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PLANAR SUBLATTICES OF A FREE LATTICE. II

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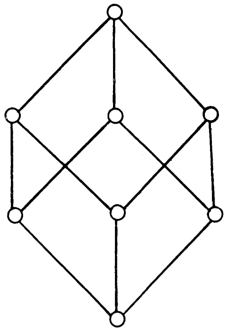
In *Planar sublattices of a free lattice, I* [**8**] we verify Jónsson's conjecture for finite planar lattices; in particular we obtain a characterization of finite planar sublattices of a free lattice among all finite lattices. In the present paper we use arguments of a quite different flavour to obtain another characterization. Let

$$\mathscr{F} = \{C_2{}^3\} \cup \{S_n | n \geqq 0\} \cup \{L_1, L_2, L_2{}^d, L_3, L_3{}^d, L_4\} \cup \{L_5, L_6\}$$

be the family of lattices illustrated in Figures 1, 2, 3, and 4. Our goal is to prove the following theorem: *a finite lattice is a planar sublattice of a free lattice if and only if it does not have a member of* $\mathscr{F}$ *as a sublattice.*



$C_2{}^3$

FIGURE 1

1. Introduction, and plan of the proof.

A lattice L is *semidistributive* if it satisfies the two conditions

$$(\mathbf{SD}_\vee) \quad a \vee b = a \vee c \quad \text{implies} \quad a \vee b = a \vee (b \wedge c)$$

and

$$(\mathbf{SD}_\wedge) \quad a \wedge b = a \wedge c \quad \text{implies} \quad a \wedge b = a \wedge (b \vee c).$$

Jónsson [**3**] has demonstrated that sublattices of a free lattice are semidistributive. Some years earlier, Whitman [**9**] showed that sublattices of a free lattice

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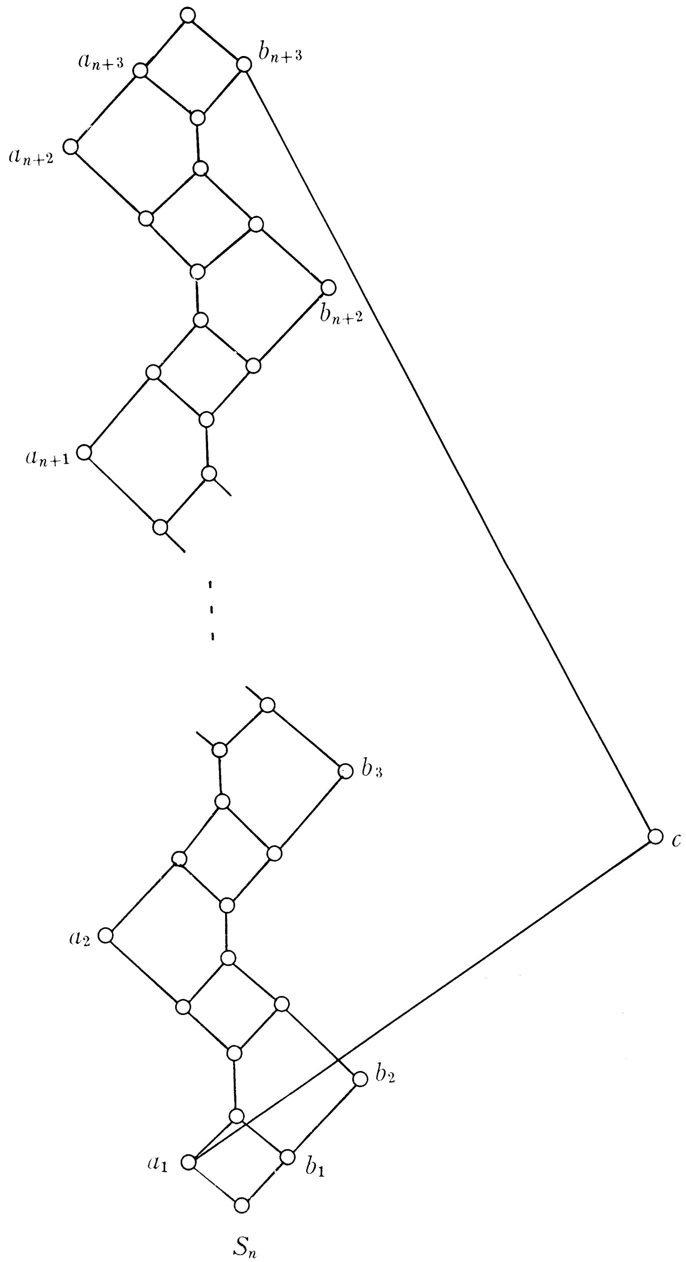

FIGURE 2

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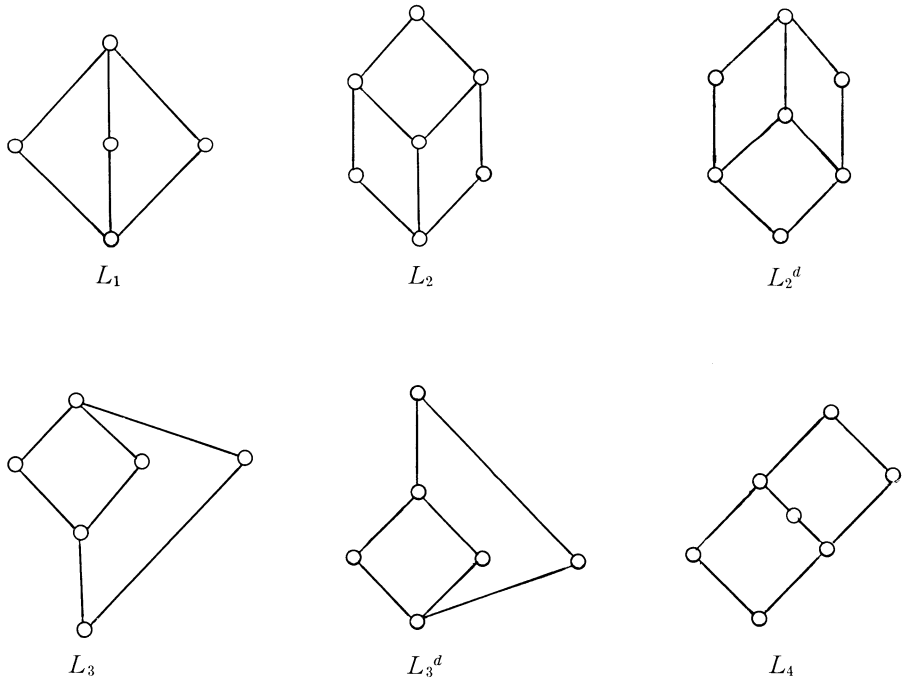


FIGURE 3

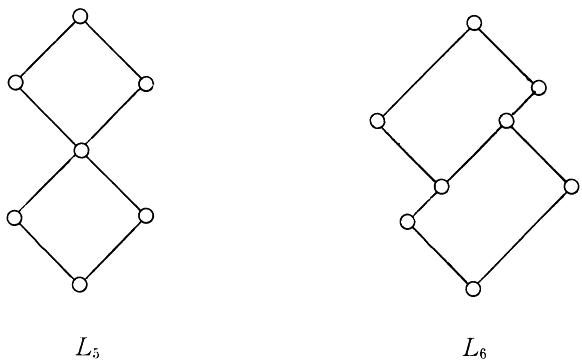
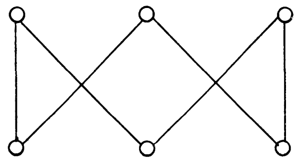


FIGURE 4

satisfy the condition

$$(\mathbf{W}) \quad a \wedge b \leqq c \vee d \quad \text{implies} \quad a \wedge b \leqq c, a \wedge b \leqq d, a \leqq c \vee d, \text{ or } b \leqq c \vee d.$$

The celebrated conjecture of Jónsson (see [**4**]), alluded to at the beginning of this paper, asserts that a finite lattice is a sublattice of a free lattice if and only if it is semidistributive and satisfies (**W**). For the history of this conjecture we refer the reader to [**8**].



A_3

FIGURE 5

Let A_3 be the partially ordered set of Figure 5, let $\mathscr{R} = \{R_n | n \geqq 0\}$ be the family of partially ordered sets illustrated in Figure 6, and let $\mathscr{S} = \{S_n | n \geqq 0\}$ be the family of lattices illustrated in Figure 2. Most of the rest of this paper is devoted to the proof of the following two results.

THEOREM 1.1. *A finite semidistributive lattice is planar if and only if it does not contain a member of* $\{A_3\} \cup \mathscr{R}$ *as a subset.*

THEOREM 1.2. *Let L be a finite semidistributive lattice satisfying* (**W**). *Then L contains a member of* $\{A_3\} \cup \mathscr{R}$ *as a subset if and only if L contains a member of* $\{C_2^3\} \cup \mathscr{S}$ *as a sublattice.*

We recall two theorems in the spirit of Theorem 1.6 below. The first is due to B. Davey, W. Poguntke, and I. Rival [**2**], and the second to R. Antonius and I. Rival [**1**].

THEOREM 1.3. *A finite lattice is semidistributive if and only if it does not contain one of the lattices of Figure* 3 *as a sublattice.*

THEOREM 1.4. *A finite semidistributive lattice satisfies* (**W**) *if and only if it does not contain one of the lattices of Figure* 4 *as a sublattice.*

Finally we quote the main result from [**8**].

THEOREM 1.5. *A finite planar lattice is a sublattice of a free lattice if and only if it is semidistributive and satisfies* (**W**).

Combining the preceding five theorems yields the promised characterization of finite planar sublattices of a free lattice.

THEOREM 1.6. *A finite lattice is a planar sublattice of a free lattice if and only if it does not contain a member of* $\mathscr{F}$ *as a sublattice.*

Observe that no member of $\mathscr{F}$ is a sublattice of another member of $\mathscr{F}$. It follows that Theorem 1.6 is best possible, in the sense that no lattice in $\mathscr{F}$ may be omitted. Also, while it is true that the lattices S_n are all sublattices of a free lattice, this observation is not essential either to the statement or the proof of the theorem.

Theorem 1.6 provides an unexpected dividend.

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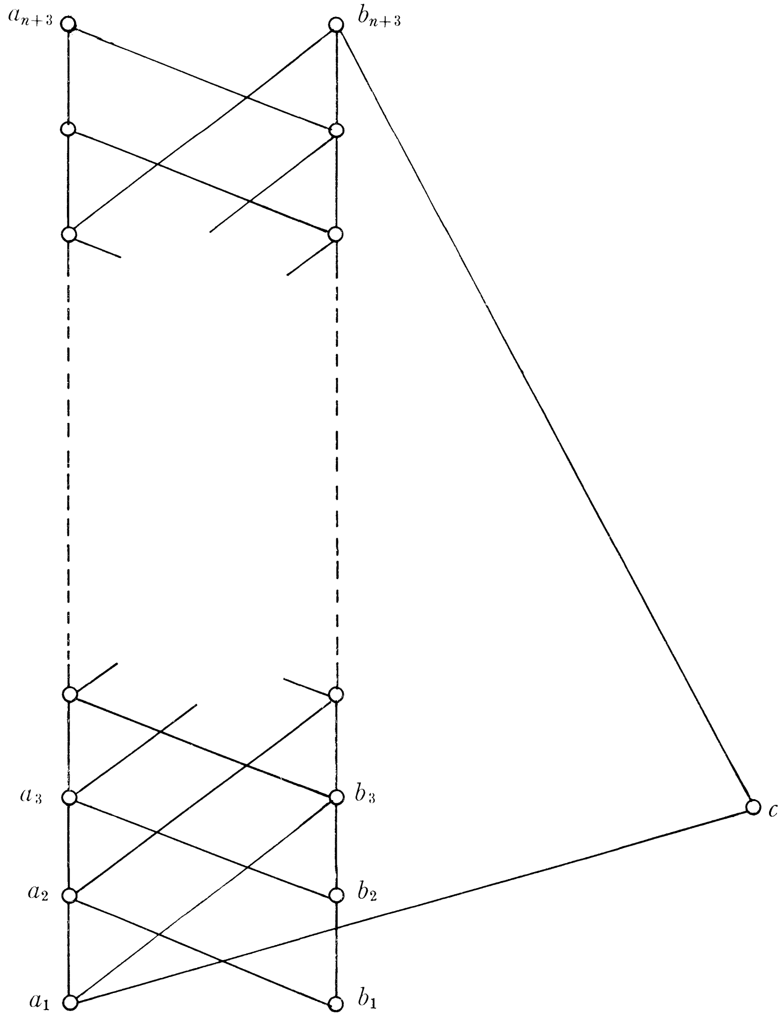

R_n

FIGURE 6

COROLLARY 1.7. *Let L be a finite semidistributive lattice satisfying* (**W**) *and of breadth at most two. If L is subdirectly irreducible then L is planar.*

Combining Corollary 1.7 with Theorem 1.5 yields

COROLLARY 1.8. *Let L be a finite subdirectly irreducible lattice of breadth at most two. Then L is a sublattice of a free lattice if and only if L is semidistributive and satisfies* (**W**).

2. Preliminaries. The *breadth* $b(L)$ of a finite lattice L is the smallest integer b such that every join $\bigvee_{i=1}^{b+1} x_i$ of elements of L is equal to a join of b of the x_i's. For any integer $n \geqq 3$, a *crown* of order $2n$ (Figure 7) is a partially ordered set $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ in which

$$x_1 < y_1, y_1 > x_2, x_2 < y_2, y_2 > x_3, \ldots, y_{n-1} > x_n, x_n < y_n, \text{ and } y_n > x_1$$

are the only comparability relations.

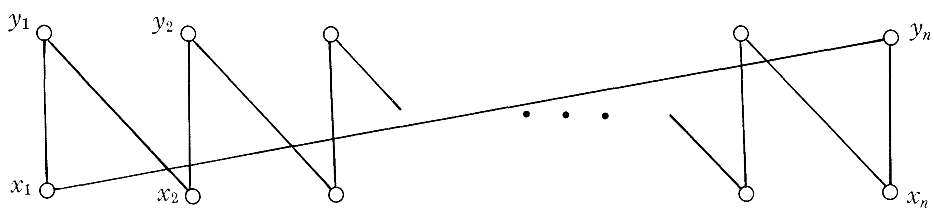

FIGURE 7. A crown of order $2n$

LEMMA 2.1. *Let L be a finite semidistributive lattice. The following are equivalent:*
(i) *L contains no crown of order six;*
(ii) $b(L) \leqq 2$;
(iii) *L contains no crown.*

Proof. (i) $\Leftrightarrow$ (ii) is Lemma 3.4 of [**5**], while (ii) $\Leftrightarrow$ (iii) is Lemma 2.4 of [**8**] together with Theorem 3.1 of [**5**].

LEMMA 2.2. *Let L be a finite semidistributive lattice of breadth at most two, and let $a, b, c \in L$.*
(i) *If $a \vee b = a \vee c = b \vee c$, then $\{a, b, c\}$ is not an antichain.*
(ii) *Either $a \vee b \geqq c$ or $a \vee c \geqq b$ or $b \vee c \geqq a$; in particular,*

$$\{a \vee b, a \vee c, b \vee c\} \textit{ is not an antichain.}$$

Proof. (i) is the dual of Lemma 2.7(ii) of [**8**]. To prove (ii), suppose that $a \vee b \ngeqq c$, $a \vee c \ngeqq b$, and $b \vee c \ngeqq a$. Then $\{a \vee b, a \vee c, b \vee c\}$ is an antichain, and the join of any pair equals $a \vee b \vee c$, contradicting (i).

Let n be a positive integer. A *down-down fence* [**6**] of length $2n + 1$ is a partially ordered set $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1}\}$ in which

$$x_1 < y_1, y_1 > x_2, x_2 < y_2, y_2 > x_3, \ldots, x_n < y_n, y_n > x_{n+1}$$

are the only comparability relations (see Figure 8).

LEMMA 2.3. *Let L be a finite semidistributive lattice of breadth at most two. Let n be an integer $\geqq 2$ and let $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1}\}$ be a down-down fence in L. Then*

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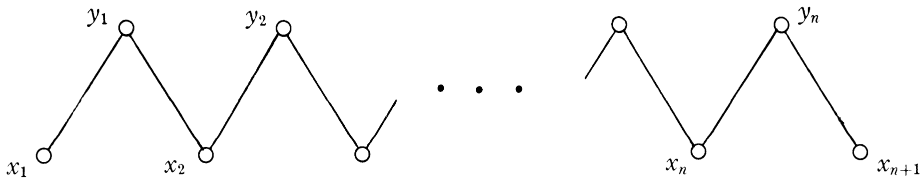

FIGURE 8. A down-down fence of length $2n + 1$

(i) $x_1 \vee x_{n+1} > x_i$ *for each* $i \in \{1, \ldots, n + 1\}$: *and*

(ii) *there exists* $i \in \{2, \ldots, n\}$ *such that* $x_1 \vee x_i < x_1 \vee x_{n+1}$ *and* $x_i \vee x_{n+1} < x_1 \vee x_{n+1}$.

Proof. (i) First let $n = 2$. Since $x_1 \vee x_2 \leqq y_1$ and $x_2 \vee x_3 \leqq y_2$, we have that $x_1 \vee x_2 \ngeqq x_3$ and $x_2 \vee x_3 \ngeqq x_1$, and therefore $x_1 \vee x_3 > x_2$ by Lemma 2.2(ii). Proceeding by induction, assume the result is true for all integers k such that $2 \leqq k \leqq n - 1$. We certainly have that $x_1 \vee x_{n+1} \ngeqq y_i$ for any $i \in \{1, \ldots, n\}$. Therefore, if $x_1 \vee x_{n+1} \ngeqq x_i$ for any $i \in \{2, \ldots, n\}$, the subset $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1}, x_1 \vee x_{n+1}\}$ of L is a crown, contradicting Lemma 2.1. Hence we can find $i \in \{2, \ldots, n\}$ such that $x_1 \vee x_{n+1} > x_i$. Now by induction we have $x_1 \vee x_{n+1} \geqq x_1 \vee x_i > x_j$ for all $j \in \{1, \ldots, i\}$, and $x_1 \vee x_{n+1} \geqq x_i \vee x_{n+1} > x_k$ for all $k \in \{i, \ldots, n + 1\}$, as claimed.

(ii) When $n = 2$, $x_1 \vee x_2 \leqq x_1 \vee x_3$ and $x_2 \vee x_3 \leqq x_1 \vee x_3$ follow from (i). Also, since $x_1 \vee x_2 \leqq y_1$ and $x_2 \vee x_3 \leqq y_2$ we have $x_1 \vee x_2 < x_1 \vee x_3$ and $x_2 \vee x_3 < x_1 \vee x_3$, as desired. Therefore let $n > 2$. As above, $x_n \vee x_{n+1} < x_1 \vee x_{n+1}$; it follows that if $x_1 \vee x_n < x_1 \vee x_{n+1}$ we are done. Hence, since $x_n < x_1 \vee x_{n+1}$ by part (i), we assume $x_1 \vee x_n = x_1 \vee x_{n+1}$. By induction we choose $j \in \{2, \ldots, n - 1\}$ such that $x_1 \vee x_j < x_1 \vee x_n = x_1 \vee x_{n+1}$ and $x_j \vee x_n < x_1 \vee x_{n+1}$. If $x_n \vee x_{n+1} \leqq x_j \vee x_n$, then $x_j \vee x_n = x_j \vee x_{n+1}$ by (i), establishing (ii). Therefore let $x_n \vee x_{n+1}$ be noncomparable to $x_j \vee x_n$. We now have that $\{x_1 \vee x_j, x_j \vee x_n, x_n \vee x_{n+1}\}$ is an antichain, and $(x_1 \vee x_j) \vee (x_j \vee x_n) = x_1 \vee x_{n+1} = (x_1 \vee x_j) \vee (x_n \vee x_{n+1})$; from Lemma 2.2(i), $x_j \vee x_{n+1} = (x_j \vee x_n) \vee (x_n \vee x_{n+1}) < x_1 \vee x_{n+1}$, and (ii) follows.

3. The proof of Theorem 1.1. By the *completion* $\mathbf{L}(P)$ of a partially ordered set P to a lattice we shall mean the construction known variously as the "normal completion", "completion by cuts", or "MacNeille completion"; recall that a partially ordered set P is a subset of a lattice L exactly when $\mathbf{L}(P)$ is a subset of L. In [**6**], D. Kelly and I. Rival defined a family $\mathscr{L}$ of lattices with the property that a finite lattice L is planar if and only if L does not contain a member of $\mathscr{L}$ as a subset. The family

$$\mathscr{P} = \{A_n|n \geqq 0\} \cup \{B, B^d, C, C^d, D, D^d\} \cup \{E_n, E_n^{\ d}, F_n, G_n, H_n|n \geqq 0\}$$

of partially ordered sets, which (up to duality) is illustrated in Figure 9,

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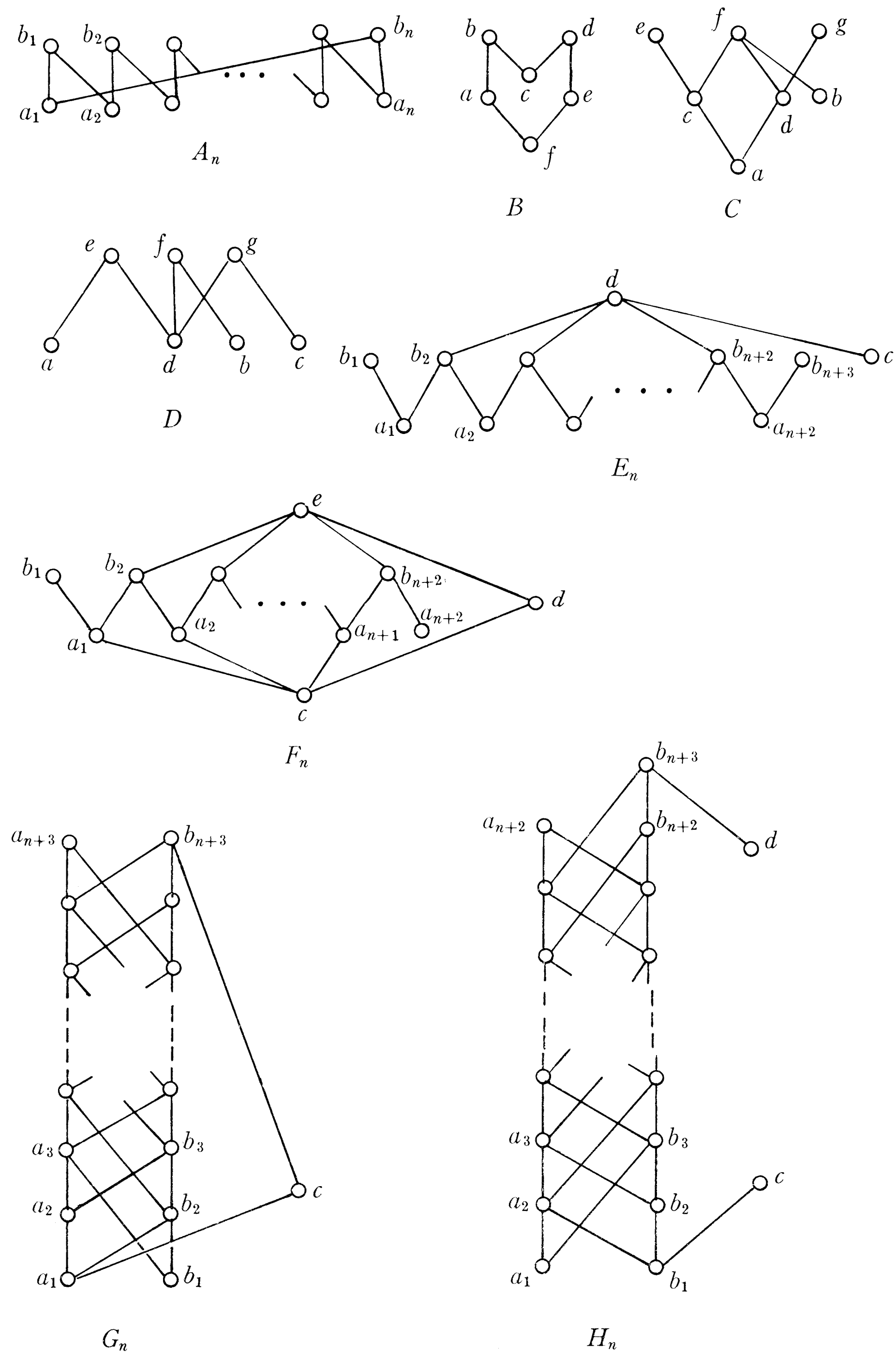

FIGURE 9

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satisfies $\{\mathbf{L}(P)|P \in \mathscr{P}\} = \mathscr{L}$ (see [**7**]). Hence the following is an alternate formulation of the Kelly-Rival result.

THEOREM 3.1. *A finite lattice is planar if and only if it does not contain a member of $\mathscr{P}$ as a subset.*

To begin the proof of Theorem 1.1, we first observe that if L is a finite planar lattice, L cannot contain a member of $\{A_3\} \cup \mathscr{R}$ as a subset. Certainly $A_3 \not\subseteq L$ by Theorem 3.1. If $R_0 = \{a_1, a_2, a_3, b_1, b_2, b_3, c\}$ is contained in L, then so is $\{a_2, a_3, b_2, b_3, c, a_1\}$, which is isomorphic to the partially ordered set B, contrary to Theorem 3.1. Finally if some

$$R_n = \{a_1, a_2, \ldots, a_{n+3}, b_1, b_2, \ldots, b_{n+3}, c\},$$

$n \geqq 1$, is contained in L, then so is $\{a_1, a_2, \ldots, a_{n+2}, b_2, b_3, \ldots, b_{n+3}, c\}$, which is isomorphic to G_{n-1}, again a contradiction. We have proven the "only if" direction of Theorem 1.1.

The converse is a little more complicated, and will be established gradually.

THEOREM 3.2. *Let L be a finite semidistributive lattice of breadth at most two.*

(a) *If L contains a member of $\{C, C^d, D, D^d\} \cup \{E_n, E_n{}^d, F_n|n \geqq 0\}$ as a subset then L contains B or B^d as a subset.*

(b) *If L contains H_n as a subset then L contains B, B^d, or G_m as a subset for some $m \leqq n$.*

Proof. (a) We proceed through the list of partially ordered sets in (a) in the order given; at each stage we will establish the existence of B or B^d in L, or (what is sufficient) we will exhibit in L a partially ordered set already considered. A similar strategy will be adopted elsewhere in this paper.

Case (i): C.

Choose $C = \{a, b, c, d, e, f, g\} \subseteq L$; observe that we may assume $e \wedge f = c$, $f \wedge g = d$, and $c \wedge d = a$. By the dual of Lemma 2.3 (i), $e \wedge g = e \wedge f \wedge g = c \wedge d = a$. Next, if $b \wedge c \nleqq d$ then $\{b, b \wedge c, e, a, d, f\}$ is a subset of L isomorphic to B^d, as desired; hence we now let $b \wedge c \leqq d$ and similarly $b \wedge d \leqq c$, which implies $b \wedge c = b \wedge d$. Thus $b \wedge g = b \wedge f \wedge g = b \wedge d = b \wedge c = b \wedge f \wedge e = b \wedge e$, and by $(\mathbf{SD}_\wedge)$ $b \wedge g = b \wedge (e \vee g)$. It follows that $e \vee g \ngeqq b$, and so $e \vee g \ngeqq f$. Hence $\{e, c, f, d, g, e \vee g\} \cong B^d$. Of course, a dual argument handles C^d.

Case (ii): D.

This one is easy. Let $D = \{a, b, c, d, e, f, g\} \subseteq L$; by Lemma 2.2(i) and the symmetry of D we may assume that $e \vee f < e \vee f \vee g$, that is, $e \vee f \ngeqq g$. Hence $\{e \vee f, g, e, f, a, d, b\} \cong C^d$, and by the previous case we are done.

Case (iii): $\{E_n|n \geqq 0\}$.

Let $n \geqq 0$ be minimal such that there is a subset of L isomorphic to either E_n or $E_n{}^d$. We first consider the case $n = 0$.

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Without loss of generality, let $E_0 = \{a_1, a_2, b_1, b_2, b_3, c, d\} \subseteq L$. We may assume that $b_1 \wedge b_2 = a_1$, $b_2 \wedge b_3 = a_2$, $a_1 \vee a_2 = b_2$, $b_2 \vee c = d$, and (from the dual of Lemma 2.3(i)) $b_1 \wedge b_3 = a_1 \wedge a_2$. If $a_1 \wedge a_2 \nleqq c$ then

$$\{a_1 \wedge a_2, c, a_1, a_2, b_1, d, b_3\} \cong C;$$

hence we let $a_1 \wedge a_2 < c$. If $a_1 \vee c > a_2$ and $a_2 \vee c > a_1$ then $a_1 \vee c = a_2 \vee c$, and by $(\mathbf{SD}_\vee)$ $a_1 \vee c = (a_1 \wedge a_2) \vee c = c$, a contradiction. By symmetry we may let $a_1 \nleqq a_2 \vee c$. If $a_1 \wedge c \nleqq a_2$ then

$$\{a_1, b_2, a_2, a_2 \vee c, c, a_1 \wedge c\} \cong B;$$

hence we assume $a_1 \wedge c < a_2$ whereupon $a_1 \wedge c = a_1 \wedge a_2 = a_1 \wedge b_2 \wedge b_3 = a_1 \wedge b_3$. By $(\mathbf{SD}_\wedge)$ $a_1 \wedge c = a_1 \wedge (c \vee b_3)$, and it follows that $a_1 \nleqq c \vee b_3$. Hence $b_2 \nleqq c \vee b_3$, and we have $\{b_2, d, c, c \vee b_3, b_3, a_2\} \cong B$.

Next assume $n = 1$, and let $E_1 = \{a_1, a_2, a_3, b_1, b_2, b_3, b_4, c, d\} \subseteq L$. We know immediately that $c < b_2 \vee b_3$, for otherwise

$$\{a_1, a_3, b_1, b_2 \vee b_3, b_4, c, d\} \cong E_0,$$

contradicting the choice of n. By Lemma 2.2(i) and the symmetry of E_1, we may assume that $b_3 \vee c < b_2 \vee b_3$, and hence $b_3 \vee c \ngeqq b_2$. But now $\{a_2, a_3, b_2, b_3, b_4, c, b_3 \vee c\} \cong E_0$, again contradicting the choice of n.

Finally suppose that $n > 1$, and let $\{a_1, \ldots, a_{n+2}, b_1, \ldots, b_{n+3}, c, d\}$ be a subset of L isomorphic to E_n. We may assume that $a_j \vee a_{j+1} = b_{j+1}$ for each $j \in \{1, \ldots, n+1\}$. Since $\{a_1, b_2, a_2, b_3, \ldots, b_{n+2}, a_{n+2}\}$ is a down-down fence, by Lemma 2.3 (ii) we may choose $i \in \{2, \ldots, n+1\}$ such that $a_1 \vee a_i < a_1 \vee a_{n+2}$ and $a_i \vee a_{n+2} < a_1 \vee a_{n+2}$. If $c \nleqq a_1 \vee a_i$ and $c \nleqq a_i \vee a_{n+2}$, then $\{a_1, a_i, a_{n+2}, b_1, a_1 \vee a_i, a_i \vee a_{n+2}, b_{n+3}, c, d\} \cong E_1$, which is a contradiction; therefore by symmetry let $c < a_1 \vee a_i$. Since $a_1 \vee a_2 = b_2$, in particular we have $i > 2$. Now set

$$k = \max\{j | 2 \leqq j \leqq n+1, a_1 \vee a_j < a_1 \vee a_{n+2}, a_j \vee a_{n+2} < a_1 \vee a_{n+2}\}.$$

By Lemma 2.3 (i), $a_j < a_1 \vee a_k$ for all $j \in \{1, \ldots, k\}$, which implies that $b_j = a_{j-1} \vee a_j < a_1 \vee a_k$ for each $j \in \{2, \ldots, k\}$; also, since $i \leqq k$ we have $c < a_1 \vee a_i \leqq a_1 \vee a_k$. If $b_{k+1} < a_1 \vee a_k$, then $a_1 \vee a_{k+1} \leqq a_1 \vee b_{k+1} \leqq a_1 \vee a_k < a_1 \vee a_{n+2}$, and $a_{k+1} \vee a_{n+2} \leqq a_k \vee a_{n+2} < a_1 \vee a_{n+2}$ by Lemma 2.3 (i), contradicting the maximality of k. Hence $b_{k+1} \nleqq a_1 \vee a_k$, and so $\{a_1, \ldots, a_k, b_1, \ldots, b_{k+1}, c, a_1 \vee a_k\} \cong E_{k-2}$. Since $k - 2 < n$, this contradicts the choice of n.

Case (iv): $\{F_n | n \geqq 0\}$.

Let $n \geqq 0$ be minimal such that there is a subset of L isomorphic to F_n.

First suppose $n = 0$, and let $F_0 = \{a_1, a_2, b_1, b_2, c, d, e\} \subseteq L$. We may assume that $a_1 \wedge d = c$, $b_2 \vee d = e$, $a_1 \vee a_2 = b_2$, and $b_1 \wedge b_2 = a_1$. If $b_1 \vee a_2 \ngeqq d$ then $\{b_1, b_1 \vee a_2, a_2, e, d, c\} \cong B$; hence let $b_1 \vee a_2 > d$, and dually $b_1 \wedge a_2 < d$. If $b_1 \vee d \ngeqq b_2$ then $\{b_1, b_1 \vee d, d, e, b_2, a_1\} \cong B$; hence

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we may let $b_1 \vee d > b_2 \vee d = e$, and dually $a_2 \wedge d < c$. But now $b_1 \vee d = b_1 \vee d \vee a_2 = b_1 \vee a_2$, and by ($\mathbf{SD}_\vee$) we have $b_1 \vee d = b_1 \vee (a_2 \wedge d) = b_1$, a contradiction.

Next suppose $n = 1$, and let $F_1 = \{a_1, a_2, a_3, b_1, b_2, b_3, c, d, e\} \subseteq L$. We may assume that $b_1 \wedge b_2 = a_1$, $a_1 \vee a_2 = b_2$, $b_2 \wedge b_3 = a_2$, and $a_2 \vee a_3 = b_3$. If $b_2 \vee d \ngeqq b_3$ then $\{a_1, a_2, b_1, b_2, b_3, d, b_2 \vee d\} \cong E_0$; hence we let $b_2 \vee d > b_3$. If $b_3 \vee d \ngeqq b_2$ then $\{a_2, a_3, b_2, b_3, c, d, b_3 \vee d\} \cong F_0$, contradicting the choice of n; hence $b_3 \vee d > b_2$, and so $b_2 \vee d = b_3 \vee d$. From Lemma 2.2(i), $b_2 \vee b_3 \ngeqq d$. But now $\{a_1, a_3, b_1, b_2 \vee b_3, c, d, e\} \cong F_0$, contradicting the choice of n.

Finally suppose $n > 1$, and let $\{a_1, \ldots, a_{n+2}, b_1, \ldots, b_{n+2}, c, d, e\}$ be a subset of L isomorphic to F_n. We may assume that $a_j \vee a_{j+1} = b_{j+1}$ for each $j \in \{1, \ldots, n + 1\}$. Since $\{a_1, b_2, a_2, b_3, \ldots, b_{n+2}, a_{n+2}\}$ is a down-down fence, by Lemma 2.3 (ii) we may choose $i \in \{2, \ldots, n + 1\}$ such that $a_1 \vee a_i < a_1 \vee a_{n+2}$ and $a_i \vee a_{n+2} < a_1 \vee a_{n+2}$. If $d \nleqq a_1 \vee a_i$ and $d \nleqq a_i \vee a_{n+2}$, then $\{a_1, a_i, a_{n+2}, b_1, a_1 \vee a_i, a_i \vee a_{n+2}, c, d, e\} \cong F_1$, which is a contradiction; therefore either $d < a_1 \vee a_i$ or $d < a_i \vee a_{n+2}$. Suppose that $d < a_1 \vee a_i$. Set

$$k = \max \{j | 2 \leqq j \leqq n + 1, a_1 \vee a_j < a_1 \vee a_{n+2}, a_j \vee a_{n+2} < a_1 \vee a_{n+2}\};$$

as in Case (iii), $b_j < a_1 \vee a_k$ for each $j \in \{2, \ldots, k\}$, and $i \leqq k$ implies that $d < a_1 \vee a_i \leqq a_1 \vee a_k$. By the maximality of k, we again conclude $b_{k+1} \nleqq a_1 \vee a_k$, and so $\{a_1, \ldots, a_k, b_1, \ldots, b_{k+1}, d, a_1 \vee a_k\} \cong E_{k-2}$. We now suppose $d < a_i \vee a_{n+2}$. Set

$$k' = \min \{j | 2 \leqq j \leqq n + 1, a_1 \vee a_j < a_1 \vee a_{n+2}, a_j \vee a_{n+2} < a_1 \vee a_{n+2}\}.$$

As before, $b_j < a_{k'} \vee a_{n+2}$ for $j \in \{k' + 1, \ldots, n + 2\}$, and since $k' \leqq i$ we have $d < a_i \vee a_{n+2} \leqq a_{k'} \vee a_{n+2}$. From the minimality of k', it follows that $b_{k'} \nleqq a_{k'} \vee a_{n+2}$, and hence

$$\{a_{k'}, \ldots, a_{n+2}, b_{k'}, \ldots, b_{n+2}, c, d, a_{k'} \vee a_{n+2}\} \cong F_{n+1-k'}.$$

Since $n + 1 - k' < n$, this contradicts the choice of n.

(b) Let $n \geqq 0$ be minimal such that there is a subset of L isomorphic to H_n.

First assume $n = 0$, and let $H_0 = \{a_1, a_2, b_1, b_2, b_3, c, d\} \subseteq L$. If $c \wedge a_2 \nleqq b_3$ then $\{c \wedge a_2, a_2, a_1, b_3, b_2, b_1\} \cong B$; hence we let $c \wedge a_2 < b_3$. Since $a_2 \wedge b_3 \geqq a_1$, we have $a_2 \wedge b_3 \nleqq c$. If $c \wedge b_3 \nless a_2$ then

$$\{b_1, d, a_2 \wedge b_3, c \wedge b_3, a_2, b_3, c\} \cong C;$$

hence assume $c \wedge b_3 < a_2$. If $b_2 \nleqq c \vee a_1$ then $\{c, c \vee a_1, a_1, b_3, b_2, b_1\} \cong B$; hence assume $b_2 < c \vee a_1 \leqq c \vee a_2$ and dually $b_2 > d \wedge a_2$. If $c > a_2 \wedge b_2$ then $c \wedge b_2 = (c \wedge b_3) \wedge b_2 \leqq a_2 \wedge b_2 \leqq c \wedge b_2$, implying that $c \wedge b_2 = a_2 \wedge b_2$. Since $c \vee a_2 > b_2$ this is a violation of ($\mathbf{SD}_\wedge$), and so $c \ngeqq a_2 \wedge b_2$. If $c \vee d \ngeqq b_2$ then $\{c, c \vee d, d, b_3, b_2, b_1\} \cong B$; hence assume $c \vee d > b_2$. From $b_2 \vee d \leqq b_3$ it follows that $b_2 \vee d \ngeqq c$ and $b_2 \vee d \ngeqq a_2$. If $c \vee d \ngeqq a_2$ then

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$\{a_2 \wedge b_2, d, a_2, b_2 \vee d, b_1, c, c \vee d\} \cong F_0$; hence assume $c \vee d > a_2$. If $c \vee a_2 \ngeqq b_3$ then $\{b_1, b_2, b_3, a_1, a_2, c \vee a_2, c\} \cong G_0$ as desired; hence assume $c \vee a_2 > b_3$. Now $c \vee a_2 \geqq c \vee b_3 \geqq c \vee d \geqq c \vee a_2$, implying $c \vee a_2 = c \vee d$. By $(\mathbf{SD}\vee)$ and the above results, $d < c \vee d = c \vee (d \wedge a_2) \leqq c \vee b_2 \leqq c \vee a_1$. But a dual argument shows that $d \wedge a_2 < c$, and hence $c \vee d = c \vee (d \wedge a_2) = c$, a contradiction.

Therefore $n > 0$. Let $H_n = \{a_1, \ldots, a_{n+2}, b_1, \ldots, b_{n+3}, c, d\} \subseteq L$. If $c \vee d \ngtr a_{n+1}$ then $\{c, c \vee d, d, b_{n+3}, a_{n+1}, b_1\} \cong B$; hence we let $c \vee d > a_{n+1}$. If $c \vee d \ngtr b_{n+2}$ then $\{c, c \vee d, d, b_{n+3}, b_{n+2}, b_1\} \cong B$; hence we let $c \vee d \geqq d \vee b_{n+2}$. If $d \vee b_{n+2} \ngeqq a_{n+1}$ then

$$\{a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1}, d \vee b_{n+2}, c, d\} \cong H_{n-1},$$

contradicting the choice of n; hence $d \vee b_{n+2} > a_{n+1}$. Next, we may assume $c \vee a_{n+2} > b_{n+2}$, for otherwise $\{b_1, \ldots, b_{n+2}, a_1, \ldots, a_{n+1}, c \vee a_{n+2}, c\} \cong G_{n-1}$, as desired. Further, we assume $c \vee a_{n+2} > b_{n+3}$, for otherwise

$$\{b_1, \ldots, b_{n+3}, a_1, \ldots, a_{n+2}, c \vee a_{n+2}, c\} \cong G_n;$$

thus we have that $c \vee a_{n+2} \geqq c \vee b_{n+3} \geqq c \vee d$. We may assume $b_{n+3} \vee c > a_{n+2}$, for otherwise $\{a_2, d, a_{n+2}, b_{n+3}, b_1, c, b_{n+3} \vee c\} \cong F_0$. It follows that $b_{n+3} \vee c = a_{n+2} \vee c$, and by $(\mathbf{SD}\vee)$ $b_{n+3} \vee c = (b_{n+3} \wedge a_{n+2}) \vee c$. We may assume $b_{n+3} \wedge a_{n+2} < c \vee d$, for otherwise

$$\{b_{n+3} \wedge a_{n+2}, b_{n+3}, d, c \vee d, c, b_1\} \cong B.$$

Thus $c \vee (b_{n+3} \wedge a_{n+2}) = c \vee d$, and by $(\mathbf{SD}\vee)$ $c \vee d = c \vee (b_{n+3} \wedge a_{n+2} \wedge d) = c \vee (a_{n+2} \wedge d)$, implying $c \ngeqq a_{n+2} \wedge d$. However, $d \wedge a_{n+2} < b_{n+3}$; therefore, letting k be minimal such that $d \wedge a_{n+2} < b_k$, we have $2 \leqq k \leqq n + 3$. If $k = n + 3$, then $d \wedge a_{n+2} \nless b_{n+2}$, and

$$\{b_{n+2}, b_{n+1}, a_{n+2}, d \wedge a_{n+2}, d, b_{n+3}\} \cong B^d;$$

hence we assume $k < n + 3$. If $d \wedge a_{n+2} < a_{k-1}$, then

$$\{d \wedge a_{n+2}, a_{k-1}, a_k, \ldots, a_{n+2}, b_{k-1}, b_k, \ldots, b_{n+3}, d\} \cong G_{n-k+2};$$

hence we assume $d \wedge a_{n+2} \nless a_{k-1}$. If $k = 2$ we have

$$\{b_1, d \wedge a_{n+2}, c, b_2, d, a_1, b_3\} \cong E_0;$$

thus we let $k > 2$. But now $\{a_1, \ldots, a_{k-1}, b_1, \ldots, b_k, c, d \wedge a_{n+2}\} \cong H_{k-3}$ where $0 \leqq k - 3 < n$, contradicting the choice of n.

As a corollary, we obtain an improvement of Theorem 3.1 for finite semidistributive lattices.

COROLLARY 3.3. *A finite semidistributive lattice L is planar if and only if it does not contain A_3, B, B^d, or G_n, $n \geqq 0$, as a subset.*

Proof. We need only prove the "if" direction. By Lemma 2.1, L contains

A_n for some $n \geqq 3$ if and only if it contains A_3. By Theorems 3.1 and 3.2 the corollary follows.

THEOREM 3.4. *Let L be a finite semidistributive lattice of breadth at most two.*

(a) *If L contains B or B^d as a subset then L contains R_0 as a subset.*

(b) *If L contains G_n as a subset for some $n \geqq 0$ then L contains R_m as a subset for some $m \leqq n + 1$.*

Proof. (a) Assume $B = \{a, b, c, d, e, f\} \subseteq L$. We first observe that we must have $a \vee e > c$, for otherwise $\{a \vee e, a \vee c, c \vee e\}$ is an antichain, contrary to Lemma 2.2(ii). If $c \wedge a \nleqq e$, then $\{f, a, b, c \wedge a, c, d, e\}$ is a subset of L isomorphic to R_0, as desired. Thus we assume $c \wedge a \leqq e$ and similarly $c \wedge e \leqq a$, which implies $c \wedge a = c \wedge e$. But since $c < a \vee e$ this is a violation of $(\mathbf{SD}_\wedge)$. Of course, a dual argument handles B^d.

(b) Let $n \geqq 0$ be minimal such that there is a subset

$$\{a_1, \ldots, a_{n+3}, b_1, \ldots, b_{n+3}, c\}$$

of L isomorphic to G_n. Since $b_{n+2} \wedge c \geqq a_1$, we have $b_{n+2} \wedge c \nleqq b_1$. Choose k minimal such that $b_{n+2} \wedge c < b_k$; then $2 \leqq k \leqq n + 2$. First we assume $k > 2$. If $b_{n+2} \wedge c \nless a_k$, then $\{a_1, \ldots, a_k, b_1, \ldots, b_k, b_{n+2} \wedge c\} \cong G_{k-3}$ where $0 \leqq k - 3 < n$; on the other hand, if $b_{n+2} \wedge c < a_k$ then

$$\{b_{n+2} \wedge c, a_k, a_{k+1}, \ldots, a_{n+3}, b_{k-1}, b_k, \ldots, b_{n+3}, c\} \cong G_{n+2-k}$$

where $0 \leqq n + 2 - k < n$. In either case we have a contradiction to the choice of n. Hence $k = 2$; that is, $b_{n+2} \wedge c < b_2$, which implies $b_{n+2} \wedge c = b_2 \wedge c$.

We first consider the case $b_2 \wedge c \nless a_2$. Assume that $b_2 \wedge c \nleqq a_2 \vee b_1$; then $\{a_2, a_2 \vee b_1, b_1, b_2, b_2 \wedge c, a_1\} \cong B$, and we are done by part (a). Hence $b_2 \wedge c \leqq a_2 \vee b_1$. Now, if $a_2 \wedge b_2 \nleqq c$ we have

$$\{a_1, b_1, a_2 \wedge b_2, b_2 \wedge c, a_2, b_2, c\} \cong C,$$

while if $a_2 \wedge b_2 < c$ we have

$$\{b_2 \wedge c, b_1, c, (b_2 \wedge c) \vee b_1, a_2 \wedge b_2, a_2, a_2 \vee b_1\} \cong F_0.$$

In either case we are done by Theorem 3.2 and part (a).

Therefore $b_{n+2} \wedge c = b_2 \wedge c < a_2$, and by duality $a_2 \vee c > b_{n+2}$. If $a_2 \wedge b_1 \leqq c$ then $a_2 \wedge b_1 \leqq c \wedge b_1 = c \wedge b_2 \wedge b_1 \leqq a_2 \wedge b_1$, implying $a_2 \wedge b_1 = c \wedge b_1$, and by $(\mathbf{SD}_\wedge)$ we have $a_2 \wedge b_1 = (a_2 \vee c) \wedge b_1 = b_1$, a contradiction. Thus $a_2 \wedge b_1 \nleqq c$ and by duality $a_{n+3} \vee b_{n+2} \ngeqq c$. Now

$$\{a_1, \ldots, a_{n+3}, a_{n+3} \vee b_{n+2}, a_2 \wedge b_1, b_1, \ldots, b_{n+3}, c\} \cong R_{n+1},$$

and Theorem 3.4 is established.

The assumption that L has breadth at most two is necessary. For example, the lattice of Figure 10 has breadth three, is semidistributive, and contains B as a subset (the shaded elements), but does not contain R_0 as a subset.

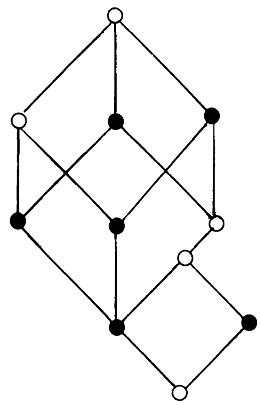

FIGURE 10

COROLLARY (THEOREM 1.1). *A finite semidistributive lattice is planar if and only if it does not contain* A_3 *or* R_n, $n \geqq 0$, *as a subset.*

Proof. Immediate from Corollary 3.3 and Theorem 3.4.

4. The proof of Theorem 1.2. It is well-known and easy to prove that if L is an arbitrary lattice, A_3 is a subset of L if and only if C_2^3 is a sublattice of L. Also, it is evident from Figures 2 and 6 that R_n is a subset of S_n for each $n \geqq 0$; thus if S_n is a sublattice of a lattice L, certainly R_n is a subset of L. This completes the "if" direction of Theorem 1.2.

For each $n \geqq 0$, let $P_n = \mathbf{L}(R_n)$, the completion of R_n (see Figure 11); recall that if R_n is a subset of a lattice L, so is P_n. We require one more lemma.

LEMMA 4.1. *Let L be a lattice satisfying* (**W**), *and let* $R_n \backslash \{c\}$ *be a subset of L for some* $n \geqq 0$. *Then* $S_n \backslash \{c\}$ *is a sublattice of L.*

Proof. Let $R_n \backslash \{c\} = \{a_1, \ldots, a_{n+3}, b_1, \ldots, b_{n+3}\} \subseteq L$. Since $\mathbf{L}(R_n \backslash \{c\}) = P_n \backslash \{c\}$, $P_n \backslash \{c\}$ is a subset of L, as indicated in Figure 11. Moreover we claim that the elements $\{a_1, \ldots, a_{n+2}, b_2, \ldots, b_{n+3}\}$ of $P_n \backslash \{c\}$ generate a sublattice of L isomorphic to $S_n \backslash \{c\}$. For simplicity, we will give the construction only in the case $n = 0$; an induction based on similar arguments will handle the general case. If $n = 0$, the required sublattice of L isomorphic to $S_0 \backslash \{c\}$ is given in Figure 12. Notice that $a_1 \vee (a_2 \wedge b_2) < a_2 \wedge (a_1 \vee b_2)$, $(a_2 \wedge b_3) \vee b_2 < (a_2 \vee b_2) \wedge b_3$, and $a_2 \wedge b_3 \nleqq a_1 \vee b_2$ hold by virtue of (**W**).

Now let L be a finite semidistributive lattice satisfying (**W**). We may assume that A_3 is not a subset of L, which implies that $b(L) \leqq 2$ by Lemma 2.1. Let $n \geqq 0$ be minimal such that there exists a subset of L isomorphic to R_n. Choose

$$R_n = \{a_1, \ldots, a_{n+3}, b_1, \ldots, b_{n+3}, c\} \subseteq [a_1 \wedge b_1, a_{n+3} \vee b_{n+3}] \subseteq L$$

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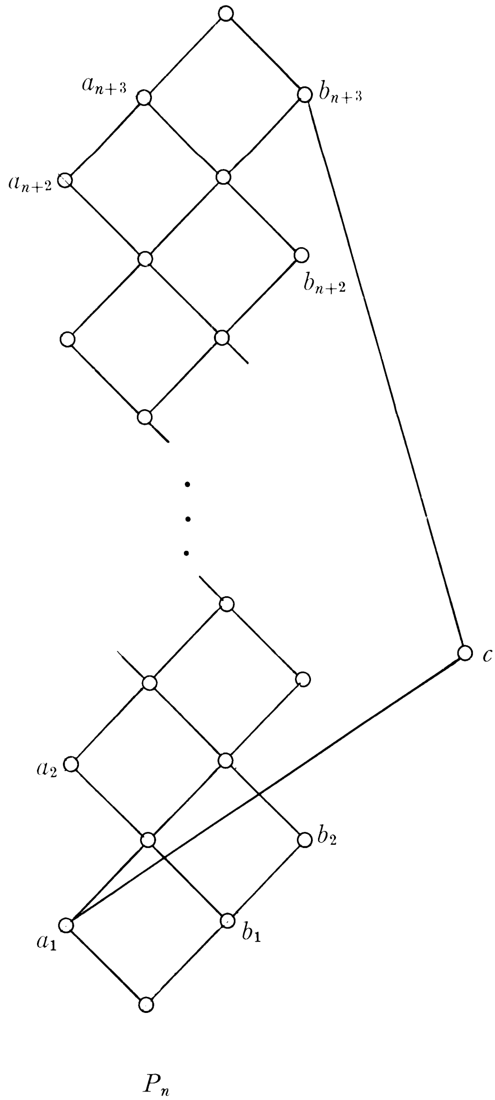

FIGURE 11

such that there does not exist a subset of L isomorphic to R_n in any proper subinterval of $[a_1 \wedge b_1, a_{n+3} \vee b_{n+3}]$. Then we have seen that we can find $S_n \subseteq L$, generated by $\{a_1, \ldots, a_{n+2}, b_2, \ldots, b_{n+3}, c\}$, such that $S_n \backslash \{c\}$ is a sublattice of L. Observe that we may assume that $a_2 \wedge b_2 = b_1$ and $a_{n+2} \vee b_{n+2} = a_{n+3}$. To complete the proof of Theorem 1.2 we need only show that $a_{n+2} \vee c = a_{n+3} \vee b_{n+3}$ and $b_1 \vee c = b_{n+3}$ (a dual argument handles the corresponding meets).

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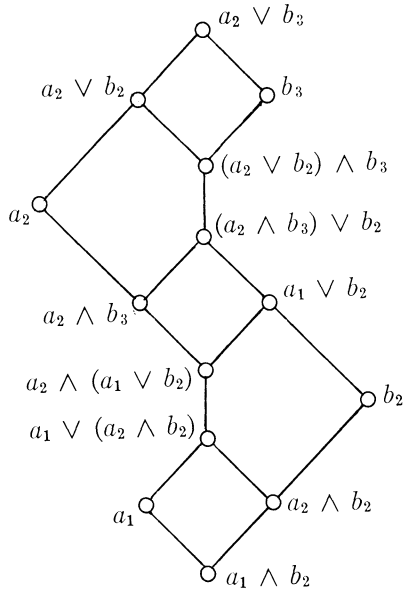

FIGURE 12

First, we may assume that $b_{n+3} = (a_{n+3} \wedge b_{n+3}) \vee c$. Hence, since

$$\{a_{n+2}, a_{n+3}, a_{n+3} \wedge b_{n+3}, b_{n+3}, c\}$$

is a down-down fence, Lemma 2.3 (i) implies that $a_{n+2} \vee c = a_{n+2} \vee (a_{n+3} \wedge b_{n+3}) \vee c = a_{n+3} \vee b_{n+3}$, as desired.

Now suppose $n = 0$. If $a_2 \wedge b_3 \nleqq b_2 \vee c$ then

$$\{a_2 \wedge b_3, a_3 \wedge b_3, b_2, b_2 \vee c, c, a_1\}$$

is a subset of L isomorphic to B, and is contained in $[a_1 \wedge b_2, b_3]$. By Theorem 3.4 there is a subset of L isomorphic to R_0 which is contained in $[a_1 \wedge b_2, b_3]$, a proper subinterval of $[a_1 \wedge b_1, a_3 \vee b_3]$, contrary to assumption. Therefore $a_2 \wedge b_3 < b_2 \vee c$, and by (**W**) we are forced to conclude that $b_3 = b_2 \vee c$. If $(a_2 \wedge b_3) \vee c \ngeqq b_2$ then $\{a_2, a_3, b_2, b_3, (a_2 \wedge b_3) \vee c, a_2 \wedge b_3\}$ is a subset of L isomorphic to B, and is contained in $[b_1, a_3 \vee b_3]$, a proper subset of $[a_1 \wedge b_1, a_3 \vee b_3]$. By Theorem 3.4 we again have a contradiction. Thus $(a_2 \wedge b_3) \vee c \geqq b_2 \vee c$ which implies $(a_2 \wedge b_3) \vee c = b_3 = b_2 \vee c$. By (**SD**$\vee$) we conclude that $b_3 = (a_2 \wedge b_3 \wedge b_2) \vee c = b_1 \vee c$, completing the case $n = 0$.

We now assume $n > 0$. As in the case $n = 0$, our first goal will be to prove that $b_2 \vee c = b_{n+3}$. Choose k maximal such that $b_2 \vee c > a_k$; it is clear that $1 \leqq k \leqq n + 1$. If $k = 1$, then $\{a_2, a_3, b_2 \vee c, c, a_1\} \cong B$, and by Theorem 3.4 L must contain a subset isomorphic to R_0. contrary to the choice of n. Assume $2 \leqq k \leqq n$. If $b_2 \vee c > b_{k+1}$, then

$$\{a_1, \ldots, a_{k+2}, b_1, \ldots, b_{k+1}, b_2 \vee c, c\} \cong R_{k-1},$$

and $k - 1 < n$, contradicting the choice of n. On the other hand, if $b_2 \vee c \ngtr b_{k+1}$ then $\{a_k, \ldots, a_{n+2}, b_{k+1}, \ldots, b_{n+3}, b_2 \vee c\} \cong G_{n-k}$; by Theorem 3.4 L must contain a subset isomorphic to R_m for some $m \leqq n - k + 1$, and since $1 \leqq n - k + 1 \leqq n - 1$ this again contradicts the choice of n. Therefore $k = n + 1$, and so $b_2 \vee c > a_{n+1}$. If $b_2 \vee c \ngeqq b_{n+2}$, then

$$\{a_{n+2}, a_{n+3}, b_{n+2}, b_{n+3}, b_2 \vee c, a_{n+1}\} \cong B,$$

which is a contradiction; hence $b_2 \vee c > b_{n+2}$. If $b_2 \vee c \ngeqq a_{n+2} \wedge b_{n+3}$ then

$$\{a_1, \ldots, a_{n+1}, a_{n+2} \wedge b_{n+3}, b_2, \ldots, b_{n+2}, b_2 \vee c, c\}$$

is a subset of L isomorphic to G_{n-1}, and is contained in $[a_1 \wedge b_2, b_{n+3}]$, a proper subinterval of $[a_1 \wedge b_1, a_{n+3} \vee b_{n+3}]$. By Theorem 3.4 $[a_1 \wedge b_2, b_{n+3}]$ contains a subset isomorphic to R_m for some $m \leqq n$. By the choice of n we must have $m = n$; but this contradicts the minimality of R_n. Thus $b_2 \vee c \geqq a_{n+2} \wedge b_{n+3}$, and by (**W**) we conclude $b_2 \vee c = b_{n+3}$.

Now if $a_2 \vee c \ngeqq b_2$, we have $\{a_2, \ldots, a_{n+3}, b_2, \ldots, b_{n+3}, a_2 \vee c\} \cong R_{n-1}$, contradicting the choice of n. Hence $a_2 \vee c \geqq b_2$, and so $a_2 \vee c = b_{n+3} = b_2 \vee c$. By (**SD**$\vee$), $b_{n+3} = (a_2 \wedge b_2) \vee c = b_1 \vee c$, and the proof of Theorem 1.2 is complete.

5. The corollaries. With Theorem 1.6 in hand the proof of Corollary 1.7 is simple although not obvious. The principal observation is this:

LEMMA 5.1. *Let L be a finite lattice satisfying* (**W**) *and let a, b be elements of L with $a < b$, a join reducible and b join irreducible. Then there exist elements a', b' of L such that $a \leqq a'$, $b' \leqq b$, b' is join irreducible, and b' is the unique cover of a'.*

Proof. Let a' be a maximal join reducible element in $\{x \in L | a \leqq x \leqq b\}$. Then $a' < b$. Since L satisfies (**W**) a' must have a unique cover b'. Evidently, $b' \leqq b$.

Let L be a finite, semidistributive lattice satisfying (**W**) and of breadth at most two. In addition, let us suppose that L is nonplanar. Then according to Theorem 1.6 L contains a sublattice isomorphic to S_n for some $n \geqq 0$ (cf Figure 2). Then

$$a_1 \vee b_1 < a_2 \wedge (a_1 \vee b_2) < b_{n+2} \vee (a_{n+2} \wedge b_{n+3}) < a_{n+3} \wedge b_{n+3}$$

(cf. Figure 12). In view of Lemma 5.1 there exists elements a_1', b_1', a_2', b_2' such that

$$a_1 \vee b_1 \leqq a_1' < b_1' \leqq a_2 \wedge (a_1 \vee b_2),$$

$$b_{n+2} \vee (a_{n+2} \wedge b_{n+3}) \leqq a_2' < b_2' \leqq a_{n+3} \wedge b_{n+3},$$

a_1', a_2' are join reducible (whence meet irreducible), b_1', b_2' are join irreducible, and b_1' covers a_1', b_2' covers a_2'. Let $\theta_1 = \theta(a_1', b_1')$, $\theta_2 = \theta(a_2', b_2')$, be the smallest congruence relations identifying a_1' with b_1', and a_2' with b_2', respectively. Then evidently $\theta_1 \neq \theta_2$ and any congruence relation θ smaller than

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either is the equality relation. In particular, L cannot be subdirectly irreducible. This establishes Corollary 1.7.

Corollary 1.8 now follows at once from Theorem 1.5.

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