A NOTE ON APPROXIMATION OF DISTRIBUTIONS BY QUASI-ANALYTIC FUNCTIONS

S. R. HARASYMIV

(Received 28 August 1967, revised 3 March 1968)

1. Introduction and notation

Throughout this note R^n denotes the n-dimensional Euclidean space. Addition and multiplication in R^n are defined component-wise. If $k \leqq n$ is a positive integer and $x \in R^n$, we write x_k for the k-th component of x. The set $\{x \in R^n : x_k \neq 0 \text{ for each } k \leqq n\}$ is designated by $R^{\#}$.

We shall use the standard notations of the calculus of n variables; see, for example, Hörmander [5], p. 4. If α is a multi-index, then j^α is the function on R^n defined by $j^\alpha(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for all $x \in R^n$.

Suppose that W is an open subset of R^n. We write $\boldsymbol{D}(W)$ for the space of functions which are indefinitely differentiable and have compact support in W; and the space of distributions with support in W is denoted by $\boldsymbol{D}'(W)$. The spaces of rapidly decreasing indefinitely differentiable functions and temperate distributions on R^n are denoted by $\boldsymbol{S}(R^n)$ and $\boldsymbol{S}'(R^n)$, respectively. In what follows, $\boldsymbol{S}'(R^n)$ is always assumed to have the strong topology $\beta(\boldsymbol{S}', \boldsymbol{S})$.

Finally, let $\phi \in \boldsymbol{D}(R^n)$. If $b \in R^n$, then the function $\phi_b \in \boldsymbol{D}(R^n)$ defined by

$$\phi_b(x) = \phi(x+b) \quad \text{for all} \quad x \in R^n$$

is called a *translate* of ϕ. If $a \in R^{\#}$, then the function $\phi^a \in \boldsymbol{D}(R^n)$ defined by

$$\phi^a(x) = \phi(ax) \quad \text{for all} \quad x \in R^n$$

is called a *dilation* of ϕ. The *translate* u_b and *dilation* u^a of an arbitrary distribution $u \in \boldsymbol{D}'(R^n)$ are defined via the adjoints of the mappings $\phi \to \phi_b$ and $\phi \to \phi^{a^{-1}}$; we write $u_b(\phi) = u(\phi_b)$ and $u^a(\phi) = |1/j(a)| \cdot u(\phi^{a^{-1}})$ for all $\phi \in \boldsymbol{D}(R^n)$. A vector subspace F of $\boldsymbol{D}'(R^n)$ is said to be *dilation-invariant* [resp. *translation-invariant*] if $u^a \in F [u_b \in F]$ for all $u \in F$ and all $a \in R^{\#} [b \in R^n]$.

In Harasymiv [3], the following problem was considered: if E is a dilation-invariant and translation-invariant locally convex space of temperate distributions and $u \in E$, what is the closed vector subspace $T[u]$ of E generated by the set of distributions $\{(u_b)^a : a \in R^{\#}, b \in R^n\}$. It was

https://doi.org/10.1017/S1446788700006923 Published online by Cambridge University Press

shown that if we make certain assumptions about the topology on E, then $T[u]$ coincides with the whole of E provided that the support of the Fourier transform of u is sufficiently 'thick'. Moreover, it was found that we could replace the parameter sets $R^{\#}$ and R^n by a dense subset A of $R^{\#}$ and a dense subset B of R^n without altering the conclusions in [3]. In this note we remark on a condition which allows us to restrict still further the size of the parameter sets A and B.

2. Preliminaries

In this section we derive some results which we shall need to prove the approximation theorem in § 3. Throughout, the term *space of temperate distributions* will mean a vector subspace of $\mathbf{S}'(R^n)$ which contains $\mathbf{S}(R^n)$. We begin with two definitions.

2.1. DEFINITION. *A locally convex space E of temperate distributions is said to be an* admissible space *of it satisfies the following two conditions.*

(i) *$\mathbf{S}(R^n)$ is dense in E.*

(ii) *The injections $\mathbf{S}(R^n) \to E \to \mathbf{S}'(R^n)$ are continuous.*

REMARK. It is very easy to verify that the topological dual space E' of an admissible space E can be identified with a space of temperate distributions in such a way that

$$\langle u, \phi\rangle = u * \phi(0) \quad \text{for all} \quad u \in E \tag{2.1}$$

$$\langle \phi, v\rangle = \phi * v(0) \quad \text{for all} \quad v \in E' \tag{2.2}$$

whenever $\phi \in \mathbf{S}(R^n)$. [If E is an admissible space, then the symbol $\langle , \rangle$ will always denote the bilinear form on $E \times E'$ induced by the natural pairing of E and E'.]

2.2 DEFINITION. *Suppose that E is an admissible space. We say that E is c-*admissible *if it satisfies conditions* (i)-(iii) *below.*

(i) *E is translation-invariant.*

(ii) *For each $x \in R^n$, the mapping $u \to u_x$ of E (with its usual topology) into E (with the weak topology $\sigma(E, E')$) is continuous.*

(iii) *For each $u \in E$ and each $v \in E'$, the mapping $x \to \langle u_x, v\rangle$ defines a continuous function which is a temperate distribution on R^n.*

A c-admissible space which satisfies conditions (iv)-(vi) *below is called a* dilation space.

(iv) *E is dilation-invariant.*

(v) *For each $x \in R^{\#}$, the mapping $u \to u^x$ of E (with its usual topology) into E (with the weak topology $\sigma(E, E')$) is continuous.*

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(vi) *For each* $u \in E$, *the mapping* $x \to u^x$ *of* $R^{\#}$ *into* E *is continuous for the* $\sigma(E, E')$ *topology on* E.

REMARK. Suppose that E is a translation-invariant barrelled admissible space such that for each $x \in R^n$, the mapping $u \to u_x$ of E (with its usual topology) into E (with the weak topology $\sigma(E, E')$) is continuous, and for each $u \in E$ the mapping $x \to u_x$ of R^n into E is continuous for the weak topology on E. In this case it can be shown that if for each $u \in E$ and each $v \in E'$ the convolution $u * v$ is defined (in the general sense of Chevalley [1]) and is a temperate distribution on R^n, then E is c-admissible.

Assume that E is a c-admissible space and that $u \in E$ and $v \in E'$. In what follows, we shall use the symbol $u \circledast v$ to denote the temperate distribution on R^n generated by the function $x \to \langle u_x, v\rangle$ $(x \in R^n)$, as in condition (iii) of Definition 2.2. If we consider $u \circledast v$ as a function, then in view of Theorem 2.2(a) in Harasymiv [3], we have

$$u \circledast v(x) = \langle u_x, v\rangle = \langle u, v_x\rangle \quad \text{for all} \quad x \in R^n.$$

If E is a dilation space, then we shall write $u \triangledown v$ for the function on $R^{\#}$ defined by $u \triangledown v(x) = \langle u^x, v\rangle (x \in R^{\#})$. By condition (vi) in Definition 2.2, $u \triangledown v$ is continuous on $R^{\#}$; and by Theorem 2.2(b) in Harasymiv [3], $u \triangledown v(x) = |1/j(x)| \cdot \langle u, v^{x^{-1}}\rangle$ for all $x \in R^{\#}$.

We now list several results about dilation spaces which we shall need in what follows.

2.3 LEMMA. (a) *Suppose that* E *is a barrelled c-admissible space and that* M *is a weakly bounded subset of* E. *Then for each* $v \in E'$ *and each compact set* $K \subset R^n$, *there exists a positive constant* m *(depending on* v *and* K*) such that*

$$|u \circledast v(x)| \leqq m \quad \textit{for all} \quad x \in K$$

simultaneously for all $u \in M$.

(b) *Suppose that* E *is a barrelled dilation space and that* M *is a weakly bounded subset of* E. *Then for each* $v \in E'$ *and each compact set* $K \subset R^{\#}$ *there exists a positive constant* M *(depending on* v *and* K*) such that*

$$|u \triangledown v(x)| \leqq m' \quad \textit{for all} \quad x \in K$$

simultaneously for all $u \in M$.

PROOF. We shall restrict ourselves to establishing (b); a very similar argument will prove (a). Thus, assume that E is a barrelled dilation space, $v \in E'$ and that K is a compact subset of $R^{\#}$. The continuity of the mapping $x \to v^{x^{-1}}$ of $R^{\#}$ into E' (for the weak topology on E') entails that the set $\{v^{x^{-1}} : x \in K\}$ is a weakly compact, and hence weakly bounded subset of E'. Theorem 7.1.1(b) in Edwards [2] now tells us that the set $\{v^{x^{-1}} : x \in K\}$ is

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equicontinuous, and so this set is uniformly bounded on each bounded subset of E. Since any weakly bounded subset of a locally convex topological vector space is necessarily bounded (Edwards [2], Theorem 8.2.2), we infer the existence of a constant $m > 0$ such that

$$(2.3) \qquad |\langle u, v^{x^{-1}}\rangle| \leqq m \quad \text{for all} \quad u \in M \quad \text{and all} \quad x \in K.$$

In view of the definition of $u \triangledown v$, relation (2.3) is easily seen to lead to the desired boundedness property.

In order to abbreviate the stements of the results below, we introduce the following terminology.

2.4. DEFINITION. *Let E be an admissible space, and suppose that F is an algebraic subspace of E which is admissible relative to some topology such that the injection $F \to E$ is continuous. We then say that F is a* subspace of type (Γ) *if for each $u \in F$ and each pair of multi-indices α and β such that $\beta \leqq \alpha$, we have $j^\beta D^\alpha u \in F$ and the following condition is satisfied.*

(i) *For each pair of multi-indices α and β such that $\beta \leqq \alpha$, the mapping $u \to j^\beta D^\alpha u$ of F into F is continuous.*

REMARK. Obviously, each admissible space contains at least one subspace of type (Γ); namely, $\boldsymbol{S}(R^n)$.

2.5. LEMMA. *Suppose that E is a barrelled dilation space and that F is a subspace of E of type (Γ). Then the following assertions are true.*

(a) *For each $u \in F$ and each $v \in E'$, the function $u \circledast v$ is indefinitely differentialbe on R^n and for each multi-index α*

$$D^\alpha(u \circledast v)(x) = (D^\alpha u) \circledast v(x) \quad \textit{for all} \quad x \in R^n.$$

(b) *For each $u \in F$ and each $v \in E'$, the function $u \triangledown v$ is indefinitely differentiable on $R^\#$ and for each multi-index α*

$$D^\alpha(u \triangledown v)(x) = [1/j^\alpha(x)] \cdot \sum_{\beta \leqq \alpha} C^\alpha_\beta (j^\beta D^\beta u) \triangledown v(x) \quad \textit{for all} \quad x \in R^\#.$$

where $C^\alpha_\beta = \alpha!/\beta!(\alpha-\beta)!$.

PROOF. Once again we content ourselves with proving (b). The proof of (a) is similar but simpler.

Assume that u and v are as in part (b) in the statement of the lemma. It is evident that our proof will be complete if we succeed in showing that if W is an arbitrary relatively compact subset of R^n such that $\overline{W} \subset R^\#$, then $u \triangledown v$ is indefinitely differentiable in W and for each multi-index α

$$(2.4) \quad D^\alpha(u \triangledown v)(x) = [1/j^\alpha(x)] \sum_{\beta \leqq \alpha} C^\alpha_\beta (j^\beta D^\beta u) \triangledown v(x) \quad \text{for all} \quad x \in W.$$

Now, in view of Théorème VII in Chapitre II of Schwartz [7] and the continuity of the functions $(j^\beta D^\beta u) \bigtriangledown v$, it is easy to see that relation (2.4) is equivalent to the demand that $D^\alpha(u \bigtriangledown v)$ and $[1/j^\alpha] \sum_{\beta \leqq \alpha} C^\alpha_\beta (j^\beta D^\beta u] \bigtriangledown v$ should coincide as distributions on W. In other words, the validity of (2.4) will be assured if we show that for each $\psi \in \boldsymbol{D}(W)$

$$(2.5) \qquad \int_W u \bigtriangledown v(x) \cdot D^\alpha \psi(-x) dx = \int_W [1/j^\alpha(x)] \sum_{\beta \leqq \alpha} C^\alpha_\beta (j^\beta D^\beta u) \bigtriangledown v(x) \cdot \psi(-x) dx.$$

With this end in view, we argue as follows. Since F is admissible, we can extract a net (ϕ_i) from $\boldsymbol{D}(R^n)$ such that $\lim_i \phi_i = u$ in F. Then, by virtue of the continuity (for each multi-index β) of the mapping $w \to j^\beta D^\beta w$ of F into F, it is also true that $\lim_i j^\beta D^\beta \phi_i = j^\beta D^\beta u$ in F for each multi-index $\beta \geqq 0$. Since the topology on F is stronger than that induced by E (see Definition 2.4), this entails that for each multi-index β

$$(2.6) \qquad \lim_i j^\beta D^\beta \phi_i = j^\beta D^\beta u \quad \text{in} \quad E.$$

Next we notice that since $\overline{W}$ is compact, the set $\{v^{x^{-1}} : x \in \overline{W}\}$ is weakly compact, and hence weakly bounded in E'. This is a consequence of the continuity (for the weak topology on E') of the mapping $x \to v^{x^{-1}}$ of $R^{\#}$ into E'. Theorem 7.1.1(b) in Edwards [2] now tells us that the set $\{v^{x^{-1}} : x \in \overline{W}\}$ is equicontinuous. If we bear this fact in mind, then the remark on p. 504 (third paragraph) of Edwards [2], together with (2.6), leads us to the conclusion that for each multi-index β

$$\lim_i \langle j^\beta D^\beta \phi_i, v^{x^{-1}} \rangle = \langle j^\beta D^\beta u, v^{x^{-1}} \rangle \quad \text{uniformly for} \quad x \in \overline{W}.$$

In view of the definition of the functions $(j^\beta D^\beta u) \bigtriangledown v$, and the fact that j is bounded away from zero on $\overline{W}$, we may now assert that for each multi-index β

$$(2.7) \qquad \lim_i j^\beta D^\beta \phi_i \bigtriangledown v(x) = j^\beta D^\beta u \bigtriangledown v(x) \quad \text{uniformly for} \quad x \in \overline{W}.$$

It is now easy to verify that (2.5) holds. Consider an arbitrary function $\psi \in \boldsymbol{D}(W)$. Then, because of (2.7), we have

$$(2.8) \qquad \begin{aligned} \int_W u \bigtriangledown v(x) \cdot D^\alpha \psi(-x) dx &= \lim_i \int_W \phi_i \bigtriangledown v(x) \cdot D^\alpha \psi(-x) dx \\ &= \lim_i \int_W D^\alpha (\phi_i \bigtriangledown v)(x) \cdot \psi(-x) dx. \end{aligned}$$

Now, each ϕ_i belongs to $\boldsymbol{D}(R^n)$. Therefore, if we use relation (3.1) in Harasymiv [4], it is easily demonstrated that for each i

$$(2.9) \qquad D^\alpha(\phi_i \bigtriangledown v)(x) = [1/j^\alpha(x)] \sum_{\beta \leqq \alpha} C^\alpha_\beta (j^\beta D^\beta \phi_i) \bigtriangledown v(x) \quad \text{for all} \quad x \in \overline{W}$$

Relations (2.7), (2.8) and (2.9) together entail that

$$\int_W u \triangledown v(x) \cdot D^\alpha \psi(-x)dx$$
$$= \lim_i \int_W [1/j^\alpha(x)] \sum_{\beta \leqq \alpha} C^\alpha_\beta (j^\beta D^\beta \phi_i) \triangledown v(x) \cdot \psi(-x)dx$$
$$= \int_W [1/j^\alpha(x)] \sum_{\alpha \leqq \beta} C^\alpha_\beta (j^\beta D^\beta u) \triangledown v(x) \cdot \psi(-x)dx.$$

This establishes (2.5), which is what we set out to do.

REMARK. If E is a B_r-complete module over $\boldsymbol{S}(R^n)$ then in part (a) of Lemma 2.5 it is sufficient to merely assume that $u \in E$ is such that $D^\alpha u \in E$ for each multi-index α; the result still holds. However, we shall nowhere make use of this fact, and mention it only in passing.

2.6. DEFINITION. *Let E be an admissible space and $(a_k)_{k=1}^\infty$ a sequence of complex numbers. For each multi-index α, let $a_\alpha = a_{\alpha_1} \cdots a_{\alpha_n}$. We shall write $M(a_k)$ for the set of all $u \in E$ which have the following properties.*

(i) *If α and β are multi-indices such that $\beta \leqq \alpha$ then $j^\beta D^\alpha u \in E$.*

(ii) *The set $\{a_\alpha j^\beta D^\alpha u : \beta \leqq \alpha, |\alpha| = 1, 2, \cdots\}$ is weakly bounded in E.*

With the above notation, we state the following corollary to Lemmas 2.3 and 2.5.

2.7. LEMMA. *Suppose that E is a barrelled dilation space and that F is a subspace of E of type (Γ). Let $u \in F$ and assume that $(a_k)_{k=1}^\infty$ is a monotonic non-increasing sequence of positive numbers such that $u \in M(a_k)$. Then the following two assertions are true.*

(a) *For each $v \in E'$ and each compact set $K \subset R^n$, there exists a positive constant m (depending on v and K) such that*

$$|D^\alpha(u \circledast v)(x)| \leqq m/a_\alpha \quad \text{for all} \quad x \in K$$

simultaneously for all multi-indices $\alpha \geqq 0$.

(b) *For each $v \in E'$ and each compact set $K \subset R^{\#}$, there exist positive constants m' and ρ (both depending on v and K) such that*

$$|D^\alpha[(D^\gamma u) \triangledown v](x)| \leqq m' \cdot \rho^{|\alpha|}/a_{\gamma+\alpha} \quad \textit{for all} \quad x \in K$$

simultaneously for all multi-indices $\alpha \geqq 0$ and $\gamma \geqq 0$.

PROOF. The proofs of parts (a) and (b) of Lemma 2.7 are very similar; we shall only give the argument for part (b).

Suppose that $v \in E'$ and that K is a compact subset of $R^{\#}$. In view of the definition of $M(a_k)$ and Lemma 2.3(b), we infer that there exists a constant $m' > 0$ (depending on v and K) such that

$$(2.10) \qquad |(j^\beta D^{\gamma+\beta} u) \triangledown v(x)| \leqq m'/a_{\gamma+\beta} \quad \text{for all} \quad x \in K$$

simultaneously for all multi-indices β and γ. Now suppose that α and γ are arbitrary, but fixed, multi-indices. Since the sequence (a_k) is non-increasing, we deduce from (2.10) that

$$|(j^\beta D^{\gamma+\beta}u) \triangledown v(x)| \leqq m'/a_{\gamma+\alpha} \quad \text{for all} \quad x \in K \tag{2.11}$$

simultaneously for all multi-indices $\beta \leqq \alpha$. Write

$$\rho = 2 \sup \{|x_i| : x \in K,\ 1 \leqq i \leqq n\}.$$

Using Lemma 2.5 and relation (2.11), it is easy to verify that for each $x \in K$

$$\begin{aligned} |D^\alpha[(D^\gamma u) \triangledown v](x)| &\leqq |1/j^\alpha(x)| \sum_{\beta \leqq \alpha} C^\alpha_\beta |(j^\beta D^{\gamma+\beta}u) \triangledown v(x)| \\ &\leqq \rho^{|\alpha|} \cdot 2^{-|\alpha|} \cdot (m'/a_{\gamma+\alpha}) \cdot \sum_{\beta \leqq \alpha} C^\alpha_\beta \\ &\leqq m' \cdot \rho^{|\alpha|}/a_{\gamma+\alpha} \end{aligned}$$

since $\sum_{\beta \leqq \alpha} C^\alpha_\beta \leqq 2^{|\alpha|}$. This completes the proof of Lemma 2.7.

The following result is a straight-forward consequence of the theorem stated at the foot of p. 75 in Mandelbrojt [6]. We omit its proof.

2.8 LEMMA. *Suppose that $(a_k)_{k=1}^\infty$ is a monotonic non-increasing sequence of positive numbers such that the sequence $(a_k^{1/k})_{k=1}^\infty$ is also monotonic non-increasing. Moreover, suppose that the series $\sum_{k=1}^\infty a_k^{1/k}$ diverges. Let W be an open subset of R^n and suppose that f is a function which is indefinitely differentiable in W and has the following properties.*

(i) *For each compact subset K of W, there exist constants $m > 0$ and $\rho > 0$ (depending on K) such that*

$$|D^\alpha f(x)| < m \cdot \rho^{|\alpha|}/a_\alpha \quad \textit{for all} \quad x \in K$$

simultaneously for all multi-indices $\alpha \geqq 0$. [Here, as elsewhere, we write $a_\alpha = a_{\alpha_1} \cdots a_{\alpha_n}$ for each multi-index α.]

(ii) *There exists a point $x_0 \in W$ such that $D^\alpha f(x_0) = 0$ for each multi-index α.*

Then f vanishes identically throughout W.

3. Some approximation results

Throughout this section, we shall adopt the following notation. Suppose that E is a dilation space and let A and B be subsets of $R^\#$ and R^n, respectively. If $u \in E$, then we denote by $T^A_B[u]$ the closed vector subspace of E generated by the set of distributions $\{(u_b)^a : a \in A,\ b \in B\}$. In the case when A coincides with $R^\#$ and B coincides with R^n, we drop the superscript and subscript, and write $T[u]$ for $T^A_B[u]$.

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The results which we derived in the last section enable us to prove the following theorems.

3.1 THEOREM. *Let E be a barrelled dilation space. Let F be a subspace of E of type (Γ) and let $u \in F$ be such that the following condition is satisfied.*

(i) *$u \in M(a_k)$ for some sequence $(a_k)_{k=1}^{\infty}$ of positive numbers such that for each integer $m \geqq 0$, the sequence $(a_{m+k}^{1/k})_{k=1}^{\infty}$ is monotonic non-increasing and the series $\sum_{k=1}^{\infty} a_{m+k}^{1/k}$ diverges.*

In the above circumstances, the following assertion is true: If H is a closed vector subspace of E such that

(ii) *$j^{\beta} D^{\alpha} u \in H$ for each pair of multi-indices α and β such that $\beta \leqq \alpha$ then $H \supset T[u]$.*

PROOF. Our proof will be complete if we succeed in showing that $H \supset T[u]$ whenever H is a closed vector subspace of E which satisfies condition (ii) in the statement of Theorem 3.1; and according to the Hahn-Banach theorem, this is equivalent to showing that

$$\langle (u_y)^x, v \rangle = 0 \quad \text{for all} \quad x \in R^{\#} \quad \text{and all} \quad y \in R^n \tag{3.1}$$

whenever $v \in E'$ is such that

$$\langle w, v \rangle = 0 \quad \text{for all} \quad w \in H. \tag{3.2}$$

Thus, suppose that H is a closed vector subspace of E which satisfies condition (ii) above; and suppose that $v \in E'$ is such that (3.2) holds. Then

$$\langle j^{\beta} D^{\alpha} u, v \rangle = 0 \tag{3.3}$$

for each pair of multi-indices α and β such that $\beta \leqq \alpha$. Let γ be an arbitrary multi-index. Then from Lemma 2.5(b) it follows that

$$\begin{aligned} D^{\alpha}[(D^{\gamma} u) \nabla v](1) &= \sum_{\beta \leqq \alpha} C_{\beta}^{\alpha} (j^{\beta} D^{\gamma+\beta} u) \nabla v(1) \\ &= \sum_{\beta \leqq \alpha} C_{\beta}^{\alpha} \cdot \langle j^{\beta} D^{\gamma+\beta} u, v \rangle \\ &= 0 \end{aligned} \tag{3.4}$$

the last equality being a consequence of relation (3.3). Next we notice that Lemma 2.7(b) ensures that if K is a compact subset of $R^{\#}$, then there exist constants $m' > 0$ and $\rho > 0$ (depending on K) such that

$$|D^{\alpha}[(D^{\gamma} u) \nabla v](x)| \leqq m' \cdot \rho^{|\alpha|} / a_{\gamma+\alpha} \quad \text{for all} \quad x \in K \tag{3.5}$$

holds simultaneously for all multi-indices $\alpha \geqq 0$. Equipped with the knowledge that (3.4) and (3.5) hold (and bearing in mind the hypotheses about the sequence $(a_k)_{k=1}^{\infty}$) we may turn to Lemma 2.8 and deduce that

$$(3.6) \qquad (D^{\gamma}u) \bigtriangledown v(x) = 0 \quad \text{for all} \quad x \in R^{\#}.$$

Relation (3.6) holds for each multi-index γ. Now consider an arbitrary, but fixed, $x \in R^{\#}$. If we refer to Lemma 2.5(a) and Theorem 2.2(b) in Harasymiv [3], we easily verify that for each multi-index γ

$$(3.7) \qquad \begin{aligned} D^{\gamma}(u \circledast v^{x^{-1}})(0) &= (D^{\gamma}u) \circledast v^{x^{-1}}(0) \\ &= \langle D^{\gamma}u, v^{x^{-1}} \rangle \\ &= |j(x)| \cdot \langle (D^{\gamma}u)^{x}, v \rangle \\ &= |j(x)| \cdot (D^{\gamma}u) \bigtriangledown v(x) \\ &= 0 \end{aligned}$$

the last equality following immediately from (3.6). Next, we appeal to Lemma 2.7(a) to assure ourselves that if K is a compact subset of R^n, then there exists a constant $m > 0$ (depending on K) such that

$$(3.8) \qquad |D^{\gamma}(u \circledast v^{x^{-1}})(y)| \leqq m/a_{\gamma} \quad \text{for all} \quad y \in K$$

simultaneously for all multi-indices $\gamma \geqq 0$. Relations (3.7) and (3.8) allow us to appeal to Lemma 2.8 and find that

$$(3.9) \qquad u \circledast v^{x^{-1}}(y) = 0 \quad \text{for all} \quad y \in R^n.$$

Now, the point $x \in R^{\#}$ which figures in (3.9) was arbitrarily chosen; hence relation (3.9) is easily seen to entail that for each $x \in R^{\#}$ and each $y \in R^n$

$$\begin{aligned} \langle (u_y)^x, v \rangle &= |1/j(x)| \cdot \langle u_y, v^{x^{-1}} \rangle \\ &= |1/j(x)| \cdot u \circledast v^{x^{-1}}(y) \\ &= 0. \end{aligned}$$

This establishes (3.1) and so completes the proof of the theorem.

3.2 Corollary. *Let E be a barrelled dilation space. Suppose that $\phi \in \boldsymbol{S}(R^n)$ is such that the set $\{(1/\alpha!)j^{\beta}D^{\alpha}\phi : \beta \leqq \alpha, |\alpha| = 1, 2, \cdots\}$ is weakly bounded in E. Then the closed vector subspace of E generated by the set of functions $\{j^{\beta}D^{\alpha}\phi : \beta \leqq \alpha, |\alpha| = 1, 2, \cdots\}$ contains the whole of $T[\phi]$.*

Proof. We recall that $\boldsymbol{S}(R^n)$ is a subspace of E of type (Γ); and the boundedness of the set $\{(1/\alpha!)j^{\beta}D^{\alpha}\phi : \beta \leqq \alpha, |\alpha| = 1, 2, \cdots\}$ *entails that* $\phi \in M(1/k!)$.

3.3 Theorem. *Let E be a barrelled dilation space. Let F be a subspace of E of type (Γ) and let $u \in F$ be such that the following condition is satisfied.*

$u \in M(a_k)$ for some sequence $(a_k)_{k=1}^{\infty}$ of positive numbers such that the sequence $(a_k^{1/k})_{k=1}^{\infty}$ is monotonic non-increasing and the series $\sum_{k=1}^{\infty} a_k^{1/k}$ diverges.

In the above circumstances, the following assertion is true: If A is a non-meagre subset of $R^{\#}$ and B is a non-meagre subset of R^n, then $T_B^A[u] = T[u]$.

PROOF. It is sufficient to show that $T^A_B[u] \supset T[u]$. Thus, suppose that $v \in E'$ is such that

$$(3.10) \qquad \langle (u_b)^a, v\rangle = 0 \quad \text{for all} \quad a \in A \quad \text{and all} \quad b \in B.$$

Now consider a fixed $a \in A$. In view of Theorem 2.2(b) in Harasymiv [3], relation (3.10) is easily seen to entail that $u \circledast v^{a^{-1}}(b) = 0$ for all $b \in B$. Since B is a non-meagre subset of R^n and the function $u \circledast v^{a^{-1}}$ is continuous, it follows that $u \circledast v^{a^{-1}}$ must vanish on some non-void open subset W of R^n. Hence there exists a point $y_0 \in W$ such that

$$(3.11) \qquad D^\alpha(u \circledast v^{a^{-1}})(y_0) = 0 \quad \text{for all multi-indices} \quad \gamma \geqq 0.$$

Secondly, we observe that if K is a compact subset of R^n, then Lemma 2.7(a) implies the existence of a constant $m > 0$ (depending on K) such that

$$(3.12) \qquad |D^\alpha(u \circledast v^{a^{-1}})(y)| \leqq m/a_\alpha \quad \text{for all} \quad y \in K$$

simultaneously for all multi-indices $\alpha \geqq 0$. In view of (3.11) and (3.12), we may apply Lemma 2.8 and deduce that

$$(3.13) \qquad (u \circledast v^{a^{-1}})(y) = 0 \quad \text{for all} \quad y \in R^n.$$

Now from (3.13) and Theorem 2.4(b) in Harasymiv [3] it follows that $u^a \circledast v(y) = 0$ for all $y \in R^n$; whence (since the point $a \in A$ is arbitrary) we infer that

$$(3.14) \qquad \langle u^a, v_y\rangle = 0 \quad \text{for all} \quad a \in A \quad \text{and all} \quad y \in R^n.$$

Choose an arbitrary, but fixed $y \in R^n$. Relation (3.14) asserts that the continuous function $u \bigtriangledown v_y$ vanishes on the non-meagre subset A of $R^{\#}$. If we now use reasoning similar to that which led to relation (3.11), we deduce the existence of a point $x_0 \in R^{\#}$ such that

$$(3.15) \qquad D^\alpha(u \bigtriangledown v_y)(x_0) = 0 \quad \text{for all multi-indices} \quad \alpha \geqq 0.$$

Moreover, Lemma 2.7(b) asserts that corresponding to each compact set $K \subset R^{\#}$, there exist constants $m' > 0$ and $\rho > 0$ (depending on K) such that the relations

$$(3.16) \qquad |D^\alpha(u \bigtriangledown v_y)(x)| \leqq m' \cdot \rho^{|\alpha|}/a_\alpha \quad \text{for all} \quad x \in K$$

hold simultaneously for all multi-indices $\alpha \geqq 0$. In view of (3.15) and (3.16), Lemma 2.8 now tells us that $u \bigtriangledown v_y(x) = 0$ for all $x \in R^{\#}$. Since $y \in R^n$ was arbitrarily chosen, it is now evident that

$$(3.17) \qquad \langle (u_y)^x, v\rangle = 0 \quad \text{for all} \quad x \in R^{\#} \quad \text{and all} \quad y \in R^n.$$

We have therefore shown that (3.17) holds whenever $v \in E'$ satisfies (3.10). An easy application of the Hahn-Banach theorem now shows that $T^A_B[u] \supset T[u]$; hence $T^A_B[u] = T[u]$.

3.4. COROLLARY. *Suppose that E is a barrelled dilation space. Let $\phi \in \boldsymbol{S}(R^n)$ be such that the set $\{(1/\alpha!)j^\beta D^\alpha \phi : \beta \leqq \alpha, |\alpha| = 1, 2, \cdots\}$ is weakly bounded in E. Then $T^A_B[\phi] = T[\phi]$ whenever A is a non-meagre subset of $R^{\#}$ and B is a non-meagre subset of R^n.*

REMARK. Suppose that $n = 1$, so that R^n reduces to the real line R. Let E be a barrelled dilation space of distributions on R, and suppose that $u \in E$ satisfies the conditions of Theorem 3.3. Since the dual of any admissible space on R contains $\boldsymbol{D}(R)$, Lemma 2.7 and relation (2.1) (together with the hypotheses about the sequence $(a_k)_{k=1}^{\infty}$ in Theorem 3.3) entail that $u * \phi$ is a quasi-analytic function (in the sense of Mandelbrojt [6]) for each $\phi \in \boldsymbol{D}(R)$. An argument similar to that used to prove Théorème XXIV in Chapitre VI of Schwartz [8] now shows that u itself must be a quasi-analytic function.

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Department of Pure Mathematics
University of Sydney