A TAUBERIAN THEOREM AND ANALOGUES OF THE PRIME NUMBER THEOREM

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1. Introduction. In 1945 Ingham **(3)** proved the following Tauberian theorem: if f is a non-decreasing, non-negative function on $[1, \infty)$ and

$$(1) \qquad \sum_{n<x} f(xn^{-1}) = cx \log x + c'x + o(x), \qquad \text{as } x \to \infty,$$

then $f(x) \sim cx$. His proof is based on the non-vanishing of the Riemann zeta-function, $\zeta(s)$, on the line $\Re(s) = 1$, and uses Pitt's form of Wiener's Tauberian theorem; **(**see, e.g., **5**, Theorem 109, p. 211**)**. By modifying Ingham's proof to take account of suitable weighting functions $\alpha(n)$, I can deduce (Theorem 1) the "fine" behaviour of a function f if its "gross" behaviour is known, and if $\sum_{n<x} \alpha(n)f(xn^{-1})$ has an estimate similar to the right-hand side of (1). In the proof of this theorem I use a modified zeta-function, $\zeta_\alpha(s)$, which for $\Re(s) > 1$ has the Dirichlet series representation

$$\zeta_\alpha(s) = \sum_1^\infty \alpha(n)n^{-s}.$$

The prime number theorem without error term can be stated in many equivalent forms, for example:

$$\sum_{n<x} \mu(n) = M(x) = o(x)$$

and

$$\sum_{n<x} \Lambda(n) = \Psi(x) \sim x,$$

where μ, Λ are the Möbius and von Mangoldt functions respectively. To obtain the analogues of these results I use properties of the Dirichlet convolution

$$f * g(n) = \sum_{d|n} f(d)g(nd^{-1})$$

of the arithmetic functions f, g, as follows. Let α be an arithmetic function (i.e. a function from the positive integers to the reals) such that $\alpha(1) \neq 0$. Define μ_α, Λ_α by

$$(2) \qquad (\mu_\alpha * \alpha)(n) = \delta(n) \qquad \text{for all } n \geqslant 1,$$

$$(3) \qquad (\Lambda_\alpha * \alpha)(n) = \alpha(n) \log n \qquad \text{for all } n \geqslant 1,$$

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where $\delta(1) = 1$, $\delta(n) = 0$ for all $n > 1$. μ_α, Λ_α can equally well be thought of as the coefficients of the formal Dirichlet series $1/\zeta_\alpha$, $\zeta'_\alpha/\zeta_\alpha$.

For all $x > 0$ define M_α, ψ_α by

$$M_\alpha(x) = \sum_{n<x} \mu_\alpha(n), \tag{4}$$

$$\Psi_\alpha(x) = \sum_{n<x} \Lambda_\alpha(n). \tag{5}$$

In Theorems 2 and 3 I state sufficient conditions under which $M_\alpha(x) = o(x)$, and $\psi_\alpha(x) \sim x$. These results are deduced (as are the results for M and ψ in Ingham's paper) from Theorem 1, and from the easily verified identities

$$\sum_{n<x} \alpha(n) M_\alpha(xn^{-1}) = 1 \qquad \text{for all } x > 1, \tag{6}$$

$$\sum_{n<x} \alpha(n) \Psi_\alpha(xn^{-1}) = \sum_{k<x} \alpha(k) \log k, \qquad \text{for all } x > 1. \tag{7}$$

I shall now briefly outline the organization of the paper. In §2 I define what is meant by a suitable weighting function α, and I state the three theorems which are proved in §§3, 4, 5 respectively. The final section (§6) is devoted to examples and some concluding remarks. The notation throughout is standard, in particular I use O, o, $\sim$ to refer to behaviour as $x \to \infty$.

2. Statement of results. Let α be an arithmetical function. For $x > 0$ put $A(x) = \sum_{n<x} \alpha(n)$. (Thus $A(x) = 0$ for $x \leqslant 1$.)

Definition. α is *admissible* if

I. $A(x) \sim ax$, where $a > 0$.

Put $R(x) = A(x) - ax$ for $x > 1$, and $R(x) = 0$ for $x \leqslant 1$.

II. The function $x^{-s}R(x) \in L^1(0, \infty)$ for all s in an open connected subset of $\mathbf{C}$ containing $\Re(s) \geqslant 1$ (i.e.

$$\int_0^\infty x^{-s} R(x) x^{-1} dx$$

is finite in such a domain). Moreover, we require that the above integral represent a function holomorphic in a domain containing $\Re(s) \geqslant 1$.

III. Put

$$\zeta_\alpha(s) = as(s-1)^{-1} + s \int_0^\infty x^{-s} R(x) x^{-1} dx.$$

We require that $\zeta_\alpha(1 + it) \neq 0$ for $t \in \mathbf{R}$. Note that for $\Re(s) > 1$,

$$\zeta_\alpha(s) = s \int_1^\infty x^{-s-1} A(x) dx$$

and so

$$\zeta_\alpha(s) = \sum_1^\infty \alpha(n) n^{-s}$$

by a routine summation.

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THEOREM 1. *Suppose that α is admissible. If f is a real-valued function on $(1, \infty)$ satisfying*

(i) *$f_1(x) = x^{-1}f(x)$ when $x > 1$, and 0 otherwise, is bounded and slowly decreasing on $(0, \infty)$,*

(ii) *$F(x) = \sum_{n<x} \alpha(n)f(xn^{-1}) = cx \log x + c'x + o(x)$, where c, c' are constants, then $f(x) \sim ca^{-1}x$.*

Remark. If f is non-negative and non-decreasing on $(1, \infty)$, and if furthermore $f(x) = O(x)$ then f_1 as defined above is bounded and slowly decreasing. (When $\alpha \equiv 1$ the fact that $f(x) = O(x)$ can be deduced from condition (ii) and the non-decreasing of f.)

THEOREM 2. *Suppose that α is admissible, and that $\alpha(1) \neq 0$. Let μ_α be defined by (2); and assume that $M_\alpha(x)$ (defined in (4)) is $O(x)$. If there is a function β with $B(x) = \sum_{n<x} \beta(n) \sim bx$ for some $b \geqslant 0$, and*

$$\sum_{n<x} \alpha * \beta(n) = abx \log x + b'x + o(x),$$

such that $\mu_\alpha(n) + K\beta(n) \geqslant 0$ for all $n \geqslant 1$ and a fixed K, then $M_\alpha(x) = o(x)$.

COROLLARY. *Let the hypothesis on α be as above. Assume in addition that $R(x) = O(x^u)$ for some $0 \leqslant u < 1$. If either $\mu_\alpha(n) = O(1)$ or $\alpha(n) \geqslant 0$ for all n and $\mu_\alpha(n) = O(\alpha(n))$, then $M_\alpha(x) = o(x)$.*

Remark. In applications we usually have $A(x) = ax + O(x^u)$ with $0 \leqslant u < 1$. In the cases detailed above we choose $\beta = 1$, $\beta = \alpha$ respectively, and apply Theorem 2.

THEOREM 3. *Suppose that α is admissible and that $\alpha(1) \neq 0$. Suppose further that $R(x) = o(x/\log x)$. Let Λ_α, Ψ_α be defined by (3), (5) respectively. If Ψ_α satisfies the hypothesis on f in Theorem 1 then $\Psi_\alpha(x) \sim x$.*

Remark. It is not difficult to give a partial converse of this result (cf. **1**, Theorem 6, 23), namely: assume α has all the properties of an admissible function except that no hypothesis is made about the non-vanishing of ζ_α. Then we have: if $\Psi_\alpha(x) \sim x$, then $\zeta_\alpha(1 + it) \neq 0$ for all $t \in \mathbf{R}$.

3. Proof of Theorem 1. We note first the trivial (i.e. purely formal) identity:

$$\int_1^x A(xv^{-1})f(v)v^{-1}dv = \int_1^x y^{-1}F(y)dy.$$

Substituting our estimate from (ii) for $F(y)$ into the right-hand side, and dividing both sides by x, we obtain

$$x^{-1} \int_1^x A(xv^{-1})f(v)v^{-1}dv = c \log x + (c' - c) + o(1).$$

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Thus, after noting that $A(xv^{-1}) = 0$ for all $v \geqslant x$, we have

$$\int_0^\infty x^{-1}vA(xv^{-1})f_1(v)v^{-1}dv = c \log x + (c' - c) + o(1). \tag{8}$$

The left-hand side of (8) is now in the form of a convolution over the topological group formed by the positive reals under multiplication, with Haar measure $v^{-1}dv$. We want to transform (8) into a form to which Pitt's theorem can be applied. To this end define A_1 for all $x > 0$ by

$$xA_1(x) = 2A(x) - r_1A(xr_1^{-1}) - r_2A(xr_2^{-1})$$

where $r_1 > 1, r_2 > 1$ are to be restricted later. It is clear that for $x > \max(r_1, r_2)$ we can replace A by R in the definition of A_1; thus $A_1 \in L^1(0, \infty)$ since $x^{-1}R(x) \in L^1(0, \infty)$ by the admissibility of α. It is straightforward to check that

$$\int_0^\infty A_1(xv^{-1})f_1(v)v^{-1}dv = c \log(r_1 r_2) + o(1). \tag{9}$$

The manipulations performed so far have depended (as far as f is concerned) only on the fact that the weighted sum F of f has a certain estimate. The function $g(x) = ca^{-1}x$ has a weighted sum which obeys a similar law of growth, but with $c + ca'a^{-1}$ in place of c', where

$$a' = \int_0^\infty x^{-1}R(x)x^{-1}dx = \int_1^\infty x^{-2}(A(x) - ax)dx.$$

Thus by replacing f by g in (9) we can evaluate the right-hand side, to obtain

$$\int_0^\infty A_1(xv^{-1})f_1(v)v^{-1}dv = ca^{-1} \int_0^\infty A_1(v)dv + o(1). \tag{10}$$

We next discuss the Fourier transform of A_1. To do so, we consider the Laplace transform of A for $\Re(s) > 1$. We have

$$\int_0^\infty v^{-s}A(v)v^{-1}dv = s^{-1}\zeta_\alpha(s)$$

by the definition of ζ_α. Thus for $\Re(s) > 0$ we have

$$\begin{aligned}\int_0^\infty v^{-s}A_1(v)v^{-1}dv &= (2 - r_1^{-s} - r_2^{-s})(1 + s)^{-1}\zeta_\alpha(1 + s)\\ &= T(s) \text{ say.}\end{aligned}$$

Both sides of this equation represent functions holomorphic in some domain of $\mathbf{C}$ containing the half-plane $\Re(s) \geqslant 0$ (by our assumption on α), and so equality still holds for $\Re(s) = 0$, with $T(0) = a \log r_1 r_2$ as the removable singularity of the right-hand side. So

$$\hat{A}_1(t) = \int_0^\infty v^{-it}A_1(v)v^{-1}dv = T(it).$$

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Now $T(0) = a \log r_1 r_2 \neq 0$ since $r_1 r_2 > 1$, $a > 0$; and $T(it) = 0$ for $t \neq 0$ if, and only if $2 - r_1^{-it} - r_2^{-it} = 0$ (since $\zeta_\alpha(1 + it) \neq 0$ for all t). To ensure the impossibility of this we choose r_1, r_2 such that $(\log r_1)/(\log r_2)$ is irrational. Thus $\hat{A}_1(t) \neq 0$ for all $t \in \mathbf{R}$ and we can apply Pitt's theorem to (10) (since f_1 is bounded and slowly decreasing by hypothesis) to obtain the result that

$$\lim_{x\to\infty} f_1(x) = ca^{-1}, \quad \text{as desired.}$$

4. Proof of Theorem 2. Let β, K be chosen as in the hypothesis of the theorem. Put

$$G(x) = \sum_{n<x} (\mu_\alpha(n) + K\beta(n)).$$

Then $G(x) = O(x)$, and G is non-negative and non-decreasing; thus by our remark in §2 following Theorem 1 we see that $G_1(x) = x^{-1}G(x)$ when $x > 1$ and 0 otherwise is bounded and slowly decreasing on $(0, \infty)$. Moreover

$$\sum_{n<x} \alpha(n)G(xn^{-1}) = \sum_{n<x} \alpha(n)M_\alpha(xn^{-1}) + K \sum_{n<x} \alpha*\beta(n)$$
$$= 1 + Kabx \log x + Kb'x + o(x)$$

for all $x > 1$. Application of Theorem 1 to G now gives us the result that $G(x) \sim Kbx$; but

$$G(x) = M_\alpha(x) + KB(x) \sim M_\alpha(x) + Kbx$$

and so $M_\alpha(x) = o(x)$.

The proof of the corollary is immediate upon noting that

$$\sum_{n<x} \alpha*1(n) = ax \log x + (a\gamma + a')x + O(x^{(1+u)/2}),$$

and

$$\sum_{n<x} \alpha*\alpha(n) = a^2 x \log x + (a^2 + 2aa')x + O(x^{(1+u)/2}).$$

5. Proof of Theorem 3. Using the identity (7), we have

$$\sum_{n<x} \alpha(n)\Psi_\alpha(xn^{-1}) = \sum_{n<x} \alpha(n) \log n.$$

We can estimate the right-hand side (cf. **2**, Theorem 421) to obtain

$$\sum_{n<x} \alpha(n)\Psi_\alpha(xn^{-1}) = ax \log x - ax + O(R(x) \log x) + o(x),$$

which by our hypothesis on R gives us (ii) of Theorem 1 for Ψ_α. Since α is admissible, and Ψ_α satisfies requirement (i) on f of Theorem 1, we deduce that $\Psi_\alpha(x) \sim aa^{-1}x = x$, as desired.

6. Examples. The function $\alpha(n) = |\mu(n)|$ is admissible since

$$\zeta_\alpha(s) = \zeta(s)/\zeta(2s)$$

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for $\Re(s) > 1/2$; and $\mu_\alpha(n) = \lambda(n)$ is the Liouville function. Clearly $\lambda(n) = O(1)$; hence by Theorem 2 we have

$$\sum_{n<x} \lambda(n) = o(x),$$

which is, of course, a well-known corollary of the prime number theorem (see, e.g., **4**, II, §167). By using $\alpha(n) = \chi_0(n)$, where χ_0 is the principal character mod k for some $k > 1$, we can deduce that

$$\sum_{\substack{n<x \\ (n,k)=1}} \mu(n) = o(x) \quad \text{and} \quad \sum_{\substack{n<x \\ (n,k)=1}} \Lambda(n) \sim x.$$

We can combine both these examples by putting $\alpha(n) = \chi_0(n)|\mu(n)|$ and deduce that

$$\sum_{\substack{n<x \\ (n,k)=1}} \lambda(n) = o(x).$$

As a final example is exhibited an α for which μ_α is unbounded, namely

$$\alpha(n) = n^{-1}\sigma(n),$$

where $\sigma(n)$ is the sum of the divisors of n. In this case

$$\mu_\alpha(p^k) = \begin{cases} -(1+p^{-1}) & \text{if } k = 1, \\ p^{-1} & \text{if } k = 2, \\ 0 & \text{if } k \geqslant 3, \end{cases}$$

and so $\mu_\alpha(n) = O(\alpha(n))$. By Theorem 2, we still have $M_\alpha(x) = o(x)$.

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References

1. R. Ayoub, *An introduction to the analytic theory of numbers* (Providence, 1963).

2. G. H. Hardy and E.M. Wright, *An introduction to the theory of numbers* (3rd ed.; Oxford, 1954).

3. A. E. Ingham, *Some Tauberian theorems connected with the prime number theorem*, J. London Math. Soc., *20* (1945), 171–180.

4. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen, I, II* (Leipzig, 1909).

5. D. V. Widder, *The Laplace transform* (Princeton, 1946).

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