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FUNCTIONALS ON REAL $C(S)$

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The maximal ideals in a commutative Banach algebra with identity have been elegantly characterized [**5**; **6**] as those subspaces of codimension one which do not contain invertible elements. Also, see [**1**]. For a function algebra A, a closed separating subalgebra with constants of the algebra of complex-valued continuous functions on the spectrum of A, a compact Hausdorff space, this characterization can be restated: Let F be a linear functional on A with the property:

(*) For each f in A there is a point s, which may depend on f, for which $F(f) = f(s)$.

Then there is a fixed point s_0 with $F(f) = f(s_0)$ for all f in A.

For the space of real-valued continuous functions on a compact Hausdorff space S, property (*) does not generally characterize the multiplicative linear functionals. For example, the functional

$$F(f) = \int_0^1 f(x)dx, \quad S = [0, 1],$$

has property (*) [**6**]. We are thereby led to characterize exactly those linear functionals which satisfy (*) on the space of real-valued continuous functions on S. We additionally consider a condition which is suggested by (*) in which the value $F(f)$ of the functional is related to the values of f at two points.

In what follows S will be a compact Hausdorff space and $C(S)$ the supremum norm Banach space of real-valued continuous functions on S. For a continuous linear functional F on $C(S)$ there is a unique associated Borel measure μ, with variation norm $|\mu| = ||F||$, $F(f) = \int f d\mu$, and with support $\sigma(\mu)$ [**3**].

THEOREM 1. *Let F be a linear functional on the real Banach space $C(S)$. Then F satisfies (*) if and only if F is a positive linear functional of norm one with the support of the associated measure contained in a connected set.*

Proof. If $\sigma(\mu)$ is contained in a connected set C, then

$$\inf \{f(s) : s \text{ in } C\} \leqq \int f d\mu \leqq \sup \{f(s) : s \text{ in } C\}.$$

Since $f(C)$ is connected, there is a point s in C with

$$f(s) = \int f d\mu = F(f).$$

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Conversely suppose that F has property (*). It is clear that F is a positive linear functional with $||F|| = F(1) = 1$. Assume that $\sigma(\mu)$, for the associated positive measure μ, is not contained in a connected set. Then there are points x and y in $\sigma(\mu)$ with disjoint connected components C_x and C_y. Recall the fact, which will frequently be useful, that a component in a compact space is the intersection of all closed and open, i.e. clopen, sets which contain it [**4**, p. 246; **2**, p. 251]. Since C_x and C_y are compact, there is a clopen set U containing C_x with the complement U^c containing C_y. The argument used to see this is a version of the standard proof of the normality of a compact Hausdorff space in which clopen sets are used to separate points in C_x and C_y. Since F satisfies (*), the values of F on the characteristic functions of the sets U and U^c, $F(\chi_U)$ and $F(\chi_{U^c})$ must be either zero or one, and as $1 = F(\chi_U) + F(\chi_{U^c})$ one of the values must be zero. Then either $\mu(U) = 0$ or $\mu(U^c) = 0$, which contradicts both x and y belonging to the support of μ.

Thinking about property (*) suggests that we consider functionals F for which $F(f) = af(s) + bf(t)$. It is too strong to let all of a, b, s, and t vary with f; for if F is any continuous linear functional, $||F|| \leqq 1$, then, as

$$f(s_0) = \inf\{f(s) : s \text{ in } S\} \leqq F(f) \leqq \sup\{f(s) : s \text{ in } S\} = f(t_0),$$

$F(f)$ is some convex combination of $f(s_0)$ and $f(t_0)$. It is too easy to fix $s = s_0$ and $t = t_0$ and let a and b vary; for then, as whenever $f(s_0) = f(t_0) = 0$, $F(f) = 0$, we must have F a linear combination of the evaluations at s_0 and at t_0 [**3**, p. 421]. The interesting problem involves those linear functionals F satisfying:

(**) Let a and b be fixed. For each f there are points s and t, which may depend on f, with $s \neq t$ and $F(f) = af(s) + bf(t)$.

The condition $s \neq t$ keeps (*) and (**) distinct.

The characterization of functionals satisfying (**) will depend on relations between a and b. The following division is necessary.

(+) $F(f) = af(s) + bf(t)$, with $a \geqq b > 0$ and $a + b = 1$,
(−) $F(f) = af(s) + bf(t)$, with $a > 0, b < 0, a + b > 0$, and $a - b = 1$,
(0) $F(f) = f(s) - f(t)$.

Any other values for a and b can be reduced to one of these three cases by dividing F by a suitable scalar.

LEMMA 1. *If $\{U_1, U_2, U_3\}$ is a partition of S into three clopen sets and F, with associated measure μ, satisfies (**), then $|\mu|(U_i) = 0$ for at least one of $i = 1, 2, 3$.*

Proof. Let χ_{U_j} be the characteristic function of U_j, and let $\alpha_1, \alpha_2, \alpha_3$ be in $\mathbf{R}$. Then $\varphi(\alpha_1, \alpha_2, \alpha_3) = F(\sum_j \alpha_j\chi_{U_j}) = \sum_j \alpha_j\mu_j$, where $\mu_j = \mu(U_j)$, and so φ is a continuous function of $(\alpha_1, \alpha_2, \alpha_3)$. Now for a fixed $(\alpha_1', \alpha_2', \alpha_3')$ and renumbering the U's, if necessary, $\varphi(\alpha_1', \alpha_2', \alpha_3') = a(\sum_j \alpha_j'\chi_{U_j}(s)) +$

$b(\sum_j \alpha_j'\chi_{U_j}(t)) = a\alpha_1' + b\alpha_2'$ or $(a+b)\alpha_1'$. Thus $\alpha_1'\mu_1 + \alpha_2'\mu_2 + \alpha_3'\mu_3 = \alpha_1'a + \alpha_2'b$ or $(a+b)\alpha_1'$ and by continuity of the left hand side of this equation, either $\mu_1 = a$, $\mu_2 = b$, $\mu_3 = 0$ or $\mu_1 = a+b$, $\mu_2 = \mu_3 = 0$. Suppose $|\mu|(U_3) \neq 0$. Choose $h \in C(S)$ with support in U_3 such that $F(h) \neq 0$ and $\|h\| \leqq 1$. Consider $g = k\chi_{U_3} + h$ where k is in $\mathbf{R}$, $k > 2(|a|+|b|)|a+b|^{-1}$. Then for $s, t, s', t' \in S$, $ah(s) + bh(t) = F(h) = F(g) = k(a+b) + ah(s') + bh(t')$. From $k|a+b| = |a(h(s) - h(s')) + b(h(t) - h(t'))| \leqq 2(|a|+|b|)$, we obtain a contradiction.

LEMMA 2. *Let F be a linear functional, on the real Banach space $C(S)$, that is given by a point mass at x.*

1. *If condition $(+)$ holds, then F satisfies (**) if and only if x is not a G_δ.*
2. *If condition (0) holds, then F cannot satisfy (**).*
3. *If condition $(-)$ holds, then F satisfies (**) if and only if one of the following hold:*
   i) *The point x is not a G_δ.*
   ii) *The point $x \neq C_x$, the component of x.*

Proof. Suppose condition $(+)$ holds. If x is not a G_δ, then for any f there is a point $t \neq x$ with $f(t) = f(x)$; thus $F(f) = af(x) + bf(t)$ with $t \neq x$. Conversely, if x is a G_δ, there is a continuous f, $0 \leqq f \leqq 1$, with $f^{-1}(0) = \{x\}$ [**2**, p. 248]; then $F(f) = f(x) = 0 \neq af(s) + bf(t)$ for any two points s and t.

Suppose condition (0) holds. F cannot satisfy (**), for $F(1) \neq 0$.

Suppose condition $(-)$ holds. If x is not a G_δ, then (**) follows as above. If $\{x\} \neq C_x$, then for $F(f) = 0$, the only difficulty occurs when $f(y) \neq 0$ for $y \neq x$. In this case $f(C_x)$ is a nondegenerate interval containing zero. For any non-zero $f(y)$ in $f(C_x)$, $(-b/a)f(y)$ also belongs to $f(C_x)$, i.e., $af(y) + bf(t) = 0 = F(f)$ for some $t \neq y$. Conversely suppose that neither i) nor ii) hold; F is a point mass at x, x is a G_δ, and $\{x\} = C_x$. Because C_x is the intersection of all the clopen sets which contain x and because x is a G_δ, there is a countable nested collection $U_1 \supseteq U_2 \supseteq U_3 \supseteq \ldots$ of clopen sets with $\bigcap U_n = \{x\}$. If $-b/a$ is rational, consider $f = \sum (1/n^2)\chi U_n - \pi^2/6$. For this f, $F(f) = f(x) = 0$ and $f(y) \neq 0$ for $y \neq x$. For any $y \neq x$, y does not belong to U_n for large n and so $f(y) = r - \pi^2/6$, r a rational number. Thus we cannot have $af(y) + bf(z) = 0$ else π^2 would be rational. In the event that $-a/b$ is irrational the function $\sum (1/2^n)\chi_{U_n} - 1$ shows similarly that (**) cannot hold.

THEOREM 2. *Let F be a linear functional, with associated measure μ, on the real Banach space $C(S)$, and suppose that F is not a point mass. If F satisfies (**), then when condition $(+)$ holds F must be a positive linear functional of norm 1; and when condition $(-)$ holds F must be a continuous linear functional with $\|F\| \leqq 1$ and $F(1) = a + b$. In either case, F will satisfy (**) if and only if, in addition, one of the following holds:*

1. *The support of μ is contained in a connected set,*
2. *The support $\sigma(\mu) \subseteq C_1 \cup C_2$, the union of two disjoint connected sets, with $\mu(C_1) = a$ and $\mu(C_2) = b$.*

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Proof. First suppose that condition $(+)$ holds. If F satisfies (**) with a and b positive and $a + b = 1$ then F is a positive linear functional with $||F|| = F(1) = 1$.

Suppose that the measure μ associated with F has support $\sigma(\mu)$ contained in a connected set C. Because F is not a point mass, for x in σ there is an open neighborhood U of x with $0 < \mu(U) < 1$. We have, by Theorem 1,

$$\int_{U \cap C} f d\mu = f(c_1)\mu(U), \quad c_1 \text{ in } C, \text{ and}$$

$$\int_{U^c \cap C} f d\mu = f(c_2)\mu(U^c), \quad c_2 \text{ in } C.$$

Thus $F(f) = \mu(U)f(c_1) + (1 - \mu(U))f(c_2)$ is a point on the line joining $f(c_1)$ to $f(c_2)$. As $f(C)$ is an interval, if $f(c_1) \neq f(c_2)$ there are points s and t in C with $F(f) = af(s) + bf(t)$. If $f(c_1) = f(c_2)$, then we have $F(f) = (1/\mu(U)) \int_U f d\mu$. If this fails to hold for any neighborhood of x of measure less than one, then we can write F in the desired form. On the other hand, if this holds for every such neighborhood of x then, by the regularity of μ, $F(f) = f(x)$. A similar argument applied to a point $y \neq x$ in $\sigma(\mu)$ shows that we are done unless we also have $F(f) = f(y)$. But in this final case, $F(f) = f(x) = f(y) = af(x) + bf(y)$.

If the condition of 2 holds, then (**) follows directly from Theorem 1.

Suppose that F satisfies (**) and that $\sigma(\mu)$ is not contained in a connected set. Assume that there are three points x, y, and z in σ with disjoint components C_x, C_y and C_z. As in Theorem 1, there is a clopen partition of S, U_x, U_y, U_z, with $C_x \subseteq U_x$, $C_y \subseteq U_y$ and $C_z \subseteq U_z$. By Lemma 1, the measure of one of U_x, U_y, U_z must be zero, which contradicts the corresponding point being in the support of μ. So, say $\sigma \subseteq C_x \cup C_y$. From (**), the only possible values for $F(\chi_{U_x})$ and $F(\chi_{U_y})$ are $0, a, b$, and 1. Since $F(1) = F(\chi_{U_x}) + F(\chi_{U_y})$, 2 follows.

Second, suppose that condition $(-)$ holds. (In this case the measure μ is not necessarily a positive measure. This creates technical problems not present under condition $(+)$.)

If F satisfies (**), then F is bounded with $||F|| \leqq a - b = 1$ and $F(1) = a + b$.

It suffices to show that (**) holds for g in the null manifold of F, since for any f, $g = f - (1/(a + b))F(f)$ is in the null manifold, and if $F(g) = 0 = ag(s) + bg(t)$, $s \neq t$, then $F(f) = af(s) + bf(t)$.

Suppose that $\sigma(\mu)$ is contained in a connected set C. Let f be given with $F(f) = 0$ and define g on $C \times C$ by $g(s, t) = af(s) + bf(t)$. Set $m = \inf \{f(s) : s \text{ in } C\}$ and $M = \sup \{f(s): s \text{ in } C\}$. Let $\mu = \mu_1 - \mu_2$ be the Hahn decomposition of μ into the difference of two positive measures with $|\mu_1| + |\mu_2| = |\mu| = ||F|| \leqq a - b = 1$, and note that $|\mu_1| - |\mu_2| = F(1) = a + b$, and so $|\mu_1| \leqq a$ and $|\mu_2| \leqq -b$. For s and t in C, $am + bM \leqq g(s, t) \leqq aM + bm$. Also $am + bM \leqq am + bM + (m - M)(|\mu_1| - a) = m|\mu_1| - M|\mu_2| \leqq$

$\int f d\mu_1 - \int f d\mu_2 \leqq M|\mu_1| - m|\mu_2| = aM + bm + (M - m)(|\mu_2| + b) \leqq aM + bm$.

Continuity of g on the connected set $C \times C$ yields s and t in C with $0 = F(f) = \int f d\mu_1 - \int f d\mu_2 = g(s, t) = af(s) + bf(t)$. If the points s and t are distinct, $F(f)$ satisfies (**). If the points s and t are not distinct, then $f(s) = 0$. If there is a point $u \neq s$ with $f(u) = 0$, then $F(f) = af(s) + bf(u)$; if there is no such u then $f(C)$ is a nondegenerate interval which contains zero and an argument as in the first part of this proof establishes (**) .

If condition 2 holds, then $a - b \geqq |\mu| = |\mu|(C_1) + |\mu|(C_2) \geqq |\mu(C_1)| + |\mu(C_2)| = a - b$, from which it follows that μ is a positive measure on C_1 and a negative measure on C_2. That is to say that μ_1 is the restriction of μ to C_1 and μ_2 is the restriction of $-\mu$ to C_2. From Theorem 1 (**) follows.

It remains to show that if F satisfies (**) and μ does not have support contained in a connected set, then condition 2 holds.

Suppose that F satisfies (**) and the measure μ does not have support contained in a connected set. Assume that x, y, and z are three points in the support of μ which belong to disjoint components C_x, C_y, and C_z. As above there is a clopen partition U_x, U_y, and U_z of S with $U_x \supseteq C_x$, $U_y \supseteq C_y$, and $U_z \supseteq C_z$. For any clopen set U, $F(\chi_U)$ must be one of the numbers 0, a, b, or $a + b$ by (**). By Lemma 1 one of sets U_x, U_y, and U_z must have variation zero, contrary to the assumption that all of the points belonged to the support of μ. So it must be that, say $\sigma(\mu) \subseteq C_x \cup C_y$; with $\mu(U_x) \neq 0 \neq \mu(U_y)$. As $a + b = \mu(U_x) + \mu(U_y)$, the restrictions on the values for the measures of the clopen sets show that, say $\mu(U_x) = a$ and $\mu(U_y) = b$. As above, since $|\mu| \leqq a - b$, we can conclude that u is positive on U_x and negative on U_y; so $\mu(C_x) = a$ and $\mu(C_y) = b$.

The last case, case (0), is quite distinctive as it has a different character on and off the real line.

THEOREM 3. *Let F be a linear functional on the real Banach space $C(S)$. Then F satisfies (**) in the case $a = 1$ and $b = -1$, i.e. for each f in $C(S)$ there are two distinct points s and t, which may depend on f, with $F(f) = f(s) - f(t)$, if and only if F is a bounded linear functional with $||F|| \leqq 2$ and $F(1) = 0$ and:*

I. *When S is not homeomorphic to a subset of the real line $\mathbf{R}$, then the additional conditions on the measure μ associated with F are either*

1. *The support $\sigma(\mu) \subseteq C_x \cup C_y$, the union of two disjoint components with $\mu(C_x) = 1$ and $\mu(C_y) = -1$, or*
2. *The μ-measure of each component is zero.*

II. *In the alternate situation where S is homeomorphic to a subset of $\mathbf{R}$, the additional conditions on μ are either*

1. *The same as* I.1 *above, or*
2. *Here the support $\sigma(\mu) \subseteq C$, C a component. The condition on μ may be phrased by identifying C with the unit interval $[0,1]$, to which it is homeomorphic.*

Then μ corresponds to a normalized function α of bounded variation on $[0, 1]$ *with* $F(f) = \int f d\mu = \int_0^1 f d\alpha$. *Such a functional F (with* $||F|| \leqq 2$ *and* $F(1) = 0$) *has the desired form if and only if either* $\alpha(x) \geqq 0$ *for all x in* $[0, 1]$ *or* $\alpha(x) \leqq 0$ *for all x in* $[0, 1]$.

Proof. From $F(f) = f(s) - f(t)$ we see that $||F|| \leqq 2$ and $F(1) = 0$.

Suppose that $\mu(C) \neq 0$ for some component C. By the regularity of μ there is a neighborhood V of C with $\mu(W - C) \leqq \mu(C)/2$ for $C \subseteq W \subseteq V$. By the usual separation argument, using the compactness of C and V^c and the fact that C is a component, there is a clopen set U between C and V, and so $\mu(U) \neq 0$. The only possible values for $F(\chi_U)$ are 0, $+1$, and -1, so $\mu(U) = 1$ or $\mu(U) = -1$. Because $F(1) = 0$, $\mu(U) = -\mu(U^c)$. Suppose then that $\mu(U) = 1$ and $\mu(U^c) = -1$. The norm of μ is bounded by two, so μ is positive on U and negative on U^c. Since μ of a clopen subset of U (or U^c) must be zero or one (zero or minus one), it follows that $\sigma(\mu) \cap U \subseteq C_x$ and $\sigma(\mu) \cap U^c \subseteq C_y$ for two disjoint components C_x and C_y, i.e. $\sigma(\mu) \subseteq C_x \cup C_y$ with $\mu(C_x) = 1$ and $\mu(C_y) = -1$. Conversely, if F has this form, then $F(f) = \int C_x f d\mu + \int C_y f d\mu = f(s) - f(t)$, with s in C_x and t in C_y, by Theorem 1.

It remains to consider μ with the property that the measure of each component is zero. For the collection $\{C_\beta\}$ of disjoint components of S, $||\mu|| \geqq \sum|\mu|(C_\beta)$, so there are only countably many components $C_1, C_2, \ldots$ with $|\mu|(C_i) \neq 0$; and $||\mu|| = \sum |\mu|(C_i)$. For f in $C(S)$, $\sum_1^n f\chi_{C_i}$ converges to f μ — a.e. Thus, given f and $\epsilon > 0$, there is an N with

$$\left| F(f) - \sum_1^N \int_{C_i} f d\mu \right| \leqq \epsilon.$$

Using the Hahn decomposition $\mu = \mu_1 - \mu_2$ for μ, $0 = \mu(C_i) = \mu_1(C_i) - \mu_2(C_i)$, so $\mu_1(C_i) = \mu_2(C_i) = |\mu|(C_i)/2$. Using Theorem 1,

$$\int_{C_i} f d\mu = \int_{C_i} f d\mu_1 - \int_{C_i} f d\mu_2 = \mu_1(C_i) f(s) - \mu_2(C_i) f(t),$$

with s and t in C_i. Then

$$\int_{C_i} f d\mu = (|\mu|(C_i)/2)(f(s) - f(t)) = (|\mu|(C_i)/2)(\xi_i);$$

ξ_i belonging to the interval $I_i = [m_i - M_i, M_i - m_i]$, where $m_i = \inf \{f(s) : s \text{ in } C_i\}$ and $M_i = \sup \{f(s) : s \text{ in } C_i\}$. Let j be chosen so that $I_j \supseteq I_i$ for $1 \leqq i \leqq N$. Noting that 0 belongs to I_j, $\sum^N (|\mu|(C_i)/2)(\xi_i)$ is a convex combination of points from I_j as $\sum_1^N |\mu|(C_i)/2 \leqq ||\mu||/2 = ||F||/2 \leqq 1$. Thus the sum $\sum_1^N (\xi_i)|\mu|(C_i)/2$ belongs to I_j and so by Theorem 2 can be written in the form $f(s) - f(t)$ for s and t in the connected set C_j. Finally for $\epsilon = 1/n$, there are points $\{s_n\}$ and $\{t_n\}$ with $|F(f) - (f(s_n) - f(t_n))| \leqq 1/n$. If s_0 is a cluster point of $\{s_n\}$ and t_0 a cluster point of $\{t_n\}$, then $F(f) = f(s_0) - f(t_0)$. If $F(f) \neq 0$, then the s and t so obtained are distinct. In general they may not be distinct if $F(f) = 0$. If $C(S)$ contains no one-to-one functions, then for any f, in particular for f with $F(f) = 0$, there are distinct points s and t with $f(s) - f(t) = 0 = F(f)$. Under these circumstances, if the measure of every component is zero, then F satisfies (**).

The case which remains is that in which $C(S)$ contains a one-to-one function, i.e., as S is compact, where S is homeomorphic to a compact subset of R. And the only measures μ of interest are those which take every component to zero. One distinguishing feature of the real line situation is that if F satisfies (**), there cannot be more than one component C with $\mu(C) = 0$ and $|\mu|(C) \neq 0$. To see this we will first show that if U is clopen and $\mu(U) = 0$, then either $|\mu|(U) = 0$ or $|\mu|(U^c) = 0$. Suppose not. Then, for $S \subseteq [c_1, c_2]$, consider $F((x - c_1)^n \chi_U)$. If this were always zero, then $F(P(x)\chi_U)$ would be zero for each polynomial P and, consequently, $|\mu|(U) = 0$. So for some n, $F((x - c_1)^n \chi_U) \neq 0$, that is to say there is a one-to-one function h_1 with $F(h_1\chi_U) \neq 0$. By symmetry there is a one-to-one function h_2 with $F(h_2\chi_{U^c}) \neq 0$. Let

$$g = (a_1 + b_1h_1)\chi_U + (a_2 + b_2h_2)\chi_{U^c},$$

where a_1, a_2, b_1, and b_2 will be chosen shortly. By hypothesis, $F(\chi_U) = 0 = F(\chi_{U^c})$, thus $F(g) = b_1F(h_1\chi_U) + b_2F(h_2\chi_{U^c})$. Since neither $F(h_1\chi_U)$ nor $F(h_2\chi_{U^c})$ are zero, there are non-zero scalars b_1 and b_2 for which $F(g) = 0$; let b_1 and b_2 be so chosen. Choosing a_1 large and positive and a_2 large and negative makes g one-to-one. We then have $F(g) = 0$ but cannot have $g(s) - g(t) = 0$ for $s \neq t$, contradicting property (**) for F. Suppose that x and y belong to different components and to the support of μ. Then we can find clopen disjoint neighborhoods U_x and U_y. The measure of the clopen set U_x must be 0, 1, or -1. It cannot be zero, for then, by what we have just shown, either $|\mu|(U_x) = 0$, and x is not in the support of μ, or $|\mu|(U_y) = 0$ and y is not in the support of μ. Thus, say, $\mu(U_x) = 1$ and $\mu(U_y) = -1$. This leads to $\mu(C_x) = 1$ and $\mu(C_y) = -1$, as in the first part of this proof; a case which we have already handled and therefore have excluded, being now interested only in those measures which are zero on each component. So we see that such a measure must have support in a single component C which, by identification via homeomorphism we may take to be the closed interval $[0, 1]$.

In the final case remaining we then have a linear functional F on the real valued continuous functions on $[0, 1]$ and we want to know under what conditions F can, for each f, be written in the form $F(f) = f(s) - f(t)$ for distinct s and t. Of course, as before, we have $||F|| \leqq 2$ and $F(1) = 0$. Given F there is a normalized function α of bounded variation on $[0, 1]$ with $F(f) = \int_0^1 f d\alpha$ [**7**]. In a previous part of the proof we have seen that if $F(f) \neq 0$, then $F(f) = f(s) - f(t)$ for some s and t, which are necessarily distinct. Thus F has the property (**) if and only if its null manifold $N(F)$ contains no one-to-one function. We will show that this holds if and only if either α is non-positive on $[0, 1]$ or non-negative on $[0, 1]$.

Suppose that $\alpha(x) \geqq 0$ for x in $[0, 1]$. A one-to-one function f on $[0, 1]$ is either increasing or decreasing; by considering f or $-f$ we may suppose that it is increasing. Let $\alpha(x) = \alpha_1(x) - \alpha_2(x)$, the difference of two normalized monotone functions. The left-continuity of α guarantees that there is an interval $(c, d]$ on which α is strictly positive, and so $\alpha_1(x) > \alpha_2(x)$ there. Then

$$F(f) = \int_0^1 f d\alpha_1 - \int_0^1 f d\alpha_2 = \int_0^1 \alpha_2 df - \int_0^1 \alpha_1 df,$$

after an integration by parts, using the information that, by the normalization, $\alpha(0) = \alpha_1(0) = \alpha_2(0) = 0$, and $F(1) = \alpha(1) = \alpha_1(1) - \alpha_2(1) = 0$. Then

$$F(f) = \int_0^1 \alpha_2 df - \int_0^1 \alpha_1 df \leqq \int_c^d (\alpha_2 - \alpha_1) df < 0.$$

Hence $N(F)$ contains no one-to-one function. And similarly if $\alpha(x) \leqq 0$ for x in $[0, 1]$.

To complete the proof suppose that F has property (**) on $C[0, 1]$. Let h be strictly positive and continuous and set $f(x) = \int_0^x h(t)dt$. Integrating by parts,

$$F(f) = \int_0^1 f d\alpha = -\int_0^1 \alpha df = -\int_0^1 \alpha(t)h(t)dt.$$

The functional value $F(f)$ cannot be zero as f is one-to-one. More is true. We cannot have $F(f_1) < 0$ and $F(f_2) > 0$ for two such functions f_1 and f_2; else $F(cf_1 + (1 - c)f_2) = 0$ for some $0 < c < 1$ and a one-to-one function $cf_1 + (1 - c)f_2$. So, say, $F(f) \leqq 0$ for all f's so given by strictly positive h's. Then the map $G(g) = \int_0^1 \alpha(t)g(t)dt$ is a positive linear functional on $C[0, 1]$. Consequently the measure $\alpha(t)dt$ is a positive measure and $\alpha(t) \geqq 0$ for all t except perhaps those in a set of Lebesgue measure zero. Because α is continuous from the left, $\alpha(t) \geqq 0$ for all t in $[0, 1]$. If $F(f) \geqq 0$ for all f of the type described, then $\alpha(t) \leqq 0$ for all t in $[0, 1]$.

There are many variations and generalizations of our considerations which lead to interesting problems in analysis. We mention characterizing those F on real $C(S)$ which satisfy, for fixed $a_1, \ldots, a_n$, $F(f) = \sum a_i f(s_i)$ for distinct points $s_1, \ldots, s_n$ which may vary with f, and characterizing those F satisfying (**) on complex $C(S)$ or on a given function algebra.

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