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ON THE NILPOTENCY OF NIL SUBRINGS

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Introduction. A famous theorem of Levitzki states that in a left Noetherian ring each nil left ideal is nilpotent. Lanski [**5**] has extended Levitzki's theorem by proving that in a left Goldie ring each nil subring is nilpotent. Another important theorem in this area which is due to Herstein and Small [**3**] states that if a ring satisfies the ascending chain condition on both left and right annihilators, then each nil subring is nilpotent. We give a short proof of a theorem (Theorem 1.6) which yields both Lanski's theorem and Herstein-Small's theorem. We make use of the ascending chain condition on principal left annihilators in order to obtain, at an intermediate step, a theorem (Theorem 1.1) which produces sufficient conditions for a nil subring to be left T-nilpotent. As a corollary of this theorem we obtain a theorem of Björk [**1**] which states that if a nil ring satisfies the ascending chain condition on principal left annihilators and has finite left dimension, then it is left T-nilpotent.

In § 2 we define an ideal L of a ring R to be essentially nilpotent if it contains a nilpotent ideal N of R which is essential in L. We show that the prime radical of an arbitrary ring is essentially nilpotent. Also we show that if R satisfies the ascending chain condition on principal left annihilators, then each nil ideal of R is essentially nilpotent.

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1. Nilpotent subrings. Throughout this paper, R will denote a ring which does not necessarily have an identity. A left (right) ideal I of R is a *left* (*right*) *annihilator* if there exists a subset S of R such that $I = \boldsymbol{l}(S) = \{x \in R: xS = 0\}$ ($I = \boldsymbol{r}(S) = \{x \in R: Sx = 0\}$). A left (right) ideal I of R is a *principal left* (*right*) *annihilator* if there exists an element $s \in R$ such that $I = \boldsymbol{l}(s)$ ($I = \boldsymbol{r}(s)$). A ring R is said to be *left T-nilpotent* if for each sequence $\{x_n\}$ of elements in R there exists an n such that $x_1x_2 \ldots x_n = 0$.

The term "ideal" will refer to a two-sided ideal unless it is adorned with the adjective "left". For $a \in R$ we will let (a) denote the principal ideal which is generated by a and let $R^1a = Ra + Za$ denote the principal left ideal which is generated by a.

We use a technique due to Björk [**1**] to prove the following theorem.

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THEOREM 1.1. *Let R satisfy the ascending chain condition on principal left annihilators and let N be a nil subring of R which is not left T-nilpotent. Then there exists a sequence $\{a_n\}$ of elements in N such that*

(1) $R^1a_1 + R^1a_2 + R^1a_3 + \ldots$ *is a direct sum of non-zero left ideals, and*

(2) $\boldsymbol{r}(\{a_k, a_{k+1}, a_{k+2}, \ldots\}) \subset \boldsymbol{r}(\{a_{k+1}, a_{k+2}, \ldots\})$ *for each k, where $\subset$ denotes strict containment.*

Proof. We say that $x_1 \in N$ has an infinite chain if there exists an infinite sequence $\{x_n\}$ in N such that $x_1x_2 \ldots x_n \neq 0$ for each n. Since N is not left T-nilpotent, there exist elements in N which have an infinite chain. Let $\boldsymbol{l}(x)$ be maximal in $\{\boldsymbol{l}(y): y \in N$ has an infinite chain$\}$. Inductively we find x_n such that $\boldsymbol{l}(x_n)$ is maximal in $\{\boldsymbol{l}(y): y \in N$ and $xx_1x_2 \ldots x_{n-1}y$ has an infinite chain$\}$. It is now easy to verify that $\boldsymbol{l}(x_i) = \boldsymbol{l}(x_ix_{i+1} \ldots x_{i+j})$ for each i and j.

We claim that $xx_1x_2 \ldots x_nx_1 = 0$ for each $n \geqq 1$. If $xx_1x_2 \ldots x_nx_1 \neq 0$, then $xx_1x_2 \ldots x_nx_1x_2 \ldots x_k \neq 0$ for each k since $\boldsymbol{l}(x_1) = \boldsymbol{l}(x_1x_2 \ldots x_k)$ for each k. Hence $xx_1x_2 \ldots x_nx_1$ has an infinite chain. Whence $\boldsymbol{l}(x_1x_2 \ldots x_nx_1) = \boldsymbol{l}(x_1)$. However, this is impossible since $x_1x_2 \ldots x_n$ is a nilpotent element. In exactly the same way we prove that $xx_1x_2 \ldots x_nx_2 = 0$ if $n \geqq 2$, and so on.

Set $a_k = xx_1x_2 \ldots x_k$. We claim that $R^1a_1 + R^1a_2 + R^1a_3 + \ldots$ is direct. In order to show that it is direct, suppose that

$$(r_1a_1 + z_1a_1) + (r_2a_2 + z_2a_2) + \ldots + (r_na_n + z_na_n) = 0,$$

where $r_i \in R$ and $z_i \in Z$. Multiply by x_2 on the right. It follows that $(r_1a_1 + z_1a_1)x_2 = 0$ or $(r_1x + z_1x)x_1x_2 = 0$. Hence $(r_1x + z_1x)x_1 = 0$. That is $r_1a_1 + z_1a_1 = 0$. Then we multiply by x_3 on the right to obtain

$$(r_2xx_1 + z_2xx_1)x_2x_3 = 0,$$

and hence $r_2a_2 + z_2a_2 = 0$. By continuing in this way, we conclude that the sum is direct. Therefore statement (1) follows.

Statement (2) follows from the fact that for each k, $a_kx_{k+1} \neq 0$ yet $a_nx_{k+1} = 0$ for each $n \geqq k + 1$.

COROLLARY 1.2. *Let R satisfy the ascending chain condition on principal left annihilators and let R have finite left dimension. Then each nil subring of R is left T-nilpotent.*

The proof is evident.

COROLLARY 1.3. *Let R satisfy the ascending chain condition on principal left annihilators and the ascending chain condition on right annihilators. Then each nil subring of R is left T-nilpotent.*

The proof is evident.

LEMMA 1.4. *Let R satisfy the descending chain condition on principal right annihilators. Then R is left T-nilpotent if and only if for each $x \in R$ there exists a positive integer $h(x)$ such that $xR^{h(x)} = 0$.*

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Proof. Suppose that there exists $x_1 \in R$ such that for each positive integer h, $x_1 R^h \neq 0$. Hence $x_1 R \neq 0$ and $\{\mathbf{r}(x_1 y): x_1 y \neq 0 \text{ and } y \in R\}$ has a minimal element, say $\mathbf{r}(x_1 x_2)$. We claim that $x_1 x_2 R \neq 0$. If $x_1 x_2 R = 0$, then $\mathbf{r}(x_1 x_2) = R$. However, in this case, the minimality of $\mathbf{r}(x_1 x_2)$ would contradict $x_1 R^2 \neq 0$. Thence $x_1 x_2 R \neq 0$ and $\{\mathbf{r}(x_1 x_2 y): x_1 x_2 y \neq 0 \text{ and } y \in R\}$ has a minimal element, say $\mathbf{r}(x_1 x_2 x_3)$. By continuing in this manner, we obtain a sequence $\{x_n\}$ such that $x_1 x_2 \ldots x_n \neq 0$ for each n. Whence R is not left T-nilpotent.

The proof in the opposite direction is obvious.

PROPOSITION 1.5. *Let R satisfy the ascending chain condition on left annihilators. Then a subring N of R is nilpotent if and only if it is left T-nilpotent.*

Proof. Suppose that N is left T-nilpotent. Since the ascending chain condition on left annihilators (equivalently the descending chain condition on right annihilators) is inherited by subrings, we have by Lemma 1.4 that for each $x \in N$ there exists a positive integer $h(x)$ such that $xN^{h(x)} = 0$. From the ascending chain condition on left annihilators, there exists an m such that $\mathbf{l}(N^m) = \mathbf{l}(N^{m+1}) = \ldots$. If $N^{m+1} \neq 0$, then there exists $x \in N$ such that $xN^m \neq 0$. However, $xN^{h(x)} = 0$. This contradicts $\mathbf{l}(N^m) = \mathbf{l}(N^{h(x)})$. Therefore $N^{m+1} = 0$ and N is nilpotent.

The proof in the opposite direction is obvious.

THEOREM 1.6. *Let R satisfy the ascending chain condition on left annihilators and let N be a nil subring of R which is not nilpotent. Then there exists a sequence $\{a_n\}$ of elements in N such that*

(1) $R^1 a_1 + R^1 a_2 + R^1 a_3 + \ldots$ *is a direct sum of non-zero left ideals, and*

(2) $\mathbf{r}(\{a_k, a_{k+1}, a_{k+2}, \ldots\}) \subset \mathbf{r}(\{a_{k+1}, a_{k+2}, \ldots\})$ *for each k.*

Proof. The result follows immediately from Theorem 1.1 and Proposition 1.5.

COROLLARY 1.7 (*Lanski*). *Let R satisfy the ascending chain condition on left annihilators and let R have finite left dimension. Then each nil subring of R is nilpotent.*

The proof is evident.

COROLLARY 1.8 (*Herstein-Small*). *Let R satisfy the ascending chain condition on both left and right annihilators. Then each nil subring of R is nilpotent.*

The proof is evident.

2. Essential nilpotency. An ideal L of R is said to be *essentially nilpotent* if it contains a nilpotent ideal N of R which is *essential* in L, i.e., N has non-zero intersection with each non-zero ideal of R which is contained in L. We notice that if an ideal K of R is contained in an essentially nilpotent ideal L, then K is essentially nilpotent. If N is a nilpotent ideal of R which is essential in L, then $N \cap K$ is a nilpotent ideal of R which is essential in K.

LEMMA 2.1. *Let L be a non-zero ideal of R and let $k \geqq 2$ be a fixed integer. If each non-zero ideal $J \subseteq L$ of R contains a non-zero nilpotent ideal whose kth power is zero, then L is essentially nilpotent.*

Proof. Let $\{N_\lambda: \lambda \in \Lambda\}$ be the collection of all the non-zero nilpotent ideals of R which are contained in L and whose kth power is zero. Let

$$\Omega = \{S \subseteq \Lambda: \textstyle\sum_{\lambda \in S} N_\lambda \text{ is direct}\}.$$

Then Ω is non-empty and inductive. Hence by Zorn's lemma there exists a maximal element T in Ω.

Consider $N = \sum_{\lambda \in T} \oplus N_\lambda$. Since each N_λ is a two-sided ideal of R, multiplication in N is componentwise. Thence $N^k = 0$. We claim that N is essential in L. If not, then there is a non-zero ideal $J \subseteq L$ of R such that $N \cap J = 0$. However there exists a non-zero $N_\lambda \subseteq J$ such that $N_\lambda{}^k = 0$. Thus $N + N_\lambda$ is direct and so the maximality of T is contradicted.

Let the *prime radical* of R, denoted by $B(R)$, be the intersection of all the prime ideals of R.

PROPOSITION 2.2. *Each non-zero ideal J of R which is contained in $B(R)$ contains a non-zero nilpotent ideal I such that $I^3 = 0$.*

Proof. From [**4**, p. 56, Proposition 1] we have that $B(R) = \{a \in R$: each sequence $a_0, a_1, a_2, \ldots$ with $a_0 = a$, $a_{n+1} \in a_nRa_n$ is ultimately zero$\}$. Let a be a non-zero element of J. If $aRa = 0$, then it can easily be shown that $(a)^3 = 0$. If $aRa \neq 0$, then there exists a non-zero $a_1 \in aRa \subseteq J$. If $a_1Ra_1 = 0$, then $(a_1)^3 = 0$. If $a_1Ra_1 \neq 0$, then we continue until ultimately we get a non-zero $a_{n+1} \in a_nRa_n \subseteq J$ such that $a_{n+1}Ra_{n+1} = 0$. Thence $(a_{n+1})^3 = 0$, and the proof is complete.

THEOREM 2.3. *Let R be an arbitrary ring. Then the prime radical $B(R)$ is essentially nilpotent.*

Proof. The result follows immediately from Lemma 2.1 and Proposition 2.2.

Remark. It follows from Proposition 2.2 and the proof of Lemma 2.1 that $B(R)$ contains a nilpotent ideal N such that $N^3 = 0$ and N is essential in $B(R)$. If R has an identity, then this is improved to $N^2 = 0$ and N is essential in $B(R)$.

THEOREM 2.4. *Let R satisfy the ascending chain condition on principal left annihilators. Then each nil ideal L of R is essentially nilpotent.*

Proof. We claim that each non-zero ideal $J \subseteq L$ of R contains a non-zero nilpotent ideal I of R such that $I^2 = 0$. Since $J \neq 0$, $\{\boldsymbol{l}(x): x \neq 0, x \in J\}$ has a maximal element, say $\boldsymbol{l}(a)$. If $a^2 \neq 0$, then the maximality of $\boldsymbol{l}(a)$ forces $\boldsymbol{l}(a) = \boldsymbol{l}(a^2)$. This is impossible since a is nilpotent. Hence $a^2 = 0$. Moreover, if $aRa \neq 0$, then there exists $r \in R$ such that $ara \neq 0$. Again $\boldsymbol{l}(a) = \boldsymbol{l}(ara)$. This is impossible since ar is nilpotent. Whence $aRa = 0$. It follows easily from $a^2 = 0$ and $aRa = 0$ that $(a)^2 = 0$. Therefore the result follows from Lemma 2.1.

Remark 1. It follows from the proofs of Lemma 2.1 and Theorem 2.4 that L contains a nilpotent ideal N such that $N^2 = 0$ and N is essential in L.

Remark 2. Theorem 2.4 can be obtained from Theorem 2.3 and a theorem of Gupta [**2**, Theorem 3] which states that if R satisfies the ascending chain condition on principal left annihilators, then each nil ideal is contained in $B(R)$.

PROPOSITION 2.5. *If an ideal L of R is left T-nilpotent, then L is contained in $B(R)$.*

Proof. Suppose that $L \not\subseteq B(R)$. Then there exists an $x_1 \in L - B(R)$. If $x_1L \subseteq B(R)$, then $x_1Rx_1 \subseteq x_1L \subseteq B(R)$. Since $B(R)$ is a semiprime ideal, we obtain $x_1 \in B(R)$. Thus $x_1L \not\subseteq B(R)$ and so there exists $x_2 \in L$ such that $x_1x_2 \in L - B(R)$. Again $x_1x_2L \not\subseteq B(R)$. Hence there exists $x_3 \in L$ such that $x_1x_2x_3 \in L - B(R)$. By continuing in this fashion we obtain $\{x_n\}$ in L such that $x_1x_2 \ldots x_n \neq 0$ for each n. This contradicts the left T-nilpotency of L. Therefore $L \subseteq B(R)$.

PROPOSITION 2.6. *If an ideal L of R is left T-nilpotent, then L is essentially nilpotent.*

Proof. Indeed L is essentially nilpotent since $L \subseteq B(R)$ (Proposition 2.5) and $B(R)$ is essentially nilpotent (Theorem 2.3).

Example 1. Essential nilpotency does not imply left T-nilpotency. If it did, then it would follow from Theorem 2.4 and Proposition 1.5 that each nil ideal is nilpotent in a ring which satisfies the ascending chain condition only on left annihilators. The following example of Sasiada (unpublished) shows that this is not the case. Let R be the ring generated over the integers by the elements $x_1, x_2, x_3, \ldots, x_n, \ldots$ subject to the conditions that $x_jx_i = 0$ for $j \geqq i$. Then R is nil and satisfies the ascending chain condition on left annihilators, yet is not nilpotent.

Note. It has been brought to my attention that by making use of different techniques, Shock [**6**] has recently obtained some beautiful new results on the nilpotency of nil subrings. Briefly, his technique has been to make use of elementwise characterizations of the prime radical and certain conditions which are equivalent to the prime radical being nilpotent.

Added in proof. We sketch the following proof of Proposition 1.5 which does not require Lemma 1.4.

Suppose that N is left T-nilpotent. Since R satisfies the ascending chain condition on left annihilators, there exists an m such that

$$\boldsymbol{l}(N^m) = \boldsymbol{l}(N^{m+1}) = \ldots .$$

If $N^{m+1} \neq 0$, then there exists $x_1 \in N$ such that $x_1N^m \neq 0$. Then $x_1N^{m+1} \neq 0$

and hence there exists $x_2 \in N$ such that $x_1x_2N^m \neq 0$. Then $x_1x_2N^{m+1} \neq 0$ and hence there exists $x_3 \in N$ such that $x_1x_2x_3N^m \neq 0$. By continuing in this fashion, we obtain $\{x_n\}$ in N such that $x_1x_2 \ldots x_n \neq 0$ for each n. This contradicts the left T-nilpotency of N. Therefore N is nilpotent.

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