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Abstract

The *Newell–Littlewood (NL) numbers* are tensor product multiplicities of Weyl modules for the classical groups in the stable range. Littlewood–Richardson (LR) coefficients form a special case. Klyachko connected eigenvalues of sums of Hermitian matrices to the saturated LR-cone and established defining linear inequalities. We prove analogues for the saturated NL-cone: a description by *Extended Horn inequalities* (as conjectured in part II of this series), where, using a result of King's, this description is controlled by the saturated LR-cone and thereby recursive, just like the Horn inequalities; a minimal list of defining linear inequalities; an eigenvalue interpretation; and a factorization of Newell–Littlewood numbers, on the boundary.

1. Introduction

Fix $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$. This is the third installment in a series [GOY20a, GOY20b] about the *Newell–Littlewood (NL) numbers* [New51, Lit58]

$$N_{\lambda,\mu,\nu} = \sum_{\alpha,\beta,\gamma} c^{\lambda}_{\alpha,\beta} c^{\mu}_{\beta,\gamma} c^{\nu}_{\gamma,\alpha}; \qquad (1)$$

the indices are partitions in $\mathrm{Par}_n = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geqslant 0} : \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n\}$. In (1), $c^{\lambda}_{\alpha,\beta}$ is the *Littlewood–Richardson (LR) coefficient*. The Littlewood–Richardson coefficients are themselves Newell–Littlewood numbers: if $|\nu| = |\lambda| + |\mu|$ then $N_{\lambda,\mu,\nu} = c^{\nu}_{\lambda\mu}$. The goal of this series is to establish analogues of results known for Littlewood–Richardson coefficients. This paper proves Newell–Littlewood generalizations of breakthrough results of Klyachko [Kly98].

The paper [GOY20a] investigated

$$\text{NL-semigroup}(n) = \{(\lambda, \mu, \nu) \in (\mathrm{Par}_n)^3 \,:\, N_{\lambda,\mu,\nu} > 0\}.$$

Indeed, an NL-semigroup is a finitely generated semigroup [GOY20a, § 5.2]. A good approximation of it is the saturated semigroup:

$$\text{NL-sat}(n) = \{(\lambda, \mu, \nu) \in (\mathrm{Par}^{\mathbb{Q}}_n)^3 \,:\, \exists t > 0\; N_{t\lambda,t\mu,t\nu} \neq 0\},$$

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where $\mathrm{Par}_n^{\mathbb{Q}} = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Q}^n : \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}$. Our main results give descriptions of NL-sat(n), including with a minimal list of defining linear inequalities.

Fix $m \in \mathbb{N}$ and consider the symplectic Lie algebra $\mathfrak{sp}(2m,\mathbb{C})$. The irreducible $\mathfrak{sp}(2m,\mathbb{C})$-representations $V(\lambda)$ are parametrized by their highest weight $\lambda \in \mathrm{Par}_m$ (see § 3.1 for details). The tensor product multiplicities $\mathrm{mult}^m_{\lambda,\mu,\nu}$ are defined by

$$V(\lambda) \otimes V(\mu) = \sum_{\nu \in \mathrm{Par}_m} V(\nu)^{\oplus \mathrm{mult}^m_{\lambda,\mu,\nu}}.$$

Since $\mathfrak{sp}(2m,\mathbb{C})$-representations are self-dual, $\mathrm{mult}^m_{\lambda,\mu,\nu}$ is symmetric in its inputs. The supports of these multiplicities (and more generally when $\mathfrak{sp}(2m,\mathbb{C})$ is replaced by any semisimple Lie algebra) are of significant interest (see, for example, the survey [Kum15] and the references therein). Consider the finitely generated semigroup

$$\mathfrak{sp}\text{-semigroup}(m) = \{(\lambda,\mu,\nu) \in (\mathrm{Par}_m)^3 : \mathrm{mult}^m_{\lambda,\mu,\nu} > 0\},$$

and the cone generated by it,

$$\mathfrak{sp}\text{-sat}(m) = \{(\lambda,\mu,\nu) \in (\mathrm{Par}_m^{\mathbb{Q}})^3 : \exists t > 0\ \mathrm{mult}^m_{t\lambda,t\mu,t\nu} > 0\}.$$

For $m \geqslant n$, by postpending 0s, Par_n embeds into Par_m. Newell–Littlewood numbers are tensor product multiplicities for $\mathfrak{sp}(2m,\mathbb{C})$ in the *stable range* [KT87, Corollary 2.5.3]:

$$\forall (\lambda,\mu,\nu) \in (\mathrm{Par}_n)^3 \quad \text{if } m \geqslant 2n \text{ then } \mathrm{mult}^m_{\lambda,\mu,\nu} = N_{\lambda,\mu,\nu}. \tag{2}$$

Now, (2) immediately implies

$$\text{NL-sat}(n) = \mathfrak{sp}\text{-sat}(m) \cap (\mathrm{Par}_n^{\mathbb{Q}})^3 \quad \text{for any } m \geqslant 2n. \tag{3}$$

Our first result says the relationship of NL-sat to $\mathfrak{sp}$-sat is *independent* of the stable range.

THEOREM 1.1. *For any $m \geqslant n \geqslant 1$,*

$$\text{NL-sat}(n) = \mathfrak{sp}\text{-sat}(m) \cap (\mathrm{Par}_n^{\mathbb{Q}})^3.$$

Theorem 1.1 has a number of consequences. Define

$$\text{LR-sat}(n) = \{(\lambda,\mu,\nu) \in (\mathrm{Par}_n^{\mathbb{Q}})^3 : \exists t > 0\ c^{t\nu}_{t\lambda,t\mu} > 0\}.$$

Klyachko [Kly98] showed that LR-sat(n) describes the possible eigenvalues λ, μ, ν of two $n \times n$ Hermitian matrices A, B, C (respectively) such that $A + B = C$. Similarly, Theorem 1.1 shows that NL-sat(n) describes solutions to a more general eigenvalue problem; see § 2.6 and Proposition 3.1.[1]

Another major accomplishment of [Kly98] was the first proved description of LR-sat(n) via linear inequalities. We have three such descriptions of NL-sat(n). We now state the first of these. Set $[n] = \{1, \dots, n\}$ and $[a,b] = \{a, a+1, \dots, b\}$ for $a \leqslant b$. For $A \subset [n]$ and $\lambda \in \mathrm{Par}_n$, let λ_A be the partition using the only parts indexed by A; namely, if $A = \{i_1 < \cdots < i_r\}$ then $\lambda_A = (\lambda_{i_1}, \dots, \lambda_{i_r})$. In particular, $|\lambda_A| = \sum_{i \in A} \lambda_i$. Using the known descriptions of $\mathfrak{sp}$-sat(n) [BK06, Res10, Res12] we deduce from Theorem 1.1 a *minimal* list of inequalities defining NL-sat(n).

THEOREM 1.2. *Let $(\lambda,\mu,\nu) \in (\mathrm{Par}_n)^3$. Then $(\lambda,\mu,\nu) \in$ NL-sat(n) if and only if*

$$0 \leqslant |\lambda_A| - |\lambda_{A'}| + |\mu_B| - |\mu_{B'}| + |\nu_C| - |\nu_{C'}| \tag{4}$$

[1] We remark that since $(\lambda,\mu,\nu) \in$ LR-sat(n) is also in NL-sat(n) if and only if $|\lambda| + |\mu| = |\nu|$ (see [GOY20a, Lemma 2.2(II)]), LR-sat(n) is a facet of the $\mathfrak{sp}_{2n}$-sat(n).

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for any subsets $A, A', B, B', C, C' \subset [n]$ *such that:*

(1) $A \cap A' = B \cap B' = C \cap C' = \emptyset$;

(2) $|A| + |A'| = |B| + |B'| = |C| + |C'| = |A'| + |B'| + |C'| =: r$;

(3) $c^{\tau^0(C,C')}_{\tau^0(A,A')^{\vee[(2n-2r)^r]}\ \tau^0(B,B')^{\vee[(2n-2r)^r]}} = c^{\tau^2(C,C')}_{\tau^2(A,A')^{\vee[r^r]}\ \tau^2(B,B')^{\vee[r^r]}} = 1.$

Moreover, this list of inequalities is irredundant.

The definition of the partitions occurring in condition (3) is in § 3.2.

The proofs of Theorems 1.1 and 1.2 use ideas of P. Belkale and S. Kumar [BK06] on their deformation of the cup product on flag manifolds, as well as the third author's work on GIT-semigroups/cones [Res10, Res12]. We interpret NL-sat(n) from the latter perspective in § 5 (see Proposition 5.2) by study of the *truncated tensor cone*. Our argument requires us to generalize [BK06, Theorem 28] and [Res10, Theorem B] (recapitulated here as Theorem 2.3); see Theorem 5.1. As an application, we obtain Theorem 1.3 below, which is a factorization of the NL-coefficients on the boundary of NL-sat(n). Let $\lambda \in \mathrm{Par}_n$ and $A, A' \subset [n]$. Write $A' = \{i'_1 < \cdots < i'_s\}$ and $A = \{i_1 < \cdots < i_t\}$ and set

$$\lambda_{A,A'} = (\lambda_{i'_1}, \ldots, \lambda_{i'_s}, -\lambda_{i_t}, \ldots, -\lambda_{i_1}) \quad \text{and} \quad \lambda^{A,A'} = \lambda_{[n]-(A\cup A')}, \text{etc.}$$

Theorem 1.3. *Let* $A, A', B, B', C, C' \subset [n]$ *such that:*

(1) $A \cap A' = B \cap B' = C \cap C' = \emptyset$;

(2) $|A| + |A'| = |B| + |B'| = |C| + |C'| = |A'| + |B'| + |C'| =: r$;

(3) $c^{\tau^0(C,C')}_{\tau^0(A,A')^{\vee[(2n-2r)^r]}\ \tau^0(B,B')^{\vee[(2n-2r)^r]}} = c^{\tau^2(C,C')}_{\tau^2(A,A')^{\vee[r^r]}\ \tau^2(B,B')^{\vee[r^r]}} = 1,$

as in Theorem 1.2. For $(\lambda, \mu, \nu) \in (\mathrm{Par}_n)^3$ *such that*

$$0 = |\lambda_A| - |\lambda_{A'}| + |\mu_B| - |\mu_{B'}| + |\nu_C| - |\nu_{C'}|, \tag{5}$$

$$N_{\lambda,\mu,\nu} = c^{\nu^*_{C,C'}}_{\lambda_{A,A'},\mu_{B,B'}} N_{\lambda^{A,A'},\mu^{B,B'},\nu^{C,C'}}. \tag{6}$$

Theorem 1.3 is analogous to [DW11, Theorem 7.4] and [KTT09, Theorem 1.4] for $c^{\nu}_{\lambda,\mu}$.

Knutson and Tao's celebrated *saturation theorem* [KT99] proves, *inter alia*, that LR-semigroup(n) is described by Horn's inequalities (see, for example, Fulton's survey [Ful00]). This posits a generalization.

Conjecture 1.4 (NL-saturation [GOY20a, Conjecture 5.5]). Let $(\lambda, \mu, \nu) \in (\mathrm{Par}_n)^3$. Then $N_{\lambda,\mu,\nu} \neq 0$ if and only if $|\lambda| + |\mu| + |\nu|$ is even and there exists $t > 0$ such that $N_{t\lambda,t\mu,t\nu} \neq 0$.

Theorem 1.2 permits us to prove Conjecture 1.4 for $n \leqslant 5$, by computer-aided calculation of Hilbert bases; see § 6. This is the strongest evidence of the conjecture to date; previously, [GOY20a, Corollary 5.16] proved the $n = 2$ case by combinatorial reasoning.

Let $\lambda_1, \ldots, \lambda_s \in \mathrm{Par}_n$ for $s \geqslant 3$. Treat the indices $1, \ldots, s$ as elements of $\mathbb{Z}/s\mathbb{Z}$. We introduce the *multiple Newell–Littlewood number* as

$$N_{\lambda_1,\ldots,\lambda_s} = \sum_{(\alpha_1,\ldots,\alpha_s)\in(\mathrm{Par}_n)^s} \prod_{i\in\mathbb{Z}/s\mathbb{Z}} c^{\lambda_i}_{\alpha_i\ \alpha_{i+1}}. \tag{7}$$

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When $s=3$, we recover the Newell–Littlewood numbers.[2] Consider the associated semigroup and cone:

$$\mathrm{NL}^s\text{-semigroup}(n)=\{(\lambda_1,\ldots,\lambda_s)\in(\mathrm{Par}_n)^s\,:\,N_{\lambda_1,\ldots,\lambda_s}>0\}$$

and

$$\mathrm{NL}^s\text{-sat}(n)=\{(\lambda_1,\ldots,\lambda_s)\in(\mathrm{Par}_n^{\mathbb{Q}})^s\,:\,\exists t>0,\ N_{t\lambda_1,\ldots,t\lambda_s}\neq 0\}.$$

For any totally ordered set $T=\{t_1<\cdots<t_m\}$ and $R=\{t_{i_1}<\cdots<t_{i_r}\}\subseteq T$, define

$$\tau(R,T)=(i_r-r\geqslant\cdots\geqslant i_1-1). \tag{8}$$

In most cases, we will be considering some finite $A\subseteq\mathbb{Z}_{>0}$; for simplicity, we denote

$$\tau(A):=\tau(A,[n])$$

for sufficiently large n.

The Horn inequalities for LR-sat(n) are recursive, as they depend on LR-sat(n') for $n'<n$ (see [Ful00]). Theorem 1.2 is not recursive. However, our next result describes the cone NL-sat(n) by inequalities depending on NL^6-sat(n') for $n'\leqslant n$.

Theorem 1.5. *Let $(\lambda,\mu,\nu)\in(\mathrm{Par}_n^{\mathbb{Q}})^3$. Then $(\lambda,\mu,\nu)\in$ NL-sat(n) if and only if*

$$0\leqslant|\lambda_A|-|\lambda_{A'}|+|\mu_B|-|\mu_{B'}|+|\nu_C|-|\nu_{C'}| \tag{9}$$

for any subsets $A,A',B,B',C,C'\subset[n]$ such that:

(1) $A\cap A'=B\cap B'=C\cap C'=\emptyset$;

(2) $|A|+|A'|=|B|+|B'|=|C|+|C'|=|A'|+|B'|+|C'|=:r$;

(3) $(\tau(A),\tau(C'),\tau(B),\tau(A'),\tau(C),\tau(B'))\in\mathrm{NL}^6\text{-sat}(r)$.

This result is proved in § 8.1. By a result of King's [Kin71] (see also [HT05]), each six-fold Newell–Littlewood coefficient is a particular Littlewood–Richardson coefficient (see § 8.1 for details). Consequently, condition (3) is equivalent to checking if some explicitly determined triple of partitions is in LR-sat$(2r)$. Since LR-sat$(2r)$ is described by the Horn inequalities, we thereby obtain a description of NL-sat(n) only involving inequalities and no tensor product multiplicities. It is in this sense that Theorem 1.5 is of the same spirit as Horn's original inequalities.

Just as the proof of Horn's conjecture depends on Knutson and Tao's saturation theorem, our proof of Theorem 1.5 uses this consequence of King's result (see § 7.2).

Proposition 1.6 (Six-fold NL-saturation). *Let $\lambda_1,\ldots,\lambda_6\in\mathrm{Par}_n$. If there exists a positive integer t such that $N_{t\lambda_1,\ldots,t\lambda_6}\neq 0$ then $N_{\lambda_1,\ldots,\lambda_6}\neq 0$.*

In [GOY20b, Conjecture 1.4], a conjectural description of NL-sat(n) was given. That conjecture subsumes both Conjecture 1.4 and a description of NL-sat using *extended Horn inequalities* [GOY20b, Definition 1.2]. Theorem 1.5 proves the latter part of the conjecture.

2. Generalities on the tensor cones

2.1 Finitely generated semigroups

A subset $\Gamma\subseteq\mathbb{Z}^n$ is a *semigroup* if $\vec{0}\in\Gamma$ and Γ is closed under addition. A finitely generated semigroup Γ *generates* a closed convex polyhedral cone $\Gamma_{\mathbb{Q}}\subseteq\mathbb{Q}^n$:

$$\Gamma_{\mathbb{Q}}=\{x\in\mathbb{Q}^n:\exists t\in\mathbb{Z}_{>0}\ tx\in\Gamma\}.$$

[2] $N_{\lambda_1,\ldots,\lambda_s}$ also has a uniform representation-theoretic interpretation. Discussion may appear elsewhere.

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The subgroup of $\mathbb{Z}^n$ generated by Γ is

$$\Gamma_{\mathbb{Z}} = \{x - y \, : \, x, y \in \Gamma\}.$$

The semigroup Γ is *saturated* if $\Gamma = \Gamma_{\mathbb{Z}} \cap \Gamma_{\mathbb{Q}}$.

2.2 GIT-semigroups

We recall the GIT-perspective of [Res10]. Let G be a complex reductive group acting on an irreducible projective variety X. Let $\mathrm{Pic}^G(X)$ be the group of G-linearized line bundles. Given $\mathcal{L} \in \mathrm{Pic}^G(X)$, let $\mathrm{H}^0(X, \mathcal{L})$ be the space of sections of $\mathcal{L}$; it is a G-module. Let $\mathrm{H}^0(X, \mathcal{L})^G$ be the subspace of invariant sections. Define

$$\text{GIT-semigroup}(G, X) = \{\mathcal{L} \in \mathrm{Pic}^G(X) \, : \, \mathrm{H}^0(X, \mathcal{L})^G \neq \{0\}\}.$$

This is a semigroup since X being irreducible says the product of two nonzero G-invariant sections is a nonzero G-invariant section. The saturated version of it is

$$\text{GIT-sat}(G, X) = \{\mathcal{L} \in \mathrm{Pic}^G(X) \otimes \mathbb{Q} : \exists t > 0, \; \mathrm{H}^0(X, \mathcal{L}^{\otimes t})^G \neq \{0\}\}.$$

2.3 The tensor semigroup

Let $\mathfrak{g}$ be a semisimple complex Lie algebra, with fixed Borel subalgebra $\mathfrak{b}$ and Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$. Denote by $\Lambda^+(\mathfrak{g}) \subset \mathfrak{t}^*$ the semigroup of the dominant weights. It is contained in the weight lattice $\Lambda(\mathfrak{g}) \simeq \mathbb{Z}^r$, where r is the rank of $\mathfrak{g}$. Given $\lambda \in \Lambda^+(\mathfrak{g})$, denote by $V_{\mathfrak{g}}(\lambda)$ (or simply $V(\lambda)$) the irreducible representation of $\mathfrak{g}$ with highest weight λ. Let $V(\lambda)^*$ be the dual representation. Consider the semigroup

$$\mathfrak{g}\text{-semigroup} = \{(\lambda, \mu, \nu) \in (\Lambda^+(\mathfrak{g}))^3 \, : \, V(\nu)^* \subset V(\lambda) \otimes V(\mu)\},$$

and the generated cone $\mathfrak{g}$-sat in $(\Lambda(\mathfrak{g}) \otimes \mathbb{Q})^3$. When $\mathfrak{g} = \mathfrak{sp}(2m, \mathbb{C})$ we have $V(\nu)^* \simeq V(\nu)$ and $\mathfrak{g}$-semigroup is what we denoted by $\mathfrak{sp}$-semigroup(m) in the introduction. The set $\mathfrak{g}$-semigroup spans the rational vector space $(\Lambda(\mathfrak{g}) \otimes \mathbb{Q})^3$, or equivalently, the cone $\mathfrak{g}$-sat has nonempty interior. The group $(\mathfrak{g}\text{-semigroup})_{\mathbb{Z}}$ is well known (see, for example, [PR13, Theorem 1.4]):

$$(\mathfrak{g}\text{-semigroup})_{\mathbb{Z}} = \{(\lambda, \mu, \nu) \in (\Lambda(\mathfrak{g}))^3 \, : \, \lambda + \mu + \nu \in \Lambda_R(\mathfrak{g})\},$$

where $\Lambda_R(\mathfrak{g})$ is the root lattice of $\mathfrak{g}$.

We now interpret $\mathfrak{g}$-semigroup in terms of § 2.2. Consider the semisimple, simply connected algebraic group G with Lie algebra $\mathfrak{g}$. Denote by B and T the connected subgroups of G with Lie algebras $\mathfrak{b}$ and $\mathfrak{t}$, respectively. The character groups $X(B) = X(T) = \Lambda(\mathfrak{g})$ of B and T coincide. For $\lambda \in X(T)$, $\mathcal{L}_\lambda$ is the unique G-linearized line bundle on the flag variety G/B such that B acts on the fiber over B/B with weight $-\lambda$.

Assume $X = (G/B)^3$. Then $\mathrm{Pic}^G(X)$ identifies with $X(T)^3$. For $(\lambda, \mu, \nu) \in X(T)^3$, define $\mathcal{L}_{(\lambda,\mu,\nu)}$. By the Borel–Weil theorem,

$$\mathrm{H}^0(X, \mathcal{L}_{(\lambda,\mu,\nu)}) = V(\lambda)^* \otimes V(\mu)^* \otimes V(\nu)^*. \tag{10}$$

In particular, GIT-semigroup$(G, X) \simeq \mathfrak{g}$-semigroup.

Given three parabolic subgroups P, Q and R containing B, we consider more generally $X = G/P \times G/Q \times G/R$. Then $\mathrm{Pic}^G(X)$ identifies with $X(P) \times X(Q) \times X(R)$ which is a subgroup of $X(T)^3$. Moreover,

$$\text{GIT-semigroup}(G, X) = \text{GIT-semigroup}(G, (G/B)^3) \cap (X(P) \times X(Q) \times X(R)).$$

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2.4 Schubert calculus

We need notation for the cohomology ring $\mathrm{H}^*(G/P,\mathbb{Z})$; $P\supset B$ being a parabolic subgroup. Let W (respectively, W_P) be the Weyl group of G (respectively, P). Let $\ell : W \longrightarrow \mathbb{N}$ be the *Coxeter length*, defined with respect to the simple reflections determined by the choice of B. Let W^P be the minimal length representatives of the cosets in W/W_P.

For a closed irreducible subvariety $Z\subset G/P$, let $[Z]$ be its class in $\mathrm{H}^*(G/P,\mathbb{Z})$, of degree $2(\dim(G/P)-\dim(Z))$. For $v\in W^P$, set

$$\sigma_v=[\overline{BvP/P}]$$

$(\dim(\overline{BvP/P})=\ell(v))$. Then

$$\mathrm{H}^*(G/P,\mathbb{Z})=\bigoplus_{v\in W^P}\mathbb{Z}\sigma_v.$$

Let w_0 be the longest element of W and $w_{0,P}$ be the longest element of W_P. Set $v^\vee=w_0vw_{0,P}$ and $\sigma_v=\sigma^{v^\vee}$; σ^v and σ_v are Poincaré dual.

Let ρ be the half sum of the positive roots of G. To any one-parameter subgroup $\tau:\mathbb{C}^*\to T$, associate the parabolic subgroup (see [MFK94])

$$P(\tau)=\left\{g\in G\,:\,\lim_{t\to 0}\tau(t)g\tau(t^{-1})\text{ exists}\right\}.$$

Fix such a τ such that $P=P(\tau)$.

For $v\in W^P$, define the *BK-degree* of $\sigma^v\in\mathrm{H}^*(G/P,\mathbb{Z})$ to be

$$\text{BK-deg}(\sigma^v):=\langle v^{-1}(\rho)-\rho,\tau\rangle.$$

Let v_1, v_2 and v_3 in W^P. By [BK06, Proposition 17], if σ^{v_3} appears in the product $\sigma^{v_1}\cdot\sigma^{v_2}$ then

$$\text{BK-deg}(\sigma^{v_3})\leqslant\text{BK-deg}(\sigma^{v_1})+\text{BK-deg}(\sigma^{v_2}). \tag{11}$$

In other words, the BK-degree filters the cohomology ring. Let $\odot_0$ denote the associated graded product on $\mathrm{H}^*(G/P,\mathbb{Z})$.

2.5 Well-covering pairs

In [Res10], GIT-sat(G,X) is described in terms of *well-covering pairs*. When $X=(G/B)^3$, it recovers the description made by Belkale and Kumar [BK06]. We now discuss the case when $X=G/P\times G/Q\times G/R$ is the product of three partial flag varieties of G.

Let τ be a dominant one-parameter subgroup of T. The centralizer G^τ of the image of τ in G is a Levi subgroup. Moreover, $P(\tau)$ is the parabolic subgroup generated by B and G^τ. Let C be an irreducible component of the fixed set X^τ of τ in X. It is well known that C is the $(G^\tau)^3$-orbit of some T-fixed point:

$$C=G^\tau u^{-1}P/P\times G^\tau v^{-1}Q/Q\times G^\tau w^{-1}R/R, \tag{12}$$

with $u\in W_P\backslash W/W_{P(\tau)}$, and similarly for v and w. Set

$$C^+=P(\tau)u^{-1}P/P\times P(\tau)v^{-1}Q/Q\times P(\tau)w^{-1}R/R.$$

Then the closure of C^+ is a Schubert variety (for G^3) in X. By [Res10, Proposition 11], the pair (C,τ) is *well covering* if and only if

$$[\overline{PuP(\tau)/P(\tau)}]\odot_0[\overline{QvP(\tau)/P(\tau)}]\odot_0[\overline{RwP(\tau)/P(\tau)}]=[pt]\in\mathrm{H}^*(G/P(\tau),\mathbb{Z}). \tag{13}$$

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It is said to be *dominant* if

$$[\overline{PuP(\tau)/P(\tau)}]\cdot[\overline{QvP(\tau)/P(\tau)}]\cdot[\overline{RwP(\tau)/P(\tau)}]\neq 0\in \mathrm{H}^*(G/P(\tau),\mathbb{Z}). \tag{14}$$

In this paper, the reader can take these characterizations as definitions of well-covering and dominant pairs. They are used in [Res10] to produce inequalities for the GIT-cones.

Proposition 2.1. *Let $(\lambda,\mu,\nu)\in X(P)\times X(Q)\times X(R)$ be dominant, and let $\langle\cdot,\cdot\rangle$ be the pairing between one parameter subgroups and characters of T. Then the following are equivalent:*

(1) $\mathcal{L}_{(\lambda,\mu,\nu)}\in$ GIT-sat(G,X);
(2) $\langle u\tau,\lambda\rangle+\langle v\tau,\mu\rangle+\langle w\tau,\nu\rangle\leqslant 0$ *for all dominant pairs* (C,τ);
(3) $\langle u\tau,\lambda\rangle+\langle v\tau,\mu\rangle+\langle w\tau,\nu\rangle\leqslant 0$ *for all well-covering pairs* (C,τ).

Remark 2.2. By the definition of $\odot$, if (C,τ) is well covering; it is also dominant. Hence, the inequalities in (3) are a subset of inequalities in (1).

In the case $P=Q=R=B$, there is a more precise statement. The fact that the inequalites define the cone is due to Belkale and Kumar [BK06, Theorem 28]. The irredundancy is [Res10, Theorem B]. Let α be a simple root of G. Denote by P^α the associated maximal parabolic subgroup of G containing B. Denote by $\varpi_{\alpha^\vee}$ the associated fundamental one-parameter subgroup of T characterized by $\langle\varpi_{\alpha^\vee},\beta\rangle=\delta_{\alpha\beta}$ (Kronecker delta) for any simple root β.

Theorem 2.3 ([BK06, Theorem 28], [Res10, Theorem B]). *Here $X=(G/B)^3$. Let $(\lambda,\mu,\nu)\in X(T)^3$ be dominant. Then, $\mathcal{L}_{(\lambda,\mu,\nu)}\in$ GIT-sat(G,X) if and only if for any simple root α, for any u,v,w in W^{P^α} such that*

$$[\overline{BuP^\alpha/P^\alpha}]\odot_0[\overline{BvP^\alpha/P^\alpha}]\odot_0[\overline{BwP^\alpha/P^\alpha}]=[pt]\in\mathrm{H}^*(G/P^\alpha,\mathbb{Z}), \tag{15}$$

$$\langle u\varpi_{\alpha^\vee},\lambda\rangle+\langle v\varpi_{\alpha^\vee},\mu\rangle+\langle w\varpi_{\alpha^\vee},\nu\rangle\leqslant 0. \tag{16}$$

Moreover, this list of inequalities is irredundant.

Theorem 2.3 can be obtained from Proposition 2.1(3) by showing that it is sufficient to consider the one-parameter subgroups τ equal to $\varpi_{\alpha^\vee}$ for some simple root α. See the proof of Theorem 5.1 below for a similar argument.

2.6 The eigencone

A relationship between $\mathfrak{g}$-sat and projections of coadjoint orbits was discovered by Heckman [Hec82]. Theorem 2.4 below interprets $\mathfrak{g}$-sat in terms of eigenvalues.

Fix a maximal compact subgroup U of G such that $T\cap U$ is a Cartan subgroup of U. Let $\mathfrak{u}$ and $\mathfrak{t}$ denote the Lie algebras of U and T, respectively. Let $\mathfrak{t}^+$ be the Weyl chamber of $\mathfrak{t}$ corresponding to B. Let $\sqrt{-1}$ denote the usual complex number. It is well known that $\sqrt{-1}\mathfrak{t}^+$ is contained in $\mathfrak{u}$ and that the map

$$\begin{array}{ccc}\mathfrak{t}^+ & \longrightarrow & \mathfrak{u}/U\\ \xi & \longmapsto & U.(\sqrt{-1}\xi)\end{array} \tag{17}$$

is a homeomorphism. Here U acts on $\mathfrak{u}$ by the adjoint action. Consider the set

$$\Gamma(U):=\{(\xi,\zeta,\eta)\in(\mathfrak{t}^+)^3\,:\,U.(\sqrt{-1}\xi)+U.(\sqrt{-1}\zeta)+U.(\sqrt{-1}\eta)\ni 0\}.$$

Let $\mathfrak{u}^*$ (respectively, $\mathfrak{t}^*$) denote the dual (respectively, complex dual) of $\mathfrak{u}$ (respectively, $\mathfrak{t}$). Let $\mathfrak{t}^{*+}$ denote the dominant chamber of $\mathfrak{t}^*$ corresponding to B. By taking the tangent map at

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the identity, one can embed $X(T)^+$ in $\mathfrak{t}^{*+}$. Note that this embedding induces a rational structure on the complex vector space $\mathfrak{t}^*$. Moreover, it allows the tensor cone $\mathfrak{g}$-sat to be embedded in $(\mathfrak{t}^{*+})^3$.

The Cartan-Killing form allows $\mathfrak{t}^+$ and $\mathfrak{t}^{*+}$ to be identified. In particular, $\Gamma(U)$ also embeds in $(\mathfrak{t}^{*+})^3$; the subset of $(\mathfrak{t}^{*+})^3$ thus obtained is denoted by $\tilde{\Gamma}(U)$ to avoid any confusion. The following result is well known; see, for example, [Kum14, Theorem 5] and the references therein.

Theorem 2.4. *The set* $\Gamma(U)$ *is a closed convex polyhedral cone. Moreover,* $\mathfrak{g}$-sat *is the set of the rational points in* $\tilde{\Gamma}(U)$.

3. The case of the symplectic group

3.1 The root system of type C

Let $V=\mathbb{C}^{2n}$ with the standard basis $(\vec{e}_1,\vec{e}_2,\dots,\vec{e}_{2n})$. Let J_n be the $n\times n$ 'anti-diagonal' identity matrix and define a skew-symmetric bilinear form $\omega(\bullet,\bullet): V\times V\to\mathbb{C}$ using the block matrix $\Omega := \begin{bmatrix} 0 & J_n \\ -J_n & 0\end{bmatrix}$. By definition, the *symplectic group* $G=\mathrm{Sp}(2n,\mathbb{C})$ is the group of automorphisms of V that preserve this bilinear form.

Given an $n\times n$ matrix $A=(A_{ij})_{1\leqslant i,j\leqslant n}$, define ${}^TA$ by $({}^TA)=A_{n+1-j\,n+1-i}$, obtained from A by reflection across the antidiagonal. The Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ is the set of matrices $M\in \mathsf{Mat}_{2n\times 2n}(\mathbb{C})$ such that ${}^tM\Omega+\Omega M=0$; namely,

$$\mathfrak{sp}(2n,\mathbb{C})=\left\{\begin{pmatrix} A & B\\ C & -{}^TA\end{pmatrix} : \begin{matrix} A,B,C \text{ of size } n\times n,\\ {}^TB=B \text{ and } {}^TC=C\end{matrix}\right\} \tag{18}$$

which has the complex dimension $2n^2+n$. The Lie algebra $\mathfrak{u}(2n,\mathbb{C})$ of the unitary group $U(2n,\mathbb{C})$ is the set of anti-Hermitian matrices. Thus, (18) gives

$$\mathfrak{sp}(2n,\mathbb{C})\cap\mathfrak{u}(2n,\mathbb{C})=\left\{\begin{pmatrix} A & B\\ -{}^t\bar{B} & -{}^TA\end{pmatrix} : {}^t\bar{A}=-A \text{ and } {}^TB=B\right\}, \tag{19}$$

which has real dimension $2n^2+n$. As a consequence, $U(2n)\cap\mathrm{Sp}(2n,\mathbb{C})$ is a maximal compact subgroup of $\mathrm{Sp}(2n,\mathbb{C})$.

Let B be the Borel subgroup of upper triangular matrices in G. Let

$$T=\{\mathrm{diag}(t_1,\dots,t_n,t_n^{-1},\dots,t_1^{-1}) : t_i\in\mathbb{C}^*\}$$

be the maximal torus contained in B. For $i\in[n]$, let ε_i denote the character of T that maps $\mathrm{diag}(t_1,\dots,t_n,t_n^{-1},\dots,t_1^{-1})$ to t_i; then $X(T)=\oplus_{i=1}^n\mathbb{Z}\varepsilon_i$. Here

$$\Phi^+=\{\varepsilon_i\pm\varepsilon_j : 1\leqslant i<j\leqslant n\}\cup\{2\varepsilon_i : 1\leqslant i\leqslant n\},$$

$$\Delta=\{\alpha_1=\varepsilon_1-\varepsilon_2,\ \alpha_2=\varepsilon_2-\varepsilon_3,\dots,\ \alpha_{n-1}=\varepsilon_{n-1}-\varepsilon_n,\ \alpha_n=2\varepsilon_n\},$$

$$X(T)^+=\left\{\sum\nolimits_{i=1}^n\lambda_i\varepsilon_i : \lambda_1\geqslant\cdots\geqslant\lambda_n\geqslant 0\right\}=\mathrm{Par}_n.$$

For $i\in[2n]$, set $\bar{i}=2n+1-i$. The Weyl group W of G may be identified with a subgroup of the Weyl group S_{2n} of $\mathrm{SL}(V)$. More precisely,

$$W=\{w\in S_{2n} : w(\bar{i})=\overline{w(i)}\quad\forall i\in[2n]\}.$$

Observe that $T\cap U(2n,\mathbb{C})$ has real dimension n and is a maximal torus of $U(2n)\cap\mathrm{Sp}(2n,\mathbb{C})$. The bijection (17) implies that any matrix $M_1=\begin{pmatrix} A & B\\ -{}^t\bar{B} & -{}^TA\end{pmatrix}$ in $\mathfrak{sp}(2n,\mathbb{C})\cap\mathfrak{u}(2n,\mathbb{C})$ (see (19)) is diagonalizable with eigenvalues in $\sqrt{-1}\mathbb{R}$. Moreover, with the eigenvalues in nonincreasing order, we get

$$\lambda(\sqrt{-1}M_1) \in \{(\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant -\lambda_n \geqslant \cdots \geqslant -\lambda_1) : \lambda_i \in \mathbb{R}\}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathrm{Par}_n$, set $\hat{\lambda} = (\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1)$. Now, Theorems 1.1 with $m = n$ and Theorem 2.4 give an interpretation of NL-sat(n) in terms of eigenvalues:

Proposition 3.1. *Let $\lambda, \mu, \nu \in \mathrm{Par}_n$. Then $(\lambda, \mu, \nu) \in$ NL-sat(n) if and only if there exist three matrices $M_1, M_2, M_3 \in \mathfrak{sp}(2n, \mathbb{C}) \cap \mathfrak{u}(2n, \mathbb{C})$ such that $M_1 + M_2 + M_3 = 0$ and*

$$(\hat{\lambda}, \hat{\mu}, \hat{\nu}) = (\lambda(\sqrt{-1}M_1), \lambda(\sqrt{-1}M_2), \lambda(\sqrt{-1}M_3)).$$

3.2 Isotropic Grassmannians and Schubert classes

Our reference for this subsection is [Res12, § 5]. For $r = 1, \ldots, n$, the one-parameter subgroup $\varpi_{\alpha_r^\vee}$ is given by

$$\varpi_{\alpha_r^\vee}(t) = \mathrm{diag}(t, \ldots, t, 1, \ldots, 1, t^{-1}, \ldots, t^{-1}),$$

where t and t^{-1} occur r times.

A subspace $W \subseteq V$ is *isotropic* if for all $\vec{v}, \vec{v}' \in W$, $\omega(\vec{v}, \vec{v}') = 0$. Given an r-subset $I \subset [2n]$, we set $F_I = \mathrm{Span}(\vec{e}_i : i \in I)$. Clearly, F_I is isotropic if and only if $I \cap \bar{I} = \emptyset$, where $\bar{I} = \{\bar{i} : i \in I\}$. Now, P^{α_r} is the stabilizer of the isotropic subspace $F_{\{1,\ldots,r\}}$. Thus, $G/P^{\alpha_r} = \mathrm{Gr}_\omega(r, 2n)$ is the *Grassmannian of isotropic r-dimensional vector subspaces of V*.

Let $\mathcal{S}(r, N)$ denote the set of subsets of $\{1, \ldots, N\}$ with r elements. Set

$$\mathrm{Schub}(\mathrm{Gr}_\omega(r, 2n)) := \{I \in \mathcal{S}(r, 2n) : I \cap \bar{I} = \emptyset\}.$$

If $I = \{i_1 < \cdots < i_r\} \in \mathrm{Schub}(\mathrm{Gr}_\omega(r, 2n))$, let $i_{\bar{k}} := \overline{i_k}$ for $k \in [r]$, and $\{i_{r+1} < \cdots < i_{\overline{r+1}}\} = [2n] - (I \cup \bar{I})$. Therefore, $w_I = (i_1, \ldots, i_{2n}) \in S_{2n}$ is the element of $W^{P^{\alpha_r}}$ corresponding to F_I; that is, $F_I = w_I P^{\alpha_r}/P^{\alpha_r}$.

Set

$$\mathrm{Schub}'(\mathrm{Gr}_\omega(r, 2n)) := \left\{ \begin{array}{ll} (A, A') : & A \in \mathcal{S}(a, n),\ A' \in \mathcal{S}(a', n) \text{ for some } a \text{ and } a' \\ & \text{s.t. } a + a' = r \text{ and } A \cap A' = \emptyset \end{array} \right\}.$$

This map is a bijection:

$$\begin{array}{ccc} \mathrm{Schub}(\mathrm{Gr}_\omega(r, 2n)) & \longrightarrow & \mathrm{Schub}'(\mathrm{Gr}_\omega(r, 2n)) \\ I & \longmapsto & (\bar{I} \cap [n], I \cap [n]). \end{array} \tag{20}$$

Recall from the introduction the definition of $\tau(I)$ and hence $\tau(A)$ and $\tau(A')$. The relationship between these three partitions is depicted in Figure 1. In particular, note that $\tau(A') \subseteq \tau(I)$.

Given $I \in \mathrm{Schub}(\mathrm{Gr}_\omega(r, 2n))$ for some $1 \leqslant r \leqslant n$, set $I^2 \in \mathcal{S}(r, 2r)$ and $I^0 \in \mathcal{S}(r, 2n - r)$ to be the unique r-element sets, such that

$$\tau(I^2) = \tau(I, I \cup \bar{I}) \text{ and } \tau(I^0) = \tau(I, [2n] - \bar{I}).$$

Example 3.2. Let $I = \{1, 3, 5\} \in \mathrm{Schub}(\mathrm{Gr}_\omega(3, 8))$. Then $w_I = 13527468 \in S_8$,

$$\tau(I^2) = \tau(\{1, 3, 5\}, \{1, 3, 4, 5, 6, 8\}) = 100 \text{ and } \tau(I^0) = \tau(\{1, 3, 5\}, \{1, 2, 3, 5, 7\}) = 110.$$

Thus $I^2 = \{1, 2, 4\}, I^0 = \{1, 3, 4\}$. By Equation (20), $A = \bar{I} \cap [n] = \{4\}$ and $A' = I \cap [n] = \{1, 3\}$.

Definition 3.3. Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \subseteq (a^b)$, that is, the rectangle with a columns and b rows. Define $\lambda^\vee$ with respect to (a^b) to be the partition $(a - \lambda_b, a - \lambda_{b-1}, \ldots, a - \lambda_1)$ where we set $\lambda_i = 0$ for $i > k$. We will denote this by $\lambda^{\vee[a^b]}$.

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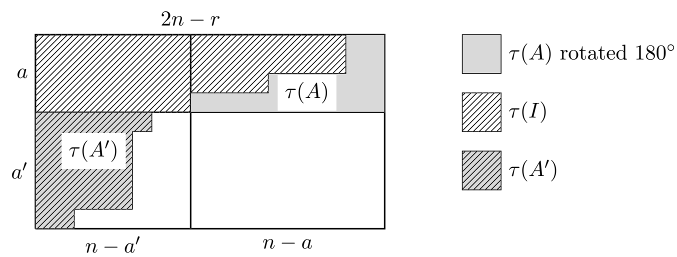

FIGURE 1. $\tau(I)$, $\tau(A)$ and $\tau(A')$ ($\tau(A') \subseteq \tau(I)$).

Now, by [BK10, Proposition 32],

$$\operatorname{codim}(\overline{BF_I}) = |\tau(I^0)^{\vee[(2n-2r)^r]}| + 1/2(|\tau(I^2)^{\vee[r^r]}| + |I \cap [n]|). \tag{21}$$

Moreover,

$$\dim(\mathrm{Gr}_\omega(r, 2n)) = r(2n-2r) + \frac{r(r+1)}{2}. \tag{22}$$

Let $I \in \mathrm{Schub}(\mathrm{Gr}_\omega(r, 2n))$ and $A = \bar{I} \cap [n]$, $A' = I \cap [n]$ be the corresponding pair in $\mathrm{Schub}'(\mathrm{Gr}_\omega(r, 2n))$. Set

$$\tau^0(A, A') = \tau(I^0), \quad \tau^2(A, A') = \tau(I^2). \tag{23}$$

For later use, observe that

$$\tau(I) = \tau(I^0) + \tau(I^2). \tag{24}$$

While the above discussion defines $\tau^0(A, A'), \tau^2(A, A')$ through the bijection (20), we emphasize that these partitions from Theorem 1.2 can be defined explicitly.

DEFINITION-LEMMA 3.4. *Set $a = |A|$ and $a' = |A'|$. Write $A = \{\alpha_1 < \cdots < \alpha_a\}$ and $A' = \{\alpha'_1 < \cdots < \alpha'_{a'}\}$. Then*

$$\begin{aligned}
\tau^2(A, A')_k &= a + |A' \cap [\alpha_k, n]| && \forall k = 1, \ldots, a;\\
\tau^2(A, A')_{l+a} &= |A \cap [\alpha'_{a'+1-l}]| && \forall l = 1, \ldots, a';\\
\tau^0(A, A')_k &= n - a - a' + |[\alpha_k, n] - (A \cup A')| && \forall k = 1, \ldots, a;\\
\tau^0(A, A')_{l+a} &= |[\alpha'_{a'+1-l}] - (A \cup A')| && \forall l = 1, \ldots, a'.
\end{aligned}$$

Proof. Write $I = A' \cup \bar{A} = \{i_1 < \cdots < i_r\}$ with $r = a + a'$. By definition,

$$\tau^2(A, A')_k := \tau(I^2)_k = |\bar{I} \cap [i_{a+a'+1-k}]| \quad \text{for } 1 \leqslant k \leqslant a + a'.$$

If $k \leqslant a$, then $i_{a+a'+1-k} \in \bar{A} \subset [n+1, 2n]$, $i_{a+a'+1-k} = \overline{\alpha_k}$ and $\bar{I} \cap [i_{a+a'+1-k}] = A \cup \overline{A' \cap [\alpha_k, n]}$. The first assertion follows.

If $k = a + l$ for some positive l, then $i_{a+a'+1-k} \in A' \subseteq [n]$, $i_{a+a'+1-k} = \alpha'_{a'+1-l}$ and $\bar{I} \cap [i_{a+a'+1-k}] = A \cap [\alpha'_{a'+1-l}]$.

Similarly,

$$\tau^0(A, A')_k := \tau(I^0)_k = |[i_{a+a'+1-k}] \cap ([2n] - (I \cup \bar{I})| \quad \text{for } 1 \leqslant k \leqslant a + a'.$$

If $k \leqslant a$, then $[i_{a+a'+1-k}] \cap ([2n] - (I \cup \bar{I})) = ([n] - (A \cup A')) \cup \overline{[\alpha_k, n] - (A \cup A')}$ (a disjoint union). This proves the third claim.

If $k = a + l$ with some positive l, then $\alpha'_{a'+1-l} = i_{a+a'+1-k} \in [n]$; the last assertion follows. □

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3.3 The parabolic subgroup P_0

Fix $m \geqslant n$. Let P_0 be the subgroup of $\mathrm{Sp}(2m,\mathbb{C})$ of matrices

$$\begin{pmatrix} T_1 & * & * \\ 0 & A & * \\ 0 & 0 & T_2 \end{pmatrix}, \tag{25}$$

where T_1 and T_2 are $n \times n$ upper-triangular matrices and A is a matrix in $\mathrm{Sp}(2m-2n,\mathbb{C})$. P_0 is the standard parabolic subgroup of $\mathrm{Sp}(2m,\mathbb{C})$ corresponding to the simple roots $\{\alpha_{n+1},\dots,\alpha_m\}$. A character $\lambda = \sum_{i=1}^m \lambda_i\varepsilon_i \in X(T)$ extends to P_0 if and only if $\lambda_{n+1} = \cdots = \lambda_m = 0$. Thus the set of dominant characters of $X(P_0)$ identifies with Par_n. Hence,

$$\mathfrak{sp}\text{-sat}(m) \cap (\mathrm{Par}_n^{\mathbb{Q}})^3 = \text{GIT-sat}(\mathrm{Sp}(2m,\mathbb{C}), (\mathrm{Sp}(2m,\mathbb{C})/P_0)^3). \tag{26}$$

Let $\mathrm{Schub}^{P_0}(\mathrm{Gr}_\omega(r,2m))$ be the set of $I \in \mathrm{Schub}(\mathrm{Gr}_\omega(r,2m))$ such that the Schubert variety $\overline{BF_I}$ is P_0-stable. Then $I \in \mathrm{Schub}^{P_0}(\mathrm{Gr}_\omega(r,2m))$ if and only if $w_I \in W^{P^{\alpha_r}}$ and $s_{\alpha_i} w_I \leqslant w_I$ (cover in Bruhat order for W) for all $i \in [n+1,m]$. Since the simple transposition s_{α_i} swaps i and $\bar{i}$ with $i+i$ and $\bar{i+1}$ respectively if $i<m$, and swaps m with $\bar{m}$ if $i=m$, we have

$$I \in \mathrm{Schub}^{P_0}(\mathrm{Gr}_\omega(r,2m)) \iff I \cap [n+1, 2m-n] = [k, 2m-n] \text{ for some } k \geqslant m+1. \tag{27}$$

4. Proof of Theorems 1.1 and 1.2

Proposition 4.1. *The inequalities (4) in Theorem 1.2 characterize* $\mathfrak{sp}\text{-sat}(n)$.

Proof. Since $\mathfrak{sp}\text{-sat}(n) = \text{GIT-sat}(\mathrm{Sp}(2n), (\mathrm{Sp}(2n)/B)^3$ (see § 2.3), we may apply Theorem 2.3. Let $(\lambda,\mu,\nu) \in (\mathrm{Par}_n)^3$. Write $\lambda = \sum_i \lambda_i\varepsilon_i$, and similarly for μ and ν.

Fix $1 \leqslant r \leqslant n$ and $\alpha = \alpha_r \in \Delta$. Given $I \in \mathrm{Schub}(\mathrm{Gr}_\omega(r,2n))$, from the description of $\varpi_{\alpha_r^\vee}$ and w_I it is easy to check that

$$\langle w_I \varpi_{\alpha_r^\vee}, \lambda\rangle = \sum_{i \in I\cap[n]} \lambda_i - \sum_{i \in \bar{I}\cap[n]} \lambda_i \tag{28}$$

Then (4) is obtained from (16) associated to the triple of Schubert classes $(I,J,K) \in \mathrm{Schub}(\mathrm{Gr}_\omega(r,2n))^3$ by setting

$$\begin{aligned} A &= \bar{I} \cap [n], & A' &= I \cap [n], \\ B &= \bar{J} \cap [n], & B' &= J \cap [n], \\ C &= \bar{K} \cap [n], & C' &= K \cap [n]. \end{aligned}$$

Since the map (20) is bijective, it suffices to show (15) from Theorem 2.3 is equivalent to:

(1) $|A'| + |B'| + |C'| = r$; and
(2) $c^{\tau^0(C,C')}_{\tau^0(A,A')^{\vee[(2n-2r)^r]},\tau^0(B,B')^{\vee[(2n-2r)^r]}} = c^{\tau^2(C,C')}_{\tau^2(A,A')^{\vee[r^r]},\tau^2(B,B')^{\vee[r^r]}} = 1$.

By [Res12, Theorem 19], condition (15) is equivalent to:

(1) $\mathrm{codim}(\overline{BF_I}) + \mathrm{codim}(\overline{BF_J}) + \mathrm{codim}(\overline{BF_K}) = \dim(\mathrm{Gr}_\omega(r,2n))$; and
(2) $c^{\tau(K^0)}_{\tau(I^0)^{\vee[(2n-2r)^r]},\tau(J^0)^{\vee[(2n-2r)^r]}} = c^{\tau(K^2)}_{\tau(I^2)^{\vee[r^r]},\tau(J^2)^{\vee[r^r]}} = 1$.

By definition, the two conditions involving Littlewood–Richardson are the same. Assuming these two Littlewood–Richardson coefficients equal to one 1, it remains to prove that $\mathrm{codim}(\overline{BF_I}) + \mathrm{codim}(\overline{BF_J}) + \mathrm{codim}(\overline{BF_K}) = \dim(\mathrm{Gr}_\omega(r,2n))$ if and only if $|A'|+|B'|+|C'| = r$. This directly follows from (21) and (22). ∎

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Proof of Theorem 1.1. By Theorem 2.4, the inclusion $\mathfrak{sp}\text{-sat}(n) \subset \mathfrak{sp}\text{-sat}(m)$ is equivalent to the inclusion $\Gamma(\mathrm{Sp}(2n,\mathbb{C}) \cap U(2n,\mathbb{C})) \subset \Gamma(\mathrm{Sp}(2m,\mathbb{C}) \cap U(2m,\mathbb{C}))$. Here we use the symplectic form defined in § 3.1 to embed $\mathrm{Sp}(2n,\mathbb{C})$ in $\mathrm{GL}(2n,\mathbb{C})$.

Clearly, the following map is well defined:

$$\begin{array}{ccc} \mathrm{Lie}(\mathrm{Sp}(2n,\mathbb{C}) \cap U(2n,\mathbb{C})) & \longrightarrow & \mathrm{Lie}(\mathrm{Sp}(2m,\mathbb{C}) \cap U(2m,\mathbb{C})) \\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \longmapsto & \tilde{M} = \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ C & 0 & D \end{pmatrix} \end{array}$$

where A, B, C and D are square matrices of size n, and the matrices of these Lie algebras are described by (19).

Let $(\hat{h}_1, \hat{h}_2, \hat{h}_3) \in \Gamma(\mathrm{Sp}(2n,\mathbb{C}) \cap U(2n,\mathbb{C}))$. Let

$$(M_1, M_2, M_3) \in (\mathrm{Sp}(2n,\mathbb{C}) \cap U(2n,\mathbb{C}))^3.(\sqrt{-1}\hat{h}_1, \sqrt{-1}\hat{h}_2, \sqrt{-1}\hat{h}_3)$$

such that $M_1 + M_2 + M_3 = 0$.

The fact that $\tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3 = 0$ implies that $(\hat{h}_1, \hat{h}_2, \hat{h}_3) \in \Gamma(\mathrm{Sp}(2m,\mathbb{C}) \cap U(2m,\mathbb{C}))$, where $\hat{h}_1, \hat{h}_2, \hat{h}_3$ are viewed as elements of Par_m by postpending $m-n$ many 0s.

To obtain the converse inclusion

$$\text{GIT-sat}(\mathrm{Sp}(2m,\mathbb{C}), (\mathrm{Sp}(2m,\mathbb{C})/P_0)^3) = \mathfrak{sp}\text{-sat}(m) \cap (\mathrm{Par}_n^{\mathbb{Q}})^3 \subset \mathfrak{sp}\text{-sat}(n),$$

we have to prove that any inequality (4) from Proposition 4.1 is satisfied by the points of $\mathfrak{sp}\text{-sat}(m) \cap (\mathrm{Par}_n^{\mathbb{Q}})^3$; here we have used (26). Fix such an inequality (A, A', B, B', C, C'). Set

$$I = A' \cup \bar{A} \subset [2n], \quad J = B' \cup \bar{B} \subset [2n], \quad K = C' \cup \bar{C} \subset [2n].$$

Similarly, for m, set

$$\tilde{I} = A' \cup \{2m+1-i \,:\, i \in A\}, \quad \tilde{J} = B' \cup \{2m+1-i \,:\, i \in B\}$$

and

$$\tilde{K} = C' \cup \{2m+1-i \,:\, i \in C\};$$

these are subsets of $[2m]$. Set also $a' = |A'|$, $b' = |B'|$ and $c = |C|$.

Notice that $(\tilde{I}^2)^0, (\tilde{J}^2)^0, (\tilde{K}^2)^0 \subseteq 0^r = \emptyset$. Thus, trivially,

$$c^{\tau((\tilde{K}^2)^0)}_{\tau((\tilde{I}^2)^0)^\vee, \tau((\tilde{J}^2)^0)^\vee} = c^{\emptyset}_{\emptyset,\emptyset} = 1. \tag{29}$$

Also, $(\tilde{I}^2)^2 = \tilde{I}^2, (\tilde{J}^2)^2 = \tilde{J}^2, (\tilde{K}^2)^2 = \tilde{K}^2$. Since $\tau(\tilde{I}^2) = \tau(I^2) := \tau^2(A, A')$, $\tau(\tilde{J}^2) = \tau(J^2) := \tau^2(B, B')$ and $\tau(\tilde{K}^2) = \tau(K^2) := \tau^2(C, C')$, we have

$$c^{\tau((\tilde{K}^2)^2)}_{\tau((\tilde{I}^2)^2)^{\vee[r^r]}, \tau((\tilde{J}^2)^2)^{\vee[r^r]}} = c^{\tau(\tilde{K}^2)}_{\tau(\tilde{I}^2)^{\vee[r^r]}, \tau(\tilde{J}^2)^{\vee[r^r]}} = c^{\tau^2(C,C')}_{\tau^2(A,A')^{\vee[r^r]}\, \tau^2(B,B')^{\vee[r^r]}} = 1. \tag{30}$$

We apply [Res12, Theorem 8.2] to $\mathrm{Gr}_\omega(r, 2r)$ and the triple $\tilde{I}^2, \tilde{J}^2, \tilde{K}^2$. Equations (29) and (30) mean that condition (iii) of the said theorem holds. Hence by part (ii) of *ibid.*,

$$[\overline{BF_{\tilde{I}^2}}] \cdot [\overline{BF_{\tilde{J}^2}}] \cdot [\overline{BF_{\tilde{K}^2}}] = [pt] \in \mathrm{H}^*(\mathrm{Gr}_\omega(r, 2r), \mathbb{Z}). \tag{31}$$

One can easily check that

$$\begin{aligned} \tau(\tilde{K}^0) &= [2(m-n)]^c + \tau(K^0), \\ \tau(\tilde{I}^0)^{\vee[(2m-2r)^r]} &= [2(m-n)]^{a'} + \tau(I^0)^{\vee[(2n-2r)^r]}, \\ \tau(\tilde{J}^0)^{\vee[(2m-2r)^r]} &= [2(m-n)]^{b'} + \tau(J^0)^{\vee[(2n-2r)^r]}. \end{aligned}$$

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The assumption $a' + b' = c$ and the semigroup property of LR-semigroup implies that

$$c^{\tau(\tilde{K}^0)}_{\tau(\tilde{I}^0)^{\vee[(2m-2r)^r]},\tau(\tilde{J}^0)^{\vee[(2m-2r)^r]}} \neq 0. \tag{32}$$

Next we apply [Res12, Proposition 8.1] to $\tilde{I}, \tilde{J}, \tilde{K}$ and the space $\mathrm{Gr}_\omega(r, 2m)$; Equations (31) and (32) mean that condition (iii) holds. Hence by (i) of *ibid.* and (27),

$$[\overline{P_0 F_{\tilde{I}}}] \odot_0 [\overline{P_0 F_{\tilde{J}}}] \odot_0 [\overline{P_0 F_{\tilde{K}}}] = d[pt] \in \mathrm{H}^*(\mathrm{Gr}_\omega(r, 2m), \mathbb{Z}), \tag{33}$$

for some nonzero d. Now use Proposition 2.1, which shows that (4) is a case of Proposition 2.1(2) which holds on $\text{GIT-sat}(\mathrm{Sp}(2m, \mathbb{C}), (\mathrm{Sp}(2m, \mathbb{C})/P_0)^3) = \mathfrak{sp}\text{-sat}(m) \cap (\mathrm{Par}_n^{\mathbb{Q}})^3$, as desired. □

Proof of Theorem 1.2. This follows from Theorem 1.1 and Proposition 4.1. □

Example 4.2. Let $n = 4, r = 3$. Let

$$A = B' = C' = \emptyset, \quad A' = \{2, 3, 4\}, \quad B = \{1, 2, 4\}, \quad C = \{1, 3, 4\},$$

giving a triple $((A, A'), (B, B'), (C, C'))$ in $(\mathrm{Schub}'(\mathrm{Gr}_\omega(3, 8)))^3$ satisfying conditions (1) and (2) from Theorem 1.2. The corresponding triple in $\mathrm{Schub}(\mathrm{Gr}_\omega(3, 8))$ is

$$I = \{2, 3, 4\}, \quad J = \{5, 7, 8\}, \quad K = \{5, 6, 8\} \subseteq [8].$$

Thus

$$\tau(I) = (1, 1, 1), \quad \tau(J) = (5, 5, 4), \quad \tau(K) = (5, 4, 4) \subseteq (5^3).$$

Now,

$$\begin{aligned} \tau(I^0) &= \tau(\{2, 3, 4\}, \{1, 2, 3, 4, 8\}) = 111, & \tau(I^2) &= \tau(\{2, 3, 4\}, \{2, 3, 4, 5, 6, 7\}) = 000, \\ \tau(J^0) &= \tau(\{5, 7, 8\}, \{3, 5, 6, 7, 8\}) = 221, & \tau(J^2) &= \tau(\{5, 7, 8\}, \{1, 2, 4, 5, 7, 8\}) = 333, \\ \tau(K^0) &= \tau(\{5, 6, 8\}, \{2, 5, 6, 7, 8\}) = 211, & \tau(K^2) &= \tau(\{5, 6, 8\}, \{1, 3, 4, 5, 6, 8\}) = 333, \end{aligned}$$

and thus

$$\begin{aligned} I^0 &= \{2, 3, 4\}, & I^2 &= \{1, 2, 3\}, \\ J^0 &= \{2, 4, 5\}, & J^2 &= \{4, 5, 6\}, \\ K^0 &= \{2, 4, 5\}, & K^2 &= \{4, 5, 6\}. \end{aligned}$$

The reader can check that

$$c^{\tau^0(C,C')}_{\tau^0(A,A')^{\vee[(2n-2r)^r]},\tau^0(B,B')^{\vee[(2n-2r)^r]}} = c^{\tau(K_0)}_{\tau(I_0)^{\vee[(2n-2r)^r]},\tau(J_0)^{\vee[(2n-2r)^r]}} = c^{(2,1,1)}_{(1,1,1),(1)} = 1,$$

$$c^{\tau^2(C,C')}_{\tau^2(A,A')^{\vee[r^r]},\tau^2(B,B')^{\vee[r^r]}} = c^{\tau(K_2)}_{\tau(I_2)^{\vee[r^r]},\tau(J_2)^{\vee[r^r]}} = c^{(3,3,3)}_{(3,3,3),(0,0,0)} = 1.$$

Hence, by Theorem 1.2, $-\lambda_2 - \lambda_3 - \lambda_4 + \mu_1 + \mu_2 + \mu_4 + \nu_1 + \nu_3 + \nu_4 \geqslant 0$ is one of the inequalities defining $\mathfrak{sp}\text{-sat}(4)$.

5. The truncated tensor cone

In this section, we characterize the *truncated tensor cone* of $\mathfrak{sp}\text{-sat}(m)$, that is, $\mathfrak{sp}\text{-sat}(m) \cap (\mathrm{Par}_n^{\mathbb{Q}})^3$ where $m > n$. By (3), this implies another set of inequalities for $\text{NL-sat}(n)$.

We first need the following result, a generalization of Theorem 2.3.

Theorem 5.1. *Here $X = G/P \times G/Q \times G/R$. Let $(\lambda, \mu, \nu) \in X(P) \times X(Q) \times X(R)$ be dominant. Then $\mathcal{L}_{(\lambda,\mu,\nu)} \in \text{GIT-sat}(G, X)$ if and only if for any simple root α, for any*

$$(u, v, w) \in W_P \backslash W / W_{P^\alpha} \times W_Q \backslash W / W_{P^\alpha} \times W_R \backslash W / W_{P^\alpha}$$

such that

$$[\overline{PuP^\alpha/P^\alpha}] \odot_0 [\overline{QvP^\alpha/P^\alpha}] \odot_0 [\overline{RwP^\alpha/P^\alpha}] = [pt] \in \mathrm{H}^*(G/P^\alpha, \mathbb{Z}), \tag{34}$$

$$\langle u\varpi_{\alpha^\vee}, \lambda\rangle + \langle v\varpi_{\alpha^\vee}, \mu\rangle + \langle w\varpi_{\alpha^\vee}, \nu\rangle \leqslant 0. \tag{35}$$

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Proof. GIT-sat(G, X) is characterized by Proposition 2.1(3); let (C, τ) and a choice of

$$(u', v', w') \in W_P\backslash W/W_{P(\tau)} \times W_Q\backslash W/W_{P(\tau)} \times W_R\backslash W/W_{P(\tau)}$$

be as in that proposition. Since every inequality (35) appears in part (3) of Proposition 2.1 with $\tau = \varpi_{\alpha^\vee}$, it suffices to show that inequalities in (3) of Proposition 2.1 are implied by the inequalities in Theorem 5.1.

Write

$$\tau = \sum_{\alpha \in \Delta} n_\alpha \varpi_{\alpha^\vee},$$

where Δ is the set of simple roots. Since τ is dominant the n_α are nonnegative. Set

$$\mathrm{Supp}(\tau) := \{\alpha \in \Delta : n_\alpha \neq 0\}.$$

Fix any $\alpha \in \mathrm{Supp}(\tau)$; $P^\alpha := P(\varpi_{\alpha^\vee})$ contains $P(\tau)$. Let

$$\pi : G/P(\tau) \longrightarrow G/P^\alpha$$

denote the associated projection. By [Ric12, Theorem 1.1 and § 1.1] (see also [Res11]), condition (13) implies there are

$$(u, v, w) \in W_P\backslash W/W_{P^\alpha} \times W_Q\backslash W/W_{P^\alpha} \times W_R\backslash W/W_{P^\alpha},$$

such that condition (34) holds and such that (u, v, w) and (u', v', w') define the same cosets in $W_P\backslash W/W_{P^\alpha} \times W_Q\backslash W/W_{P^\alpha} \times W_R\backslash W/W_{P^\alpha}$. Therefore, inequality (35) is the same as

$$\langle u'\varpi_{\alpha^\vee}, \lambda\rangle + \langle v'\varpi_{\alpha^\vee}, \mu\rangle + \langle w'\varpi_{\alpha^\vee}, \nu\rangle \leqslant 0.$$

Therefore each of the inequalities (3) can be written as a linear combination of (35). Hence, the inequalities of the theorem imply and are implied by the inequalities of Proposition 2.1 (3), so the result follows. □

We now deduce from Theorem 5.1 the following statement.

PROPOSITION 5.2. *Let (λ, μ, ν) in Par_n and $m \geqslant n$. Then $(\lambda, \mu, \nu) \in \mathfrak{sp}\text{-sat}(m)$ if and only if*

$$|\lambda_{I\cap[n]}| - |\lambda_{\bar{I}\cap[n]}| + |\mu_{J\cap[n]}| - |\mu_{\bar{J}\cap[n]}| + |\nu_{K\cap[n]}| - |\nu_{\bar{K}\cap[n]}| \leqslant 0, \tag{36}$$

for any $1 \leqslant r \leqslant m$ and $(I, J, K) \in \mathrm{Schub}^{P_0}(\mathrm{Gr}_\omega(r, 2m))^3$ such that:

(1) $|I \cap [m]| + |J \cap [m]| + |K \cap [m]| = r$; *and*
(2) $c^{\tau(K_0)}_{\tau(I_0)^{\vee[(2m-2r)^r]}, \tau(J_0)^{\vee[(2m-2r)^r]}} = c^{\tau(K_2)}_{\tau(I_2)^{\vee[r^r]}, \tau(J_2)^{\vee[r^r]}} = 1.$

Proof. We already observed (28) that inequality (36) is inequality (3) in our context. Regarding Theorem 5.1, the only thing to prove is that condition (13) associated to (I, J, K) is equivalent to the two conditions of the proposition. This is [Res12, Theorem 8.2]. □

A priori, Proposition 5.2 could contain redundant inequalities. In view of Theorem 1.2, an affirmative answer to this question would imply irredundancy:

Question 1. Does any $(I, J, K) \in \mathrm{Schub}^{P_0}(\mathrm{Gr}_\omega(r, 2m))^3$ occurring in Proposition 5.2 satisfy

(1) $I \cap [n+1, 2m-n] = J \cap [n+1, 2m-n] = K \cap [n+1, 2m-n] = \emptyset$,
(2) $c^{\tau(\hat{K}_0)}_{\tau(\hat{I}_0)^{\vee[(2n-2r)^r]}, \tau(\hat{J}_0)^{\vee[(2n-2r)^r]}} = 1$,

where $\hat{I} = I \cap [n] \cup \{i - 2(m-n) : i \in \bar{I} \cap [m+1, 2m]\}$, and $\hat{J}$ and $\hat{K}$ are defined similarly?

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Proof of Theorem 1.3. Fix an inequality (A, A', B, B', C, C') from (4). It is irredundant for the full-dimensional cone

$$\text{NL-sat}(n) = \mathfrak{sp}\text{-sat}(2n) \cap (\text{Par}_n)^3 \subset \mathbb{R}^{3n}.$$

Thus, it has to appear in Proposition 5.2 for $m = 2n$. Let $(\tilde{I}, \tilde{J}, \tilde{K}) \in \text{Schub}^{P_0}(\text{Gr}_\omega(\tilde{r}, 4n))^3$ be the associated Schubert triple. Set $\tilde{A}' = \tilde{I} \cap [2n]$, $\tilde{A} = \bar{\tilde{I}} \cap [2n]$, etc. Since $(\tilde{I}, \tilde{J}, \tilde{K}) \in \text{Schub}^{P_0}(\text{Gr}_\omega(\tilde{r}, 4n))^3$, $\tilde{A}', \tilde{B}', \tilde{C}' \subset [n]$ (by (27)). Thus, comparing (4) and (36), we have

$$\begin{aligned} A &= \tilde{A} \cap [n], & B &= \tilde{B} \cap [n], & C &= \tilde{C} \cap [n], \\ A' &= \tilde{A}' \cap [n] = \tilde{A}', & B' &= \tilde{B}' \cap [n] = \tilde{B}', & C' &= \tilde{C}' \cap [n] = \tilde{C}'. \end{aligned}$$

Now, Proposition 5.2(1) and Theorem 1.2(2) imply that $r = \tilde{r}$. In particular, $|\tilde{A}| + |\tilde{A}'| = |A| + |A'| = r$ and $A = \tilde{A}$. Similarly, $B = \tilde{B}$ and $C = \tilde{C}$.

Let α be the simple root of $\text{Sp}(4n, \mathbb{C})$ associated to r. Observe that the Levi subgroup of P^α has type $A_{r-1} \times C_{2n-r}$. Let $u, v, w \in W^{P^\alpha}$ corresponding to $(\tilde{A}', \tilde{A}), (\tilde{B}', \tilde{B})$ and $(\tilde{C}', \tilde{C})$, respectively. Proposition 5.2 and its proof show that (34) holds with $P = Q = R = P_0$. In particular, one can apply the reduction rule proved in [Roth11, Theorem 3.1] or [Res21, Theorem 1]: $\text{mult}^{2n}_{\lambda,\mu,\nu}$ is a tensor multiplicity for the Levi subgroup of P^α of type $A_{r-1} \times C_{2n-r}$. The factor $c^{\nu^*_{C,C'}}_{\lambda_{A,A'},\mu_{B,B'}}$ in the theorem corresponds to the factor of type A_{r-1}. Adding zeros, consider λ as an element of Par_{2n}. Then the dominant weights to consider for the factor C_{2n-r} are $\lambda_{[2n]-(\tilde{A}\cup\tilde{A}')}, \mu_{[2n]-(\tilde{B}\cup\tilde{B}')}, \nu_{[2n]-(\tilde{C}\cup\tilde{C}')}$. Since these partitions have length at most $n - r$, the tensor multiplicity for the factor of C_{2n-r} is a Newell–Littlewood coefficient. The theorem follows. ☐

6. Application to Conjecture 1.4

COROLLARY 6.1 (Of Theorem 1.2). *Conjecture 1.4 holds for $n \leqslant 5$.*

The proof is computational and uses the software Normaliz [BIS].

Fix $n \geqslant 2$ and consider the cone $\mathfrak{sp}$-sat(n). Consider the two lattices $\Lambda = \mathbb{Z}^{3n}$ and

$$\Lambda_2 = \{(\lambda, \mu, \nu) \in (\mathbb{Z}^n)^3 : |\lambda| + |\mu| + |\nu| \text{ is even}\}.$$

Then NL-semigroup$(n) \subset \Lambda_2 \cap \mathfrak{sp}$-sat$(n)$. Conjecture 1.4 asserts that the converse inclusion holds. The set $\Lambda_2 \cap \mathfrak{sp}$-sat$(n)$ is a semigroup of Λ_2 defined by a family of linear inequalities (explicitly given by Theorem 1.2). Using Normaliz [BIS] one can compute (for small n) the minimal set of generators, that is, the *Hilbert basis*, for this semigroup. Hence, to prove Corollary 6.1 one can proceed as follows.

(1) Compute the list of inequalities given by Theorem 1.2.
(2) Compute the Hilbert basis of $\Lambda_2 \cap \mathfrak{sp}$-sat$(n)$ using Normaliz.
(3) Check $N_{\lambda,\mu,\nu} > 0$ for any (λ, μ, ν) in the Hilbert basis.

Table 1 summarizes our computations; see [GO21].

In the column 'No. facets' there are the number of partition inequalities (like $\lambda_1 \geqslant \lambda_2$) plus the number of inequalities (4) given by Theorem 1.2. The next column counts the inequalities (9) given by applying Theorem 1.5. The number of extremal rays of the cone $\mathfrak{sp}$-sat(n) is also given. The two last column are the cardinalities of the Hilbert bases of the two semigroups $\Lambda_2 \cap \mathfrak{sp}$-sat$(n)$ and $\Lambda \cap \mathfrak{sp}$-sat$(n)$.

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Table 1. Data for $\Lambda_2 \cap \mathfrak{sp}\text{-sat}(n)$

n	No. facets	No. EHI	No. rays	No. Hilb $\Lambda_2 \cap \mathfrak{sp}$-sat	No. Hilb $\Lambda \cap \mathfrak{sp}$-sat
2	$6+18$	18	12	13	20
3	$9+93$	100	51	58	93
4	$12+474$	662	237	302	451
5	$15+2421$	5731	1122	1598	2171

7. Littlewood–Richardson coefficients

We recall material [Ful97, FH91] on Littlewood–Richardson coefficients and their role in representation theory of the general linear group.

7.1 Representations of $\mathbf{GL}(n, \mathbb{C})$

The irreducible rational representations $V(\lambda)$ of $\mathrm{GL}(n,\mathbb{C})$ are indexed by their highest weight

$$\lambda \in \Lambda_n^+ = \{(\lambda_1 \geqslant \cdots \geqslant \lambda_n) \,:\, \lambda_i \in \mathbb{Z}\} \supset \mathrm{Par}_n.$$

One has tensor product multiplicities $c_{\lambda,\mu}^{\nu}$ defined for any $\lambda, \mu, \nu \in \Lambda_n^+$ by

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu \in \Lambda_n^+} V(\nu)^{\oplus c_{\lambda,\mu}^{\nu}}. \tag{37}$$

When $\lambda, \mu, \nu \in \mathrm{Par}_n$, $c_{\lambda,\mu}^{\nu}$ is the Littlewood–Richardson coefficient (which is why we use the same notation).

The dual representation $V(\lambda)^*$ has highest weight

$$\lambda^* = (-\lambda_n \geqslant \cdots \geqslant -\lambda_1) \in \Lambda_n^+.$$

Moreover, for any $a \in \mathbb{Z}$,

$$V(\lambda + a^n) = (\det)^a \otimes V(\lambda). \tag{38}$$

Consequently, for any $\lambda, \mu, \nu \in \Lambda_n^+$,

$$c_{\lambda,\mu}^{\nu} = c_{\lambda+a^n,\mu+b^n}^{\nu+(a+b)^n} = c_{\lambda^*+a^n,\mu^*+b^n}^{\nu^*+(a+b)^n}; \tag{39}$$

this is [BOR15, Theorem 4]. For a and b big enough, formula (39) implies that $c_{\lambda,\mu}^{\nu}$ is a Littlewood–Richardson coefficient.

Let ν^t denote the conjugate of ν. Since $c_{\lambda,\mu}^{\nu} = c_{\lambda^t,\mu^t}^{\nu^t}$, by (39),

$$c_{\lambda,\mu}^{\nu} = c_{\lambda^t,\mu^t}^{\nu^t} = c_{(\lambda^t)^{\vee[n^{a+b}]},(\mu^t)^{\vee[m^{a+b}]}}^{(\nu^t)^{\vee[(n+m)^{a+b}]}} = c_{(\lambda^{\vee[(a+b)^n]})^t,(\mu^{\vee[(a+b)^m]})^t}^{(\nu^{\vee[(a+b)^{n+m}]})^t} = c_{\lambda^{\vee[(a+b)^n]},\mu^{\vee[(a+b)^m]}}^{\nu^{\vee[(a+b)^{n+m}]}}, \tag{40}$$

for any $m \geqslant \ell(\mu)$.

7.2 Six-fold Newell–Littlewood coefficients

Let p, q and m be positive integers such that $p+q \leqslant m$. Following R. Howe, Tan and Willenbring [HT05], to any $\lambda^+ \in \mathrm{Par}_p$ and $\lambda^- \in \mathrm{Par}_q$ we associate the following element in Λ_m^+:

$$[\lambda^+, \lambda^-]_m = (\lambda_1^+, \lambda_2^+, \ldots, \lambda_p^+, \underbrace{0, \ldots, 0}_{m-p-q}, -\lambda_q^-, \ldots, -\lambda_1^-).$$

Let $V(\lambda) \boxtimes V(\mu)$ be the irreducible representation of $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$, where $\boxtimes$ refers to the external tensor product. View $\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ under the diagonal

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embedding. The associated *branching coefficient* is

$$[V(\nu):V(\lambda)\boxtimes V(\mu)] := \dim \mathrm{Hom}_{\mathrm{GL}(n,\mathbb{C})}(V(\nu), V(\lambda)\boxtimes V(\mu)|_{\mathrm{GL}(n,\mathbb{C})}).$$

Proposition 7.1 ([HT05, § 2.1.1], [Kin71]). *Let $\lambda^\pm$, $\mu^\pm$ and $\nu^\pm$ be six partitions. Let p, q, r and s be four nonnegative integers such that*

$$\begin{array}{lll} \ell(\lambda^+)\leqslant p, & \ell(\mu^+)\leqslant r, & \ell(\nu^+)\leqslant p+r, \\ \ell(\lambda^-)\leqslant q, & \ell(\mu^-)\leqslant s, & \ell(\nu^-)\leqslant q+s. \end{array}$$

Let m be a positive integer such that $m\geqslant p+q+r+s$. Then

$$\begin{aligned} N_{\mu^+,\nu^+,\lambda^+,\mu^-,\nu^-,\lambda^-} &= \dim \mathrm{Hom}_{\mathrm{GL}(n,\mathbb{C})}(V(\nu), V(\lambda)\boxtimes V(\mu)) \\ &= c^{[\nu^+,\nu^-]_m}_{[\lambda^+,\lambda^-]_m,[\mu^+,\mu^-]_m}. \end{aligned}$$

Proof. The first equality is in [HT05, § 2.1.1], which credits [Kin71]. The second statement follows from [FH91, p. 427].[3] □

Conversely, any Littlewood–Richardson coefficient is a six-fold Newell–Littlewood coefficient. More precisely, $c^\nu_{\lambda,\mu} = N_{\mu,\nu,\lambda,\emptyset,\emptyset,\emptyset}$, which corresponds to the case when $\lambda^- = \mu^- = \nu^- = \emptyset$ in Proposition 7.1.

We now use Proposition 7.1 to rephrase Theorem 1.5. Fix A, A', B, B', C and C' subseets of $[n]$ satisfying the two first conditions of Theorem 1.5. Set $p=|B'|$, $r=|C'|$, $q=n-p$ and $s=n-q$. Observe that $|A|=p+r$, $|B|\leqslant q$, $|C|\leqslant s$ and $|A'|\leqslant n-p-r\leqslant p+q$. Finally, set $m=2n=p+q+r+s$. Since $\ell(\tau(A))\leqslant |A|$, Proposition 7.1 implies that

$$N_{\tau(A),\tau(C'),\tau(B),\tau(A'),\tau(C),\tau(B')} = c^{[\tau(A),\tau(A')]_m}_{[\tau(B'),\tau(B)]_m,[\tau(C'),\tau(C)]_m}$$

is a Littlewood–Richardson coefficient for $\mathrm{GL}_{2n}(\mathbb{C})$. In particular, in Theorem 1.5, condition (3) can be replaced by:

(3′) $c^{[\tau(A),\tau(A')]_m}_{[\tau(B'),\tau(B)]_m,[\tau(C'),\tau(C)]_m} > 0$.

We now observe that Proposition 7.1 and Knutson–Tao saturation [KT99] imply the saturation result for the six-fold Newell–Littlewood-coefficients from the introduction (Proposition 1.6).

8. Extended Horn inequalities and the proof of Theorem 1.5

8.1 Extended Horn inequalities

We recall the following notion from [GOY20b].

Definition 8.1. An extended Horn inequality on Par_n^3 is

$$0\leqslant |\lambda_A| - |\lambda_{A'}| + |\mu_B| - |\mu_{B'}| + |\nu_C| - |\nu_{C'}| \tag{41}$$

where $A, A', B, B', C, C'\subseteq [n]$ satisfy the following assertions:

(I) $A\cap A' = B\cap B' = C\cap C' = \emptyset$;

(II) $|A| = |B'|+|C'|$, $|B| = |A'|+|C'|$, $|C| = |A'|+|B'|$;

(III) there exists $A_1, A_2, B_1, B_2, C_1, C_2 \subseteq [n]$ such that

 (i) $|A_1| = |A_2| = |A'|$, $|B_1| = |B_2| = |B'|$, $|C_1| = |C_2| = |C'|$,

[3] Using (38) one can equate this with a Littlewood–Richardson coefficient $c^{\tilde\nu}_{\tilde\lambda,\tilde\mu}$ where $\tilde\lambda, \tilde\mu, \tilde\nu \in \mathrm{Par}_m$.

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(ii) $c^{\tau(A')}_{\tau(A_1),\tau(A_2)}, c^{\tau(B')}_{\tau(B_1),\tau(B_2)}, c^{\tau(C')}_{\tau(C_1),\tau(C_2)} > 0$,
(iii) $c^{\tau(A)}_{\tau(B_1),\tau(C_2)}, c^{\tau(B)}_{\tau(C_1),\tau(A_2)}, c^{\tau(C)}_{\tau(A_1),\tau(B_2)} > 0$.

Definition 8.2. The extended Horn cone is

$$\mathrm{EH}(n) := \{(\lambda, \mu, \nu) \in (\mathrm{Par}_n^{\mathbb{Q}})^3 : \text{inequalities (41) are satisfied}\}. \tag{42}$$

Let

$$\overline{\mathrm{EH}}(n) = \mathrm{EH}(n) \cap \{(\lambda, \mu, \nu) \in (\mathrm{Par}_n)^3 : |\lambda| + |\mu| + |\nu| \text{ is even}\}.$$

Conjecture 8.3 [GOY20b, Conjecture 1.4]. If $(\lambda, \mu, \nu) \in \overline{\mathrm{EH}}(n)$ then $N_{\lambda,\mu,\nu} > 0$.

We will prove a weakened version of Conjecture 8.3.

Theorem 8.4 (Cf. [GOY20b, Conjecture 1.4]). *We have* $\mathrm{EH}(n) = \mathrm{NL\text{-}sat}(n)$.

Consequently, we are able to answer an issue raised in [GOY20b, § 1].

Corollary 8.5. *Conjecture 1.4 implies Conjecture 8.3.*

Corollary 8.5 is analogous to the situation in Zelevinsky's [Zel99], before [KT99].
The following shows that Theorem 1.5 is equivalent to Theorem 8.4.

Lemma 8.6. *A sextuple* (A, A', B, B', C, C') *of subsets of* $[n]$ *parametrizes an extended Horn inequality if and only if it appears in Theorem 1.5.*

Proof. Definition 8.1 implies

$$c^{\tau(A)}_{\tau(B_1),\tau(C_2)} c^{\tau(C')}_{\tau(C_2),\tau(C_1)} c^{\tau(B)}_{\tau(C_1),\tau(A_2)} c^{\tau(A')}_{\tau(A_2),\tau(A_1)} c^{\tau(C)}_{\tau(A_1),\tau(B_2)} c^{\tau(B')}_{\tau(B_2),\tau(B_1)} > 0.$$

Since $\tau(A_1)$, $\tau(A_2)\ldots$ have length at most n, this implies $N_{\tau(A),\tau(C'),\tau(B),\tau(A'),\tau(C),\tau(B')} \neq 0$, and thus $(\tau(A), \tau(C'), \tau(B), \tau(A'), \tau(C), \tau(B')) \in \mathrm{NL}^6\text{-sat}(r)$.

Conversely, if $(\tau(A), \tau(C'), \tau(B), \tau(A'), \tau(C), \tau(B')) \in \mathrm{NL}^6\text{-sat}(r)$, by Proposition 1.6, $N_{\tau(A),\tau(C'),\tau(B),\tau(A'),\tau(C),\tau(B')} \neq 0$. Therefore, there exists $\alpha_1, \alpha_2, \ldots, \alpha_6 \in \mathrm{Par}_n$ such that

$$c^{\tau(A)}_{\alpha_1,\alpha_2} c^{\tau(C')}_{\alpha_2,\alpha_3} c^{\tau(B)}_{\alpha_3,\alpha_4} c^{\tau(A')}_{\alpha_4,\alpha_5} c^{\tau(C)}_{\alpha_5,\alpha_6} c^{\tau(B')}_{\alpha_6,\alpha_1} > 0.$$

Set $a = |A|$. Then, the Young diagram of $\tau(A)$ is contained in the rectangle $a \times (n - a)$. But the nonvanishing of $c^{\tau(A)}_{\alpha_1,\alpha_2}$ implies that $\alpha_1 \subset \tau(A)$. Hence, there exists $B_1 \subseteq [n]$ such that $\tau(B_1) = \alpha_1$. Similarly, we can pick $C_2, C_1, A_2, A_1, B_2 \subseteq [n]$ such that $\tau(C_2) = \alpha_2$, $\tau(C_1) = \alpha_3$, etc., that satisfy Definition 8.1. □

8.2 Proof of Theorem 1.5

(⇒) By Lemma 8.6, $\mathrm{EH}(n)$ is the cone defined by the inequalities in Theorem 1.5. Now, $\mathrm{NL\text{-}sat}(n) \subseteq \mathrm{EH}(n)$ is immediate from [GOY20b, Theorem 1].

(⇐) Fix an inequality (4) associated to (A, A', B, B', C, C') appearing in Theorem 1.2. We now show the even stronger statement that

$$N_{\tau(A),\tau(C'),\tau(B),\tau(A'),\tau(C),\tau(B')} \neq 0. \tag{43}$$

This would imply that the inequality appears in Theorem 1.5, completing the proof.

Set $a = |A|, a' = |A'|, b = |B|, b' = |B'|, c = |C|, c' = |C'|$ and $r = |A| + |A'|$. Let $I \in \mathrm{Schub}(\mathrm{Gr}_\omega(r, 2n))$ be associated to $(A, A') \in \mathrm{Schub}'(\mathrm{Gr}_\omega(r, 2n))$, under (20). Similarly define J and K. By (24) and condition (3) in Theorem 1.2, the semigroup property of nonzero

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Littlewood–Richardson coefficients implies

$$c^{\tau(K)}_{\tau(I)^{\vee[(2n-r)^r]},\tau(J)^{\vee[(2n-r)^r]}} \neq 0. \tag{44}$$

Fix a nonnegative integer k. Note that $c = a' + b'$ by condition (2) in Theorem 1.2, and $c^{(c^k)}_{(a')^k,(b')^k} = 1$. Using the semigroup property once more, one gets

$$c^{\tau(K)+(c^k)}_{\tau(I)^{\vee[(2n-r)^r]}+((a')^k),\tau(J)^{\vee[(2n-r)^r]}+((b')^k)} > 0. \tag{45}$$

Observing Figure 1,

$$(\tau(I)^\vee)^t = [\tau(A)^t, \tau(A')^t]_{2n-r} + ((a')^{2n-r}). \tag{46}$$

Similarly,

$$(\tau(I)^\vee + (a')^k)^t = [\tau(A)^t, \tau(A')^t]_m + ((a')^m) \tag{47}$$

and

$$(\tau(K) + (c^k))^t = [\tau(C')^t, \tau(C)^t]_m + (c^m), \tag{48}$$

where $m = 2n - r + k$. Now, by (39), (40), (47) and (48), condition (45) implies

$$c^{[\tau(C')^t,\tau(C)^t]_m}_{[\tau(A)^t,\tau(A')^t]_m,[\tau(B)^t,\tau(B')^t]_m} > 0. \tag{49}$$

On the other hand, for k (and hence m) that is big enough, we can apply Proposition 7.1 to get

$$N_{\tau(A)^t,\tau(C')^t,\tau(B)^t,\tau(A')^t,\tau(C)^t,\tau(B')^t} = c^{[\tau(C')^t,\tau(C)^t]_m}_{[\tau(A)^t,\tau(A')^t]_m,[\tau(B)^t,\tau(B')^t]_m}. \tag{50}$$

Since the six-fold Newell–Littlewood coefficient are invariant by conjugating the partitions, (49) and (50) imply (43) as expected. □

Remark 8.7. The earlier version of this work (arXiv:2107.03152v1) did not use Proposition 7.1. We gave a combinatorial proof, perhaps of independent interest, that connects the celebrated Robinson–Schensted–Knuth algorithm to the 'demotion' algorithm of [GOY20a].

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Conflicts of interest

None.

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Data availability

The computations in § 6 were based on Normaliz [BIS]. The computed data using that software is [GO21]. Both the software and the data are freely available for download at the indicated locations.

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