ON THE WEAKNESS OF SOME BOUNDARY COMPONENT

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1. Let D be a domain in the complex z-plane and γ be a boundary component of D consisting of a single point. The component γ is said to be weak if its image under any conformal mapping of D consists of a single point. If γ is not weak, then we say that γ is unstable (Sario [3], [4]).

Let $S_n (n = 1, 2, \ldots)$ be a sequence of slits being symmetric and orthogonal to the positive real axis of the complex z-plane and converging to the origin $O : z = 0$. We delete the set $\bigcup_{n=1}^{\infty} S_n \cup \{O\}$ from the z-plane and denote by D the resulting domain. In this note, we treat the weakness of the boundary component O of the domain D.

2. First we prove the following

LEMMA 1. *Consider two slits:* $x = a_j (> 0), |y| \leqq h_j \ (j = 1, 2), (a_2 < a_1)$ *which are symmetric and orthogonal to the positive real axis and satisfy the equality* $\frac{h_1}{a_1} = \frac{h_2}{a_2} = k$. *Construct a doubly connected domain B bounded by two circular arcs* $C_j : |z| = \sqrt{a_j^2 + h_j^2} \ \alpha \leqq, \arg z \leqq 2\pi - \alpha, (j = 1, 2)$, *where* $0 < \alpha = \tan^{-1} k \left(< \frac{\pi}{2} \right)$, *and slits* S_1, S_2. *Let* μ *be the module of B and* μ^* *be the module of the ring domain* $R : \sqrt{a_2^2 + h_2^2} < |z| < \sqrt{a_1^2 + h_1^2}$. *Then it holds*

$$\frac{1}{M(\alpha)} \mu^* \leqq \mu \leqq M(\alpha) \mu^*,$$

where $M(\alpha)$ *is a constant depending only on* α.

Proof. Let $z_j = a_j + ih_j$ and $z_j' = a_j - ih_j$ be two endpoints S_j $(j = 1, 2)$. We map the trapezoid $T : (z_2, z_2', z_1', z_1)$ onto the quadrilateral (z_2, z_2', z_1', z_1) bounded by two minor circular arcs $\widehat{z_2 z_2'}$ on $|z| = \sqrt{a_2^2 + h_2^2}$, $\widehat{z_1 z_1'}$ on $|z| = \sqrt{a_1^2 + h_1^2}$ and two rectilinear segments $\overline{z_2 z_1}$, $\overline{z_2' z_1'}$ under the topological mapping $\zeta(z) = \sqrt{(1+k^2)x^2 - y^2} + iy = \sqrt{x^2 \sec^2 \alpha - y^2} + iy$, $z = x + iy$. It is obvious that $|\zeta(z)| = \sqrt{1+k^2} x$.

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Now we put

$$p=\frac{1}{2}\left(\frac{\partial\zeta}{\partial x}-i\frac{\partial\zeta}{\partial y}\right),\qquad q=\frac{1}{2}\left(\frac{\partial\zeta}{\partial x}+i\frac{\partial\zeta}{\partial y}\right).$$

By an easy computation, we have

$$\frac{|p|+|q|}{|p|-|q|}\leqq\frac{(x\sec^2\alpha+\sqrt{x^2\sec^2\alpha-y^2})^2+y^2}{x\sec^2\alpha\sqrt{x^2\sec^2\alpha-y^2}}\leqq\frac{(\sec^2\alpha+\sec\alpha)^2+\tan^2\alpha}{\sec^2\alpha}$$
$$\leqq\frac{4\sec^4\alpha+\sec^2\alpha}{\sec^2\alpha}=4\sec^2\alpha+1.$$

If we put $4\sec^2\alpha+1=M(\alpha)$, then

$$\sup_{z\in T}\frac{|p|+|q|}{|p|-|q|}\leqq M(\alpha).$$

Hence $\zeta(z)$ is a quasiconformal mapping with bounded dilatation. Therefore, if we define

$$\zeta=\varphi(z)=\begin{cases}\zeta(z), & z\in T\\ z, & z\in B-T,\end{cases}$$

then $\zeta=\varphi(z)$ is a quasiconformal mapping of B onto R with bounded dilatation. Thus we have the required inequality

$$\frac{1}{M(\alpha)}\mu^*\leqq\mu\leqq M(\alpha)\mu^*.$$

3. Suppose that S_n $(n=1,2,\ldots)$ are segments : $x=a_n(>0)$, $|y|\leqq h_n$ satisfying $0<a_{n+1}<a_n$, $\lim_{n\to\infty}a_n=0$ and

$$h_n\leqq a_n\tan\alpha=h'_n$$

for some fixed α $\left(0<\alpha<\frac{\pi}{2}\right)$.

Let S'_n be a segments $x=a_n$, $|y|=h'_n$. Denote by D (or D') the domain obtained by deleting segments S_n(or S'_n) $(n=1,2,\ldots)$ and the origin $z=0$ from the complex z-plane. It is obvious that $D\supset D'$.

We construct doubly connected domains B_n $(n=1,2,\ldots)$ in D' bounded by S'_n, S'_{n+1} and by two circular arcs $C_j:|z|=\sqrt{a_j^2+h_j'^2}$, $\alpha\leqq$ arg $z\leqq2\pi-\alpha$, $(j=n,n+1)$. Evidently, $B_n\subset D$ and $B_n\cap B_m=\varphi$ if $n\neq m$. Let μ_n be the module of B_n. By Lemma 1, we have

$$\frac{1}{M(\alpha)}\mu_n^* \leqq \mu_n \leqq M(\alpha)\mu_n^*,$$

where $\mu_n^* = \log \dfrac{a_n}{a_{n+1}}$ is the module of the ring domain $\sqrt{a_{n+1}^2 + h_{n+1}'^2} < |z| < \sqrt{a_n^2 + h_n'^2}$. Hence it follows that

$$\sum_{n=1}^{\infty}\mu_n^* \leqq M(\alpha)\sum_{n=1}^{\infty}\mu_n.$$

Since $\lim_{n\to\infty} a_n = 0$, the left hand side of the above inequality is divergent. By Savage's criterion [5] we see that the origin O is a weak boundary component of D. Thus we obtain the following

THEOREM 1. *If S_n $(n = 1, 2, \ldots)$ are segments: $x = a_n(>0)$, $|y| \leqq h_n$ satisfying $0 < a_{n+1} < a_n$, $\lim_{n\to\infty} a_n = 0$ and*

$$(*)\qquad h_n \leqq a_n \tan \alpha = h_n'$$

for some fixed α $\left(0 < \alpha < \frac{\pi}{2}\right)$, then O is a weak boundary component of the domain obtained by deleting $\bigcup_{n=1}^{\infty} S_n \cup \{O\}$ from the z-plane.

4. Here we show that in the case when segments in our Theorem 1 do not satisfy the condition $(*)$ the origin O is not always weak.

First we prove the following

LEMMA 2. *Consider two slits $S_j : x = a_j(>0)$, $|y| \leqq h_j$ $(j = 1, 2)$, $(a_2 < a_1)$ which are symmetric and orthogonal to the real axis. Let Ω be the doubly connected domain obtained by deleting S_1 and S_2 from the z-plane and let Q be the rectangle: $(a_2 + ih,\ a_2 - ih,\ a_1 - ih,\ a_1 + ih)$, where $h = \mathrm{Min}(h_1, h_2)$. If μ is the module of Ω, then it holds*

$$\mu \leqq \frac{\pi(a_1 - a_2)}{h}.$$

Proof. We denote by $\{\gamma\}$ a family of rectifiable curves in Ω separating S_1 from S_2 and by $\{\gamma'\}$ a family whose elements consist of rectifiable curves joining the upper side $\overline{a_2 + ih,\ a_1 + ih}$ to the lower side $\overline{a_2 - ih,\ a_1 - ih}$ of Q in Q. It is obvious that each $\gamma \in \{\gamma\}$ contains a curve $\gamma' \in \{\gamma'\}$. Denoting by $\lambda\{\gamma\}$, $\lambda\{\gamma'\}$ the extremal lengths of these families in the sense of Ahlfors-Beurling [1], we get the following inequality:

$$\lambda\{\gamma'\} \leqq \lambda\{\gamma\}.$$

From the relation $\lambda\{\gamma\} = \frac{2\pi}{\mu}$ and $\lambda\{\gamma'\} = \frac{2h}{a_1 - a_2}$, we have

$$\mu \leqq \frac{\pi(a_1 - a_2)}{h}.$$

Now we denote by S_n $(n = 1, 2, \ldots)$ segments in the z-plane

$$x = \frac{1}{n}, \quad |y| \leqq h_n = c\left(\frac{1}{n-1}\right)^p, \qquad (0 < p < 1),$$

where c is a positive constant and $z = x + iy$. Let D be a domain obtained by deleting $\bigcup_{n=1}^{\infty} S_n \cup \{O\}$ from the z-plane and let B_j $(j = 1, 2, \ldots)$ be any sequence of doubly connected domains in D separating O from the infinity and converging to O. We suppose that B_{j+1} lies in a domain G_j which is a component, containing O, of the complementary sets of $\overline{B_j}$ with respect to the z-plane.

Let $S_{m(j)}$ be the segment such that, for any $n > m(j)$, $S_n \subset G_j$ and that $S_{m(j)} \not\subset G_j$. Then B_j separates S_n $(n > m(j))$ from $S_{m(j)}$.

Without loss of generality, we may assume that $\{B_j\}$ $(j = k_l + 1, \ldots, k_{l+1})$ are all the doubly connected domains separating $S_{m(k_l+1)} (= \cdots = S_{m(k_{l+1})})$ from $S_{m(k_l+1)+1}$, where $k_0 = 0$, $m(1) = 1$ and

$$m(k_{l+1}) < m(k_l + 1) + 1 \leqq m(k_{l+1} + 1).$$

Denote by Ω_l the domain obtained by deleting $S_{m(k_l+1)}$ and $S_{m(k_l+1)+1}$ from the z-plane. Then B_j $(j = k_l + 1, \ldots, k_{l+1})$ are contained in Ω_l. The well-known Teichmüller's inequality implies that

$$\sum_{j=k_l+1}^{k_{l+1}} \mu_j \leqq \mu_l^*,$$

where μ_j $(j = k_l + 1, \ldots, k_{l+1})$ are the moduli of B_j $(j = k_l + 1, \ldots, k_{l+1})$ and μ_l^* is that of Ω_l.

Thus, using Lemma 2, we obtain

$$\sum_{j=1}^{\infty} \mu_j = \sum_{l=0}^{\infty} \sum_{j=k_l+1}^{k_{l+1}} \mu_j \leqq \sum_{l=0}^{\infty} \mu_l^*$$

$$\leqq \pi \sum_{l=0}^{\infty} \frac{\frac{1}{m(k_l+1)} - \frac{1}{m(k_l+1)+1}}{h_{m(k_l+1)+1}} \leqq \pi \sum_{n=1}^{\infty} \frac{\frac{1}{n} - \frac{1}{n+1}}{h_{n+1}} \leqq \frac{\pi}{c} \sum_{n=1}^{\infty} n^{p-2}.$$

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Since $\sum_{n=1}^{\infty} n^{p-2}$ $(0<p<1)$ is convergent, we see that the series $\sum_{j=1}^{\infty}\mu_j$ is convergent for any sequence $\{B_j\}$. By using Oikawa's theorem [2], i.e., the converse of Savage's criterion, we have the following

THEOREM 2. *If* S_n $(n=1, 2, \ldots)$ *are segments:* $x=\frac{1}{n}$, $|y| \leqq c\left(\frac{1}{n-1}\right)^p$, $(0<p<1)$, *then the origin O is an unstable boundary component of the domain obtained by deleting* $\bigcup_{n=1}^{\infty} S_n \cup \{O\}$ *from the z-plane.*

Recently Oikawa has treated the case that the number of boundary components converging to the origin is not countable and obtained interesting results, some of which contain our results.

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