau) \right\} \\ & + \sin^4(i/2) \left\{ \cos 2(\omega - \Omega) \cos 2(f + \tau) - \sin 2(\omega - \Omega) \sin 2(f + \tau) \right\} \Big] \end{aligned} \tag{14}$$

which isolates the true anomaly (f) and time (τ) terms together. Note that the pericenter passage occurs at $f = \tau = 0$, and thus the angles ω and Ω represent the argument of pericenter and the ascending node in the body-fixed coordinate frame at pericenter passage.

Directly applying Equation 13 to C, G, and H yields:

$$\begin{aligned} \Delta C = -\frac{6\tilde{C}_{22}}{p^3} \Big[& \frac{3e}{8} \sin^2 i \sin 2\Omega \left(I_1^2 - I_{-1}^2 \right) \\ & + \cos^4(i/2) \sin 2(\omega + \Omega) \left(I_2^3 + \frac{3e}{4} \left\{ I_3^2 - I_1^2 \right\} \right) \\ & + \sin^4(i/2) \sin 2(\omega - \Omega) \left(I_{-2}^3 + \frac{3e}{4} \left\{ I_{-3}^2 - I_{-1}^2 \right\} \right) \Big] \end{aligned} \tag{15}$$

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$$\Delta G = -\frac{6\tilde{C}_{22}}{p^{3/2}} \left[\cos^4(i/2)\sin 2(\omega+\Omega) I_2^1 + \sin^4(i/2)\sin 2(\omega-\Omega) I_{-2}^1\right] \quad (16)$$

$$\Delta H = -\frac{6\tilde{C}_{22}}{p^{3/2}} \left[\frac{1}{2}\sin^2 i \sin 2\Omega I_0^1 + \cos^4(i/2)\sin 2(\omega+\Omega) I_2^1 - \sin^4(i/2)\sin 2(\omega-\Omega) I_{-2}^1\right] \quad (17)$$

These formula allow for explicit computation of the expected change in orbit energy, angular momentum and inclination given the basic parameters of the orbit.

The integrals I_m^n represent the interaction of the rotating body with the particle as it passes through one orbit. They are defined as:

$$I_m^n(e,q) = \int_{-\theta_\infty}^{\theta_\infty} (1+e\cos f)^n \cos(mf - 2\tau)\, df \quad (18)$$

$$\tau = \begin{cases} a^{3/2}(E - e\sin E), \quad \tan(E/2) = \sqrt{\frac{1-e}{1+e}}\tan(f/2) & e < 1 \\ \sqrt{2}q^{3/2}\left[\tan(f/2) + \frac{1}{3}\tan^3(f/2)\right] & e = 1 \\ |a|^{3/2}(e\sinh F - F), \quad \tanh(F/2) = \sqrt{\frac{e-1}{e+1}}\tan(f/2) & e > 1 \end{cases} \quad (19)$$

where $\theta_\infty = \pi$ if $e \leq 1$, and $\theta_\infty = \arccos(-1/e)$ if $e > 1$. Note that the independent variable of integration is the true anomaly.

5.3. Alternate derivation of ΔC

An alternate derivation of ΔC exists which provides a simpler form for the expression and reduces the number of I_m^n integrals that must be computed. This derivation uses the Jacobi integral as stated in Equation 9. Evaluating the integral at two subsequent apocenters (or infinities), allowing C and H to change but keeping the items in U_{22} fixed (in accord with the assumptions in deriving Equations 15 – 17), and noting that the Jacobi integral remains constant, yields:

$$\Delta C = \Delta H + \Delta U_{22} \quad (20)$$

$$\Delta U_{22} = U_{22}(\theta_\infty) - U_{22}(-\theta_\infty) \quad (21)$$

Some simple algebraic manipulation will show that:

$$\Delta U_{22} = \frac{6\tilde{C}_{22}}{p^{3/2}} I \left[\frac{1}{2}\sin^2 i \sin 2\Omega + \cos^4(i/2)\sin 2(\omega+\Omega) - \sin^4(i/2)\sin 2(\omega-\Omega)\right] \quad (22)$$

$$I = \begin{cases} \left(\frac{1-e}{1+e}\right)^{3/2} \frac{\sin(2\pi a^{3/2})}{a^{3/2}} & e < 1 \\ 0 & e \geq 1 \end{cases} \quad (23)$$

which leads to a simplified form of ΔC:

$$\Delta C = -\frac{6\tilde{C}_{22}}{p^{3/2}} \left[\frac{1}{2}\sin^2 i \sin 2\Omega \left(I_0^1 - I\right) + \cos^4(i/2)\sin 2(\omega+\Omega)\left(I_2^1 - I\right) - \sin^4(i/2)\sin 2(\omega-\Omega)\left(I_{-2}^1 - I\right)\right] \quad (24)$$

6. Elementary Properties of I_m^n

The integrals I_m^n have a few simple properties that should be discussed. First, the integrals are completely independent of the central body properties, and thus need only be computed once as a function of non-dimensional q and e to cover all cases of the central body mass, rotation rate, and C_{22}.

Second, the integrals are finite and bounded. This is clear by inspection of Equation 18:

$$|I_m^n(e,q)| \leq 2\theta_\infty(1+e)^n \quad (25)$$

Third, in Equations 16, 17 and 24 we see that the integrals are defined for both positive and negative values of the integer m. There is a marked difference in the values of the integrals for these two cases, and in most situations of interest we find that:

$$|I_{-m}^n| \ll |I_m^n| \quad (26)$$

Exceptions to this occur at some small values of q and e. This is a significant result as it clearly explains why particles in retrograde orbit about a uniformly rotating body experience relatively small fluctuations in energy and angular momentum, as compared to particles in direct orbits (see Scheeres, 1994; 1995, and Scheeres *et al.*, 1996, for discussions of this effect).

Inequality 26 is most easily understood by noting that at pericenter passage the argument of the $\cos(mf - 2\tau)$ term will remain small over a longer time interval if m is positive, allowing the integrand to contribute more to the integral while in the

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neighborhood of its maximum value. This effect becomes largest when the angular rate at pericenter passage is equal to the body rotation rate, which occurs at:

$$q^3 = (1+e)\frac{m^2}{4} \tag{27}$$

Along the lines defined by this condition one sees that the integrals I_m^n take on a larger value. For negative m this condition has no special significance.

Finally, the I_m^n integrals are computed by first recasting them as a differential equation:

$$\frac{dI_m^n}{df} = (1 + e\cos f)^n \cos(mf - 2\tau) \tag{28}$$

with initial condition evaluated at $f = 0$. The equation is then numerically integrated, with error control, from $f = 0$ to $f = \theta_\infty$, the full integral being obtained by doubling the integrated result (since the equation is even about pericenter). In all cases of conic motion this procedure is seen to work well. In the case of parabolic and hyperbolic orbits accurate results are obtained despite the infinite variations in the integrand as f approaches θ_∞. To understand this we note two items. First, that the contributions of the integrand decrease to zero with increasing true anomaly. Second, for true anomaly close to θ_∞ the contribution of the integrand to the total integral, taken over any finite interval of true anomaly, rapidly approaches a zero mean due to the swift oscillation of the time argument. These factors combine to allow the differential equation approach to computing the integrals to truncate the tails of the integration as appropriate. The accuracy of this integration method can be checked by comparing the computed value of ΔC using both Equations 15 and 24, since we know independently that these combinations of integrals should be equal. Performing this comparison we find agreement to within the specified numerical error of the integration.

7. Example Computation

To illustrate the utility of this theory we present contour plots showing the normalized change in energy (C) and angular momentum projected onto the rotation axis (H) as a function of dimensionless pericenter radius and eccentricity. The specific results plotted in Figures 1 – 3 are $\frac{6}{p^{3/2}}(I_2^1 - I)$ and $\frac{6}{p^{3/2}}I_2^1$, which correspond to the terms that contribute the most to the change in C and H, respectively. To scale these results to a specific body, inclination, and argument of pericenter passage, multiply the contour values by $-\tilde{C}_{22}\cos^4(i/2)\sin 2(\omega + \Omega)$.

In Figures 1 and 2 the contour values for the elliptic case ($e \leq 1$) are plotted for ΔC and ΔH, respectively. In Figure 3 the changes for the hyperbolic case ($e \geq 1$) are plotted, where we recall from Equation 20 that $\Delta C = \Delta H$ when $e \geq 1$.

The results found from this approach have been compared with numerical integrations. We find that agreement is good for high eccentricity elliptic orbits,

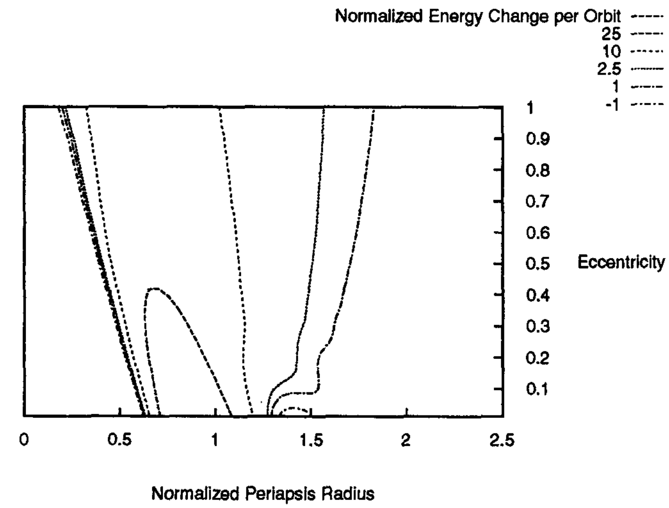

Fig. 1. Normalized ΔC per orbit for elliptic orbits (dominant terms only). Multiply contour values by $-\tilde{C}_{22}\cos^4(i/2)\sin\left[2(\omega+\Omega)\right]$ to scale to an arbitrary flyby.

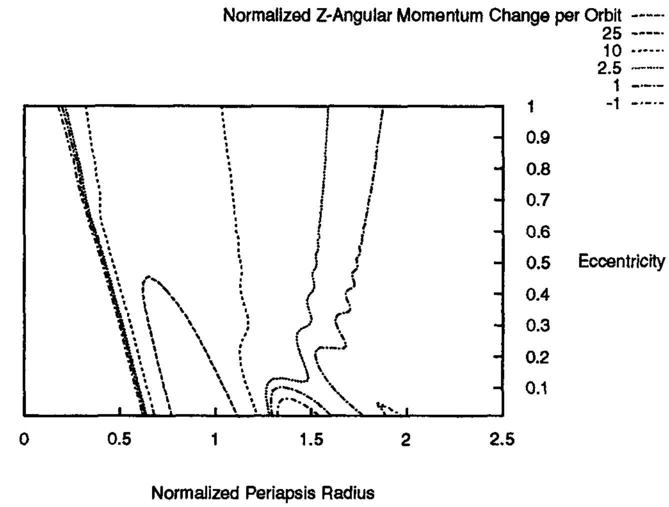

Fig. 2. Normalized ΔH per orbit for elliptic orbits (dominant terms only). Multiply contour values by $-\tilde{C}_{22}\cos^4(i/2)\sin\left[2(\omega+\Omega)\right]$ to scale to an arbitrary flyby.

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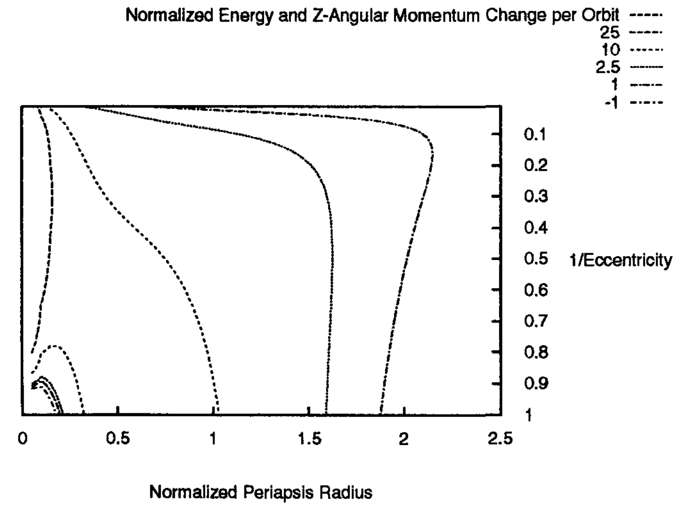

Fig. 3. Normalized ΔC and ΔH per orbit for hyperbolic orbits (dominant terms only). Note that $1/e$ is plotted. Multiply contour values by $-\tilde{C}_{22}\cos^4(i/2)\sin\left[2(\omega+\Omega)\right]$ to scale to an arbitrary flyby.

parabolic orbits and hyperbolic orbits, but that accuracy begins to degrade sharply once the eccentricity of an elliptic orbit falls below a few tenths. Possible remedies to this will be investigated in the future.

8. Conclusions

The theory presented in this paper applies to all cases of uniformly rotating bodies with a C_{22} gravity term. There are many applications for the theory as derived here. These include the computation of capture and ejection radius about a body, mission design and trajectory planning considerations for a spacecraft mission about an asteroid or comet, and long-term investigations of particle and ejecta dynamics about asteroids and comets. These applications, and others, will be detailed in future papers and reports.

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