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Lie Powers and Pseudo-Idempotents

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Abstract. We give a new factorisation of the classical Dynkin operator, an element of the integral group ring of the symmetric group that facilitates projections of tensor powers onto Lie powers. As an application we show that the iterated Lie power $L_2(L_n)$ is a module direct summand of the Lie power L_{2n} whenever the characteristic of the ground field does not divide n. An explicit projection of the latter onto the former is exhibited in this case.

1 Lie Powers and the Dynkin Operator

Let K be a commutative ring with one, V a free K-module, and let $T = T(V)$ denote the tensor algebra on V. Thus $T = \bigoplus_{n\geq 0} T_n$, where

$$T_n = \underbrace{V \otimes \cdots \otimes V}_{n}$$

is the n-th tensor power of V. By defining the Lie bracket $[u, v] = u \otimes v - v \otimes u$ for all $u, v \in T$, the tensor algebra is turned into a Lie algebra. It is well known that the Lie subalgebra generated by V in T is the free Lie alebra $L = L(V)$ on V. We let $L_n = L \cap T_n$ denote the degree n homogeneous component of L, also known as the n-th Lie power of V. The natural action of $GL(V)$ on V extends to the whole of T, turning T, L and their respective homogeneous components into $GL(V)$-modules. As such the T_n and the L_n are referred to as the tensor and Lie representations of $GL(V)$, respectively. One of the main problems in the theory of Lie powers is to determine the $GL(V)$-module structure of L_n. When working over a field of characteristic zero, this problem is comparatively well-understood. However, if the characteristic of the ground field is prime, things become much more difficult (see [2] for a survey of results and further references), and very little is known about integral Lie representations.

For each n, the symmetric group S_n acts on the tensor power T_n by the Polya action, that is, by place permutations:

$$(v_1 \otimes \cdots \otimes v_n)\sigma = v_{1\sigma^{-1}} \otimes \cdots \otimes v_{n\sigma^{-1}}, \quad (v_1, \ldots, v_n \in V,\ \sigma \in S_n).$$

Note that the $GL(V)$- and the S_n-actions on T_n centralize one another. We call an element Ψ in the integral group ring $\mathbb{Z}S_n$ a pseudo-idempotent if $\Psi^2 = m\Psi$ for some integer m, and we refer to m as the coefficient of the pseudo-idempotent. The usefulness of a pseudo-idempotent lies in the obvious but important fact that if we are working over a field K in which the integer m is invertible, then Ψ can be replaced

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by the genuine idempotent $\frac{1}{m}\Psi \in KS_n$, which gives rise to a projection of T_n onto its $GL(V)$-submodule $T_n\Psi$. Hence $T_n\Psi$ is a direct summand of the tensor power T_n as a $GL(V)$-module. A prominent example of a pseudo-idempotent is the Dynkin operator,

$$\Omega_n = (1-(1,2))(1-(1,2,3))\cdots(1-(1,2,\ldots,n)).$$

It is well known (see, *e.g.*, [4, Section 5.9]) that

$$\Omega_n^2 = n\Omega_n. \tag{1.1}$$

Moreover, since the n-th Lie power L_n is spanned by the left-normed Lie products $[v_1, v_2, \ldots, v_n]$ with $v_i \in V$, and (as is well known and easily verified),

$$[v_1, v_2, \ldots, v_n] = (v_1 \otimes v_2 \otimes \cdots \otimes v_n)\Omega_n, \tag{1.2}$$

we have

$$L_n = T_n\Omega_n. \tag{1.3}$$

Together with (1.1), (1.3) implies that, over a field K in which n is invertible, the Lie power L_n is a module direct summand of the tensor power T_n. This is precisely the reason why the representation theory of Lie powers becomes, by a magnitude, harder if the degree is divisible by the characteristic.

Using the Dynkin operator one can easily construct pseudo-idempotents for iterated Lie powers of the form $L_{n_1}(L_{n_2}(\cdots(L_{n_k})\cdots))$, that is $\Omega_{n_1,\ldots,n_k} \in \mathbb{Z}S_{n_1\cdots n_k}$ such that $T_{n_1\cdots n_k}\Omega_{n_1,\ldots,n_k} = L_{n_1}(L_{n_2}(\cdots(L_{n_k})\cdots))$ with

$$\Omega_{n_1,\ldots,n_k}^2 = n_1 n_2^{n_1} \cdots n_k^{n_1 n_2 \cdots n_{k-1}} \Omega_{n_1,\ldots,n_k}.$$

Indeed, write $T_{n_1\cdots n_k} = T_{n_1}(T_{n_2}(\cdots(T_{n_k})\cdots))$, and define $\Omega_{n_1,\ldots,n_k}$ inductively by setting

$$\Omega_{n_1,\ldots,n_k} = \underbrace{(\Omega_{n_2,\ldots,n_k} \otimes \cdots \otimes \Omega_{n_2,\ldots,n_k})}_{n_1}\Omega_{n_1}$$

with the obvious convention that Ω_{n_1} acts on $T_{n_1}(U)$ with $U = T_{n_2\cdots n_k}$. It is not hard to verify that this gives pseudo-idempotents with the required properties (full details for $\Omega_{2,n}$ are given in Section 2.2). However, when working over a field of characteristic p, these pseudo-idempotents can only be used to show that the iterated Lie power is a module direct summand of the corresponding tensor power if p does not divide $n_1 n_2 \cdots n_k$.

In this note we give a new factorisation of the Dynkin operator in which all elements of S_n involved are involutions.

Theorem 1.1 *For $k \geq 2$, let*

$$\alpha_k = (1,k)(2,k-1)\cdots\left(\left[\tfrac{k}{2}\right], k-\left[\tfrac{k}{2}\right]+1\right).$$

Then $\Omega_n = (1-\alpha_2)(1+\alpha_3)\cdots(1+(-1)^{n-1}\alpha_n)$ for all $n \geq 2$.

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As an application we derive a pseudo-idempotent $\Gamma_{2n} \in \mathbb{Z}S_{2n}$ for the iterated Lie power $L_2(L_n)$ such that

$$\Gamma_{2n}^2 = (-1)^{n-1} 2n^2 \Gamma_{2n} \quad \text{and} \quad T_{2n}\Gamma_{2n} = L_2(L_n), \tag{1.4}$$

but with the important additional property that

$$\Omega_{2n}\Gamma_{2n} \in 2\mathbb{Z}S_{2n}. \tag{1.5}$$

The advantage here is that the restriction of the operator Γ_{2n} on T_{2n} to the submodule $L_{2n} = T_{2n}\Omega_{2n}$ is divisible by 2. It follows that the map

$$[v_1, \dots, v_{2n}] \mapsto \tfrac{1}{2}(v_1 \otimes \cdots \otimes v_{2n})\Omega_{2n}\Gamma_{2n} \tag{1.6}$$

extends to a well-defined $GL(V)$-module homomorphism $L_{2n} \to L_2(L_n)$ such that the restriction to the submodule $L_2(L_n) = T_{2n}\Gamma_{2n}$ amounts to multiplication by $(-1)^{n-1}n^2$. This yields the following result.

Theorem 1.2 *If K is a field in which n is invertible, then $L_2(L_n)$ is a direct summand of L_{2n} as a $GL(V)$-module.*

Theorem 1.2 is a special case of a much more general result by Bryant and Schocker [2], see also Erdmann and Schocker [3] and Bryant [1]. The advantage of our approach (apart from being an illustration of the usefulness of Theorem 1.1) is that our projection $L_{2n} \to L_2(L_n)$ is completely explicit, see (2.7). We believe it would be rather difficult to try to extract an explicit projection from [2]. Of course, the most interesting instance of Theorem 1.2 is when K is a field of characteristic 2, since in this case the result cannot be proved using the pseudo-idempotent $\Omega_{2,n}$. Finally, we hope that Theorem 1.1will find further applications in the future.

2 Proofs

2.1 Proof of Theorem 1.1

Let $\rho_k = (1, 2, \dots, k)$ denote the standard k-cycle. The key observation for the proof of Theorem 1.1 is that

$$\rho_k = \alpha_{k-1}\alpha_k \tag{2.1}$$

for all $k \geq 2$ with the convention that $\alpha_1 = 1$. Since α_k is an involution, one has

$$(1 + (-1)^j\alpha_k)\alpha_k = (-1)^j(1 + (-1)^j\alpha_k). \tag{2.2}$$

With (2.1) and (2.2) at our disposal it remains to carry out a straightforward induction. If $n = 2$, we have that $\rho_2 = \alpha_2$, and the theorem holds trivially. For $n > 2$ we have $\Omega_n = \Omega_{n-1}(1 - \rho_n)$ and by induction we have

$$\Omega_n = (1 - \alpha_2)\cdots(1 + (-1)^{n-2}\alpha_{n-1})(1 - \rho_n).$$

Now by (2.1) and (2.2) we have that

$$\begin{aligned}(1 + (-1)^{n-2}\alpha_{n-1})(1 - \rho_n) &= (1 + (-1)^{n-2}\alpha_{n-1})(1 - \alpha_{n-1}\alpha_n) \\ &= (1 + (-1)^{n-2}\alpha_{n-1})(1 + (-1)^{n-1}\alpha_n),\end{aligned}$$

as required. This completes the proof of Theorem 1.1. ■

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2.2 Proof of Theorem 1.2

We need to construct a pseudo-idempotent Γ_{2n} satisfying (1.4) and (1.5). We introduce yet another involution by setting

$$\beta_{2n} = (1, n+1)(2, n+2)\cdots(k, n+k)\cdots(n, 2n) \in S_{2n}.$$

Then, for any permutation σ of $\{1, 2, \dots, n\}$, the conjugate $\beta_{2n}\sigma\beta_{2n}$ will act on the set $\{n+1, n+2, \dots, 2n\}$ in the same way as σ acts on $\{1, 2, \dots, n\}$, that is, if $1 \le k \le n$, then $(n+k)\beta_{2n}\sigma\beta_{2n} = n + k\sigma$. Hence, using (1.2), the Lie product $[[v_1, \dots, v_n], [v_{n+1}, \dots, v_{2n}]] \in L_2(L_n)$ may be conveniently written as

$$\big[[v_1, \dots, v_n], [v_{n+1}, \dots, v_{2n}]\big] = (v_1 \otimes \cdots \otimes v_{2n})\Omega_n(\beta_{2n}\Omega_n\beta_{2n})(1 - \beta_{2n}).$$

In fact, the element $\Omega_n(\beta_{2n}\Omega_n\beta_{2n})(1 - \beta_{2n}) \in \mathbb{Z}S_{2n}$ coincides with the element $\Omega_{2,n}$ that was mentioned in Section 1. Hence we have

$$T_{2n}\Omega_n(\beta_{2n}\Omega_n\beta_{2n})(1 - \beta_{2n}) = L_2(L_n), \tag{2.3}$$

and that $\Omega_n(\beta_{2n}\Omega_n\beta_{2n})(1 - \beta_{2n})$ is a pseudo-idempotent with coefficient $2n^2$. The latter follows immediately on noting that Ω_n commutes with $\beta_{2n}\Omega_n\beta_{2n}$ (as they involve mutually disjoint permutations), that β_{2n} commutes with $\Omega_n(\beta_{2n}\Omega_n\beta_{2n})$, and recalling that both Ω_n and $1 - \beta_{2n}$ are pseudo-idempotents with coefficients n and 2, respectively:

$$\begin{aligned} (\Omega_n(\beta_{2n}\Omega_n\beta_{2n})(1 - \beta_{2n}))^2 &= (\Omega_n(\beta_{2n}\Omega_n\beta_{2n}))^2(1 - \beta_{2n})^2 \\ &= \Omega_n^2(\beta_{2n}\Omega_n^2\beta_{2n})(1 - \beta_{2n})^2 \\ &= 2n^2\Omega_n(\beta_{2n}\Omega_n\beta_{2n})(1 - \beta_{2n}) \end{aligned} \tag{2.4}$$

Hence, up to sign, the element $\Omega_n(\beta_{2n}\Omega_n\beta_{2n})(1 - \beta_{2n})$ satisfies the conditions (1.4). However, it does not satisfy (1.5). In order to achieve this we require another twist. Set

$$\Gamma_{2n} = \alpha_n\Omega_n(\beta_{2n}\Omega_n\beta_{2n})(1 - \beta_{2n}).$$

Since α_n acts as an automorphism on T_{2n}, we see that (2.3) gives that $T_{2n}\Gamma_{2n} = L_2(L_n)$. Moreover, Theorem 1.1 together with (2.2) yields

$$\Omega_n\alpha_n = (-1)^{n-1}\Omega_n. \tag{2.5}$$

This is actually well known, see [5, Section 1.3, Lemma 1.7]. Using (2.5) and the fact that α_n commutes with $\beta_{2n}\Omega_n\beta_{2n}$ (since the permutations involved in the latter are disjoint with α_n), as well as (2.4), we obtain, writing temporarily β instead of β_{2n} to

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save space,

$$\begin{aligned}\Gamma_{2n}^2 &= \alpha_n\Omega_n(\beta\Omega_n\beta)(1-\beta)\alpha_n\Omega_n(\beta\Omega_n\beta)(1-\beta)\\ &= \alpha_n(\Omega_n(\beta\Omega_n\beta)\alpha_n - \Omega_n\beta\Omega_n\alpha_n)\Omega_n(\beta\Omega_n\beta)(1-\beta)\\ &= \alpha_n(\Omega_n\alpha_n(\beta\Omega_n\beta) - \Omega_n(\beta\Omega_n\alpha_n\beta)\beta)\Omega_n(\beta\Omega_n\beta)(1-\beta)\\ &= (-1)^{n-1}\alpha_n(\Omega_n(\beta\Omega_n\beta) - \Omega_n(\beta\Omega_n\beta)\beta)\Omega_n(\beta\Omega_n\beta)(1-\beta)\\ &= (-1)^{n-1}\alpha_n(\Omega_n(\beta\Omega_n\beta)(1-\beta))^2\\ &= (-1)^{n-1}2n^2\alpha_n\Omega_n(\beta\Omega_n\beta)(1-\beta)\\ &= (-1)^{n-1}2n^2\Gamma_{2n}.\end{aligned}$$

Hence Γ_{2n} satisfies the two conditions in (1.4). Finally, note that $\alpha_{2n}\alpha_n = \alpha_n\beta_{2n}$. Using this, Theorem 1.1, and once more the facts that β_{2n} commutes with $\Omega_n(\beta_{2n}\Omega_n\beta_{2n})$ and that $1-\beta_{2n}$ is a pseudo-idempotent with coefficient 2, we get (again writing β instead of β_{2n} for short)

$$\begin{aligned}\Omega_{2n}\Gamma_{2n} &= \Omega_{2n-1}(1-\alpha_{2n})\alpha_n\Omega_n(\beta\Omega_n\beta)(1-\beta)\\ &= \Omega_{2n-1}\alpha_n(1-\beta)\Omega_n(\beta\Omega_n\beta)(1-\beta)\\ &= \Omega_{2n-1}\alpha_n\Omega_n(\beta\Omega_n\beta)(1-\beta)^2\\ &= 2\Omega_{2n-1}\alpha_n\Omega_n(\beta\Omega_n\beta)(1-\beta) \in 2\mathbb{Z}S_{2n}.\end{aligned} \tag{2.6}$$

Hence the pseudo-idempotent Γ_{2n} satisfies (1.5), and this completes the proof of Theorem 1.2. ■

We conclude by exhibiting our projection of L_{2n} onto $L_2(L_n)$ explicitly, assuming that we now work over a field of characteristic not dividing n. In view of (1.6) and (2.6), the projection is given by the map

$$[v_1,\dots,v_{2n}] \mapsto \tfrac{1}{n^2}(v_1\otimes\cdots\otimes v_{2n})\Omega_{2n-1}\alpha_n\Omega_n(\beta_{2n}\Omega_n\beta_{2n})(1-\beta_{2n}), \tag{2.7}$$

where $v_1,\dots,v_{2n}\in V$.

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