BULL. AUSTRAL. MATH. SOC. 58F15, 54H20
VOL. 43 (1991) [225–232]

HYPERBOLIC FLOWS ARE TOPOLOGICALLY STABLE

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We show that any hyperbolic flow (X, π) on a metric space X is topologically stable by showing that it is expansive and has the chain-tracing property.

1. INTRODUCTION

In this paper we show that the following theorem:

THEOREM A. *Any hyperbolic flow (X, π) on a metric space X is topologically stable.*

This is an attempt to approach some problems of smooth dynamical systems theory from a non-differential point of view.

Let (X, π) be a flow on a connected metric space (X, d). For brevity we denote $xt = \pi(x, t)$ for all $x \in X$ and $t \in \mathbb{R}$. For a point x in X and a number $a > 0$, we define subsets of X:

$$W^+(x, a) = \{y \in X : d(xt, yt) \leqslant a \text{ for all } t \in \mathbb{R}^+\}$$

and

$$W^-(x, a) = \{y \in X : d(xt, yt) \leqslant a \text{ for all } t \in \mathbb{R}^-\}.$$

A flow (X, π) is called *hyperbolic* if there are positive constants a_0, b_0, c, r such that

(i) $W^+(x, a_0) = \{y \in X : d(xt, yt) \leqslant ce^{-rt} d(x, y) \text{ for all } t \in \mathbb{R}^+\}$,
$W^-(x, a_0) = \{y \in X : d(xt, yt) \leqslant ce^{rt} d(x, y) \text{ for all } t \in \mathbb{R}^-\}$;

(ii) for any $(x, y) \in D(b_0) = \{(x, y) \in X \times X : d(x, y) < b_0\}$, there exists a unique element $\langle x, y\rangle \in X$ such that

$$W^+(xv(x, y), a_0) \cap W^-(y, a_0) = \{\langle x, y\rangle\},$$

where $v: D(b_0) \to \mathbb{R}$ and $\langle\, ,\, \rangle: D(b_0) \to X$ are continuous maps.

Received 3 April 1990
The present studies were supported by the Basic Science Research Institute Program, Ministry of Education, Korea, 1989.

A flow (X, π) is said to be *expansive* if for any constant $a > 0$ there exists a constant $b > 0$ with the property that if for all $t \in \mathbf{R}$, $d(xt, yf(t)) < b$ for a pair of points $x, y \in X$ and a continuous map $f: \mathbf{R} \to \mathbf{R}$ with $f(0) = 0$, then $y = xt$, where $|t| \leqslant a$.

Let a and p be positive numbers. A sequence $(x_i, t_i)_{i=-m}^{n}$, $0 \leqslant m, n < \infty$, in $X \times \mathbf{R}$ is called an *(a, p)-chain* if $t_i \geqslant p$, $-m \leqslant i \leqslant n$ and $d(x_i t_i, x_{i+1}) < a$, $-m \leqslant i \leqslant n$.

Let $(x_i, t_i)_{i=-m}^{n}$ be an (a, p)-chain in $X \times \mathbf{R}$. We assume that

$$T(j, k) = \begin{cases} \sum_{i=j}^{k} t_i & \text{if } j < k \\ 0 & \text{if } j > k \end{cases}$$

as notation. For a point $x_0 \in X$ and t with $-T(-m, -1) \leqslant t \leqslant T(0, n)$, we define

$$x_0 * t = \begin{cases} x_{-j}(t + T(-j, -1)) & \text{if } -T(-j, -1) \leqslant t < -T(-j+1, -1), \\ x_k(t - T(0, k-1)) & \text{if } T(0, k-1) \leqslant t < T(0, k), \\ x_n t_n & \text{if } t = T(0, n). \end{cases}$$

For a number $b > 0$, an (a, p)-chain $(x_i, t_i)_{i=-m}^{n}$ in $X \times \mathbf{R}$ is *b-traced* if there exists a monotone increasing continuous map $f: [-T(-m, -1), T(0, n)] \to \mathbf{R}$ satisfying

(i) $f(0) = 0$,
(ii) $d(xf(t), x_0 * t) < b$ for all $t \in [-T(-m, -1), T(0, n)]$.

A flow (X, π) has a *chain tracing property with respect to $p > 0$* if for any $b > 0$ there is an $a > 0$ such that every (a, p)-chain is b-traced by some point in X. (X, π) has a *chain tracing property* if it has a chain tracing property with respect to every positive number. If (X, π) has a chain tracing property with respect to time 1, then it has a chain tracing property [**5**].

A flow (X, π) is called *topologically stable* if for any $a > 0$ there exists a $b > 0$ such that for every other flow (X, π') with $d(\pi_t, \pi_{t'}) = \sup_{x \in X} d(\pi_t(x), \pi_{t'}(x))$, where $\pi_t(x) = \pi(t, x)$, for all $t \in [0, 1]$, then there exists a continuous map $h: X \to X$ such that $d(h, \text{id}) < a$ and $h(\text{orbit of } \pi') \subseteq (\text{orbit of } \pi')$, where id is the identity homeomorphism.

Now, we list well-known results ([**1**] and [**5**]).

THEOREM B. (Bowen and Walters). *If a flow (X, π) is expansive, then all fixed points are isolated.*

THEOREM C. (Thomas). *Every continuous expansive flow without fixed points which has the chain tracing property is topologically stable.*

Then Theorem A follows from the following theorems.

Theorem D. *Any hyperbolic flow is expansive.*

Theorem E. *Any hyperbolic flow has the chain tracing property.*

Thus it suffices to prove Theorems D and E, and we need some lemmas in the next section to prove these theorems.

For any homeomorphism on a compact metric space, Ombach [3] showed further relations: pseudo-orbit tracing property, expansiveness and hyperbolicity.

Basic terminologies are followed from [4].

2. Two Lemmas

Lemma 1. *Let (X, π) be a hyperbolic flow. If, for any $a < a_0$, there exists a number $b > 0$ with $d(x, y) < b$, then*

(i) $|v(x, y)| \leqslant a$,
(ii) $W^+(xv(x, y), a) \cap W^-(y, a) = \{\langle x, y\rangle\}$.

Proof: Since $v(x, x) = 0$ and $\langle x, x\rangle = x$, there is a number $b < b_0$ such that $d(x, y) < b$ implies $|v(x, y)| \leqslant a$ and

$$d(x, \langle x, y\rangle) \leqslant a/2c,\ d(y, \langle x, y\rangle) \leqslant a/c$$

and

$$d(x, xv(x, y)) \leqslant a/2c$$

by the uniform continuity. Since $\langle x, y\rangle \in W^+(xv(x, y), a_0) \cap W^-(y, a_0)$, we have

$$d(x(v(x, y) + t), \langle x, y\rangle t) \leqslant ce^{-rt} d(xv(x, y), \langle x, y\rangle) \leqslant a,$$
$$d(y(-t), \langle x, y\rangle(-t)) \leqslant ce^{-rt} d(y, \langle x, y\rangle) \leqslant a$$

for all $t \in \mathbf{R}$. Thus

$$\langle x, y\rangle \in W^+(xv(x, y), a) \cap W^-(y, a) \subset W^+(xv(x, y), a_0) \cap W^-(y, a_0).$$

□

Another important property of hyperbolic flows is the following.

Lemma 2. *Let (X, π) be a hyperbolic flow and $a > 0$ be a constant. Suppose that there exists a constant $b > 0$ such that for all $t \in \mathbf{R}$,*

$$d(xt, y(t + f(t))) \leqslant b,$$

where $f: \mathbf{R} \to \mathbf{R}$ is a continuous map with $f(0) = 0$. Then we have

(i) $|v(x, y)| \leqslant a$,
(ii) $y = xv(x, y)$.

PROOF: We can choose $a < \min\{a_0/8,\, a_0/2c\}$ and

$$\max\{d(x,\, xt) : x \in X,\, |t| \leqslant 4a\} \leqslant a_0/8.$$

By Lemma 1, there is a constant $b > 0$ such that $d(x,\, y) \leqslant b$ implies

$$|v(x,\, y)| \leqslant a$$

and

$$W^+(xv(x,\, y),\, a) \cap W^-(y,\, a) = \{\langle x,\, y\rangle\}.$$

Let $v = v(x,\, y)$ and $z = \langle x,\, y\rangle$. Clearly $d(x,\, y) \leqslant b$ since $f(0) = 0$. Put

$$U = \{t \in \mathrm{R}^+ : |f(t)| \geqslant 3a \text{ or } d(yt,\, zt) \geqslant a_0/2\}$$

and

$$V = \{t \in \mathrm{R}^- : |f(t)| \geqslant 3a \text{ or } d(x(v+t),\, zt) \geqslant a_0/2\}.$$

There exists an $s = \min U$ if $U \neq \emptyset$. Moreover, $0 \notin U$ since $|f(0)| < 3a$ and $d(y,\, z) < a_0/2$. It follows that $s > 0$.

We claim that $d(y(s-t),\, z(s-t)) \leqslant a_0/2$ for all $t \in \mathrm{R}^+$. If $0 < t \leqslant s$, then $d(y(s-t),\, z(s-t)) < a_0/2$ since $0 \leqslant s - t < s$ and so $s - t \notin U$. Thus we have $d(ys,\, zs) \leqslant a_0/2$ if $t \to 0$. If $s < t$, then

$$d(y(s-t),\, z(s-t)) \leqslant ce^{r(s-t)} d(y,\, z) < a_0/2.$$

It is clear that $|f(s)| \leqslant 4a$. For all $t \in \mathrm{R}^+$, we have

$$\begin{aligned} &d(y(s+f(s)-t),\, z(s+f(s)-t)) \\ &\quad \leqslant d(y(s+f(s)-t),\, y(s-t)) + d(y(s-t),\, z(s-t)) \\ &\qquad + d(z(s-t),\, z(s+f(s)-t)) \\ &\quad < a_0. \end{aligned}$$

This means that $z(s+f(s)) \in W^-(y(s+f(s)),\, a_0)$. Also, $z(s+f(s)) \in W^+(x(s+f(s)+v),\, a_0)$ because

$$\begin{aligned} &d(x(s+f(s)+v+t),\, z(s+f(s)+t)) \\ &\quad \leqslant d(x(s+f(s)+v+t),\, x(s+v+t)) + d(x(s+v+t),\, z(s+t)) \\ &\qquad + d(z(s+t),\, z(s+f(s)+t)) \\ &\quad < a_0 \end{aligned}$$

for all $t \in \mathrm{R}^+$. Since $|f(s)+v| \leqslant |f(s)| + |v| \leqslant 4a$ and $d(xs,\, y(s+f(s))) \leqslant b$, we have $|v(xs,\, y(s+f(s)))| = |f(s)+v| \leqslant a$ and $\langle xs,\, y(s+f(s))\rangle = z(s+f(s))$. Furthermore, we have

$$\begin{aligned} d(ys,\, zs) &\leqslant d(ys,\, y(s+f(s))) + d(y(s+f(s)),\, z(s+f(s))) \\ &\qquad + d(z(s+f(s)),\, zs) \\ &< a_0/2 \end{aligned}$$

since $d(y(s+f(s)), z(s+f(s))) \leqslant a$ and $|f(s)| < |f(s)+v| + |v| \leqslant 2a$. This contradicts the fact that $s \in U$. Hence $U = \emptyset$. Also, we obtain $V = \emptyset$ by a similar method.

Now, let $A > 0$ be any number and $t \in \mathbf{R}^-$. When $t \geqslant -A$,

$$d(y(A+t), z(A+t)) \leqslant a_0/2$$

and
$$d(y(A+t), z(A+t)) \leqslant ce^{r(A+t)} d(y, z) < a_0/2$$

when $t \leqslant -A$. Therefore $zA \in W^-(yA, a_0/2)$. It follows that

$$d(y, z) = d((yA)(-A), (zA)(-A)) \leqslant ce^{-rA} d(yA, zA) \leqslant ca_0 e^{-rA}/2.$$

For any $t \in \mathbf{R}^+$, we have

$$d(x(v-A+t), z(-A+t)) \leqslant a_0/2$$

when $t \leqslant A$ and

$$d(x(v-A+t), z(-A+t)) \leqslant ce^{r(A-t)} d(xv, z) \leqslant a_0/2$$

when $t \geqslant A$. Thus $z(-A) \in W^+(x(v-A), a_0/2)$. This implies that

$$\begin{aligned} d(xz, v) = d(x(v-A), (z(-A))A) &\leqslant ce^{-rA} d(x(v-A), z(-A)) \\ &\leqslant ca_0 e^{-rA}/2. \end{aligned}$$

Consequently, we have

$$d(xv, y) \leqslant d(xv, z) + d(z, y) \leqslant ca_0 e^{-rA}$$

and hence $d(xv, y) = 0$ when $A \to \infty$. This completes the proof. □

3. Two Theorems

Theorem D. *Any hyperbolic flow (X, π) is expansive.*

Proof: For any $a > 0$, we can choose a number $b > 0$ by Lemma 2. We define $g: \mathbf{R} \to \mathbf{R}$ by $g(t) = f(t) - t$. Then we have $g(0) = 0$ and

$$d(xt, y(t+g(t))) = d(xt, yf(t)) < b.$$

Also, by Lemma 2, we have $|v(x, y)| \leqslant a$ and $y = xv(x, y)$. This means that (X, π) is expansive. □

Theorem E. *Any hyperbolic flow (X, π) has the chain tracing property.*

Proof: For any $a > 0$, there is a $p > 0$ such that $d(x, xt) < a/3$ for all $|t| \leqslant p$ and $x \in X$. Also, there is a $q_1 > 0$ such that $d(xf(t), yg(t)) < q_1$ for all $A \leqslant t \leqslant B$, $A < 0 < B$ and continuous maps $f, g\colon \mathrm{R} \to \mathrm{R}$ with $f(0) = 0 = g(0)$, implying $|f(t) - g(t)| < p/2$.

Putting $q = \min\{q_1/2, a/3\}$ there is a $b > 0$ such that for any $(b, 1)$-chain $(z_i, s_i)_{i=-m}^{n}$, $0 \leqslant m, n \leqslant \infty$ in $X \times \mathrm{R}$, there are a monotone increasing continuous map $g\colon [-S(-m, -1), S(0, n)] \to \mathrm{R}$ and a point $x \in X$ such that $g(0) = 0$,

$$3t/4 - S(-m, -1)/2 - 1 < g(t) < 5t/4 + S(-m, -1)/2 + 1,$$

and
$$d(zg(t), z_0 * t) < q$$

for all $t \in [-S(-m, -1), S(0, n)]$.

Now, let $(x_i, t_i)_{i=-\infty}^{\infty}$ be a $(b, 1)$-chain and $n_1 = 1$. We can choose $n_{k+1} > n_k$ so that

$$T(0, n_{k+1}) > 5T(0, n_k)/3 + 2/3.$$

Since $(x_i, t_i)_{i=-n_k}^{n_k}$ is also a $(b, 1)$-chain, there are $y_k \in X$ and a monotone increasing continuous function $g_k\colon [a_k, b_k] \to \mathrm{R}$ such that

$$3t/4 + a_k/2 - 1 < g_k(t) < 5t/4 - a_k/2 + 1,$$

where $a_k = -T(-n_k, -1)$ and $b_k = T(0, n_k)$, and

$$g(y_k g_k(t), x_0 * t) < q.$$

We may assume that $y_k \to x$ as $k \to \infty$. Since

$$d(y_k g_k(t), y_{k+1} g_{k+1}(t)) \leqslant d(y_k g_k(t), x_0 * t) + d(x_0 * t, y_{k+1} g_{k+1}(t)) < 2q$$

for all $t \in [a_k, b_k] \subset [a_{k+1}, b_{k+1}]$, we have $|g_k(t) - g_{k+1}(t)| < p/2$. Therefore

$$|g_k(a_k) - g_{k+1}(a_k)| < p/2 \text{ and } |g_k(b_k) - g_{k+1}(b_k)| < p/2.$$

Since

$$\begin{aligned} g_{k+1}(a_{k+1}) &< 5a_{k+1}/4 - a_{k+1}/2 + 1 \\ &< 3a_k/4 + a_k/2 - 1 < g_k(a_k) \end{aligned}$$

and
$$\begin{aligned} g_k(b_k) &< 5b_k/4 - a_k/2 + 1 \\ &< 3b_{k+1}/4 + a_{k+1}/2 - 1 < g_{k+1}(b_{k+1}), \end{aligned}$$

there exist monotone increasing continuous functions

$$f_k^-: [a_{k+1}, a_k] \to \mathbb{R} \text{ and } f_k^+: [b_k, b_{k+1}] \to \mathbb{R}$$

satisfying

$$f_k^-(a_{k+1}) = g_{k+1}(a_{k+1}), \quad f_k^-(a_k) = g_k(a_k),$$
$$f_k^+(b_k) = g_k(b_k), \qquad f_k^+(b_{k+1}) = g_{k+1}(b_{k+1}),$$
$$\left|f_k^-(t) - g_{k+1}(t)\right| < p/2, \quad a_{k+1} \leqslant t \leqslant a_k,$$
$$\left|f_k^+(t) - g_{k+1}(t)\right| < p/2, \quad b_k \leqslant t \leqslant b_{k+1}.$$

Now, if we define $f: \mathbb{R} \to \mathbb{R}$ by

$$f = g_1 \cup \left(\bigcup_{k=1}^{\infty} (f_k^- \cup f_k^+) \right),$$

then $f(0) = 0$ and it is monotone increasing continuous. For any $t \in \mathbb{R}$, there is an $i > k+1$ such that $d(y_i f(t), xf(t)) < a/3$ whenever $a_{k+1} \leqslant t \leqslant a_k$ since $y_k f(t) \to zf(t)$. Note that

$$\begin{aligned} |f(t) - g_i(t)| &= \left|f_k^-(t) - g_i(t)\right| \\ &\leqslant \left|f_k^-(t) - g_{k+1}(t)\right| + |g_{k+1}(t) - g_i(t)| \\ &< p. \end{aligned}$$

Therefore

$$\begin{aligned} d(xf(t), x_0 * t) &\leqslant d(xf(t), y_i f(t)) + d(y_i f(t), y_i g_i(t)) \\ &\qquad + d(y_i g_i(t), x_0 * t) \\ &< a. \end{aligned}$$

The case $b_k \leqslant t \leqslant b_{k+1}$, $d(xf(t), x_0 * t) < a$ follows in the same manner. It completes the proof. □

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